MOTIVIC MIRROR SYMMETRY FOR HIGGS BUNDLES

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Abstract

We prove that the (orbifold) motives of the moduli spaces of SL_n and PGL_n-Higgs bundles of coprime rank and degree on a smooth projective curve over an algebraically closed field of characteristic zero are isomorphic in Voevodsky's triangulated category of motives. The equality of (orbifold) Hodge numbers of these moduli spaces was conjectured by Hausel and Thaddeus and recently proven by Groechenig, Ziegler and Wyss via p-adic integration and then by Maulik and Shen using the decomposition theorem, an analysis of the supports of D-twisted Hitchin fibrations and vanishing cycles. Our proof combines the geometric ideas of Maulik and Shen with the conservativity of the Betti realisation on abelian motives; to apply the latter, we prove that the relevant motives are abelian. In particular, we prove that the motive of the SL_n-Higgs moduli space is abelian, building on our previous work in the GL_n-case.

Contents

1. Introduction

Let C be a smooth projective geometrically connected genus g curve over a field k. The moduli space $\mathcal{M} := \mathcal{M}_{n,d}(C)$ of stable Higgs bundles $(E, \Phi : E \to E \otimes \omega_C)$ of coprime rank n and degree d is a smooth quasi-projective variety of dimension $2(n^2(g-1)+1)$. Taking the characteristic polynomial of the Higgs field Φ defines the Hitchin fibration, a morphism $h : \mathcal{M} \to \mathcal{A}$ to an affine space $\mathcal{A}$ called the Hitchin base [37]. The variety $\mathcal{M}$ admits an algebraic symplectic form, and the morphism h is a proper Lagrangian fibration with respect to that symplectic structure; the generic fibres of h are torsors under the Jacobians of the corresponding spectral curves. Over the complex numbers, $\mathcal{M}$ is a non-compact hyperkähler manifold and is diffeomorphic to both the moduli space of holomorphic flat connections and the character variety of topological local systems on C by the non-abelian Hodge correspondence [62] (more precisely to variants of those moduli spaces taking into account the non-zero degree d).

More generally, for a reductive group G/k, there is a notion of G-Higgs bundles and corresponding moduli spaces $\mathcal{M}_G$ of semistable G-Higgs bundles, which in the case of $G = \mathrm{GL}_n$ coincides with $\mathcal{M}$. These G-Higgs moduli spaces are algebraic symplectic and also come with (proper Lagrangian) Hitchin fibrations; when $k = \mathbb{C}$, the non-abelian Hodge correspondence extends to G-Higgs bundles. Given two Langlands dual reductive groups G and ${}^LG$, the corresponding Hitchin fibrations have (almost) canonically isomorphic bases and it is expected that $\mathcal{M}_G$ and $\mathcal{M}_{{}^LG}$ are closely related via these Hitchin fibrations. This relationship can be understood heuristically as a "limit" of the geometric Langlands correspondence and also as a form of mirror symmetry. The first concrete statement in that direction is a relationship between the generic fibres of the two fibrations which has been established by Hausel and Thaddeus [35] in the case of $G = \mathrm{SL}_n$ and by Donagi-Pantev [24] and Derryberry [23] in general: these generic fibres are torsors under dual abelian varieties.

1.1. Mirror symmetry for SL and PGL-Higgs bundles.

In this paper, we study G-Higgs bundles for the Langlands dual groups SL_n and PGL_n. We fix a degree d coprime to n and choose a degree d line bundle L and let $\mathcal{M}_L := \mathcal{M}_{n,L}(C)$ denote the moduli space of stable Higgs bundles of rank n with determinant L and trace-free Higgs field; we refer to these as SL_n-Higgs bundles[1]. The SL_n-Higgs moduli space $\mathcal{M}_L$ is a smooth closed subvariety of $\mathcal{M} = \mathcal{M}_{n,d}(C)$, which can be realised as a fibre of the map $(\det, \mathrm{tr}) : \mathcal{M}_{n,d}(C) \to \mathcal{M}_{1,d}(C)$. The Jacobian $\mathrm{Jac}(C)$ acts on $\mathcal{M}$ by tensorisation and the action of the n-torsion subgroup $\Gamma := \mathrm{Jac}(C)[n]$ restricts to $\mathcal{M}_L$. The corresponding PGL_n-Higgs moduli space $\overline{\mathcal{M}}$ is singular, but smooth as an orbifold: we can identify it with the following quotients

$$\overline{\mathcal{M}} \simeq [\mathcal{M}_L / \Gamma] \simeq [\mathcal{M} / T^* \mathrm{Jac}(C)].$$

Consequently $\overline{\mathcal{M}}$ can be viewed as a smooth Deligne–Mumford stack, which inherits a μ_n-gerbe δ_L from the μ_n-gerbe on $\mathcal{M}_L$ coming from its corresponding moduli stack [35, §3].

The corresponding Hitchin fibrations for these SL-Higgs and PGL-Higgs moduli spaces have canonically isomorphic Hitchin bases $\mathcal{A}_L \simeq \overline{\mathcal{A}}$ and the generic fibres of the SL-Hitchin (resp. PGL-Hitchin) fibration are torsors under Prym varieties (resp. Γ-quotients of Prym varieties) [37].

Over $k = \mathbb{C}$, Hausel and Thaddeus [35] predicted a"topological mirror symmetry" relation between the E-polynomial of the SL_n-Higgs moduli space and the stringy E-polynomial of the

[1] Strictly speaking, these Higgs bundles should not quite be called SL-Higgs bundles, as specialising the general definition of G-Higgs bundle to $G = \mathrm{SL}_n$ forces the determinant to be trivial and $d = 0$ (and leads to singular moduli spaces). This variant of SL_n-Higgs bundles turns out to be the right thing to consider in the context of the Hausel-Thaddeus conjecture.

PGL_n-Higgs moduli space, which encodes the twisted orbifold Hodge numbers with respect to the gerbe δ_L; as we will see below this stringy E-polynomial has a more concrete description in terms of the Γ-action on $\mathcal{M}_L$. The conjecture of Hausel and Thaddeus was proved by Groechenig, Wyss and Ziegler [32] using p-adic integration. Recently, Maulik and Shen [50] upgraded the agreement of (stringy) E-polynomials to an agreement of (orbifold) Hodge structures (see (3) below) using perverse sheaves, the decomposition theorem, support theorems for Hitchin fibrations and vanishing cycles.

In this paper, we build on the ideas and techniques of [50] to prove a motivic version of mirror symmetry, Theorem 1.1, relating the (orbifold) Voevodsky motives of the SL_n-Higgs and the PGL_n-Higgs moduli spaces. The Voevodsky motive $M(X)$ of a smooth k-variety X with coefficient in a $\mathbb{Q}$-algebra Λ is an object of the triangulated category $\mathrm{DM}(k,\Lambda)$ of mixed motives over k with coefficients in Λ. It is a very fine cohomological invariant of X, which contains information both about the cohomology of X with its mixed Hodge structure (when $k \subset \mathbb{C}$) but also about the rational Chow groups and rational algebraic K-theory groups of X. If X is also projective, $M(X)$ contains the same information as the perhaps more familiar Chow motive of X. However, Voevodsky motives are much more flexible than Chow motives and admit a fully-fledged theory of "motivic sheaves" on schemes and stacks with a six operation formalism and vanishing cycles functors which are crucial to our results.

A virtual motivic version of topological mirror symmetry has already been established by Loeser and Wyss [48] who prove an equality of (orbifold) virtual Chow motives in the Grothendieck ring of Chow motives using motivic integration and the ideas of [32]. Our result implies and can be thought of as a categorification of the main theorem in [48], see Corollary 6.20 (i).

Another incarnation of mirror symmetry in this context is an expected derived equivalence

$$D^b_{\mathrm{coh}}(\mathcal{M}_L) \simeq D^b_{\mathrm{coh}}(\overline{\mathcal{M}}, \delta_L) \tag{1}$$

induced by a twisted Fourier-Mukai kernel relative to their common Hitchin base $\mathcal{A}_L \simeq \overline{\mathcal{A}}$ whose restriction to smooth Hitchin fibres should be the Mukai derived equivalence between dual abelian varieties (such a derived equivalence was proven for the stack of all G-Higgs bundles over an open subset of the Hitchin base in [24, Corollary 5.5]). The Chern character of this Fourier-Mukai kernel should not induce an isomorphism of motives, in the same way that for an abelian variety A, the Chern character of the Poincaré line bundle inducing the Mukai equivalence $D^b_{\mathrm{coh}}(A) \simeq D^b_{\mathrm{coh}}(\widehat{A})$ does not induce an isomorphism $M(A) \simeq M(\widehat{A})$ of rational Chow motives, but rather the Fourier transform on rational Chow groups $\mathrm{CH}^*(A,\mathbb{Q}) \simeq \mathrm{CH}^*(\widehat{A},\mathbb{Q})$, which does not preserve the cohomological grading. However in the case of abelian varieties, we know that any fixed isogeny $A \to \widehat{A}$ induces an isomorphism $M(A) \simeq M(\widehat{A})$. A bold conjecture of Orlov [58] suggests that this holds much more generally: if X and Y are two derived equivalent smooth projective varieties, there is a non-canonical isomorphism $M(X) \simeq M(Y)$ which is generally not induced by the Chern character of the given Fourier-Mukai kernel. Orlov's conjecture can also reasonably be extended to smooth proper Deligne-Mumford stacks, replacing Chow motives with orbifold Chow motives [27, Definition 2.5], in which case it is related to the generalized McKay correspondence and the motivic hyperkähler resolution conjecture [27, Conjecture 3.6]. One might speculate that Orlov's conjecture could be extended even further to a non-proper, twisted set-up like (1), and thus that the main result of [50] and our Theorem 1.1 is a natural prediction of an extension of Orlov's conjecture applied to (1).

1.2. **Γ-action on cohomology and motives.** In both [32] and [50], a crucial ingredient was the Γ-action on cohomology and its isotypical decomposition. The corresponding motivic decomposition will also play a fundamental rôle in our proof. Let us first discuss the the parallel and much simpler situation in the case of moduli of vector bundles. For the moduli spaces $\mathcal{N} := \mathcal{N}_{n,d}(C)$ (resp. $\mathcal{N}_L$) of stable vector bundles (resp. of fixed determinant) of rank n and coprime degree d, Harder and Narasimhan [33] showed that Γ-action on the ℓ-adic cohomology of $\mathcal{N}_L$ is trivial and conclude that the ℓ-adic cohomology of $\mathcal{N}$ is isomorphic to the tensor product of the cohomology of $\mathcal{N}_L$ and $\mathrm{Jac}(C)$. In joint work with Fu [25, Theorem 1.1], we lifted this to an isomorphism of Chow motives. However, the situation for Higgs bundles is very different. In rank $n = 2$, Hitchin [37] already observed that Γ acts non-trivially on the Betti cohomology of $\mathcal{M}_L$. The Γ-fixed piece of the cohomology of $\mathcal{M}_L$ is just the cohomology of the PGL_n-Higgs moduli space, but in the isotypical decomposition of $H^*(\mathcal{M}_L, \mathbb{C})$ there are non-zero pieces indexed by non-trivial characters $\kappa \in \widehat{\Gamma}$, which need to be understood.

Another difference with the case of vector bundles concerns tautological generation of the cohomology ring. For n and d coprime, Atiyah and Bott [6] showed the cohomology of the moduli space $\mathcal{N}$ of stable vector bundles is generated by tautological classes (the Künneth components of the Chern classes of the universal bundle) and by the result of Harder and Narasimhan [33], the same is true for the moduli space $\mathcal{N}_L$ of stable vector bundles of fixed determinant. Markman proved the cohomology of the GL-Higgs moduli space is also generated by the tautological classes [49]. However, the cohomology of the SL-Higgs moduli space is not generated by tautological classes. On a motivic level, this difference between the case of moduli of vector bundles and Higgs bundles (with fixed determinant) is illustrated by the fact that the motives of the vector bundle moduli spaces $\mathcal{N}$ and $\mathcal{N}_L$ are both generated by the motive of C (see [25, Proposition 4.1]), whereas in the case of Higgs bundles, although the motive of the GL-Higgs moduli space $\mathcal{M}$ is generated by the motive of C by [39], this is not true for the SL-Higgs moduli space $\mathcal{M}_L$ for a general complex curve C by [26, Proposition 5.7].

The isotypical pieces in the decomposition associated to the Γ-action on the cohomology of the SL-Higgs moduli space $\mathcal{M}_L$ were described in [50] in terms of isotypical pieces in the cohomology of the fixed locus $\mathcal{M}_\gamma := (\mathcal{M}_L)^\gamma$ for elements $\gamma \in \Gamma$. This fixed locus comes with a restricted Hitchin map $h_\gamma : \mathcal{M}_\gamma \to \mathcal{A}_\gamma := h_L(\mathcal{M}_\gamma)$. The Weil pairing yields identifications

$$\widehat{\Gamma} \cong \Gamma := \mathrm{Jac}(C)[n] \cong H^1(C, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(C), \mathbb{Z}/n\mathbb{Z})$$

between Γ and its character group $\widehat{\Gamma}$, and the Abel-Jacobi map relates Γ and cyclic covers of C of degree dividing n. For $\gamma \in \Gamma$, we let $\kappa = \kappa(\gamma) \in \widehat{\Gamma}$ denote the corresponding character and $\pi := \pi_\gamma : C_\gamma \to C$ denote the corresponding cyclic cover of degree m with $n = n_\gamma m$. One of the central results of Maulik and Shen (see [50, Theorem 0.5]) is the existence of an isomorphism of pure Hodge structures

$$\nu_\gamma : H^*(\mathcal{M}_L, \mathbb{C})_\kappa \cong H^{*-2d_\gamma}(\mathcal{M}_\gamma, \mathbb{C})_\kappa\{-d_\gamma\}, \tag{2}$$

where $d_\gamma = n(n - n_\gamma)(g-1)$ is the codimension of $i_\gamma : \mathcal{A}_\gamma \hookrightarrow \mathcal{A}_L$ or equivalently half the codimension of $\mathcal{M}_\gamma$ in $\mathcal{M}_L$ (see Lemma 5.5) and we write $\{j\} := (j)[2j]$ for the pure Tate twist associated to an integer $j \in \mathbb{Z}$. Note that, because d_γ is only half the codimension of $\mathcal{M}_\gamma$ in $\mathcal{M}_L$, the isomorphism ν_γ is not induced by a Gysin morphism. When summing these isomorphisms over all $\gamma \in \Gamma$, the left hand side sums to the cohomology of $\mathcal{M}_L$ while the right hand side sums to the orbifold cohomology of $\overline{\mathcal{M}}$ twisted with respect to its natural gerbe δ_L by [35, 48], and Maulik and Shen obtain a (twisted orbifold) Hodge structure version of topological mirror symmetry

$$H^*(\mathcal{M}_L, \mathbb{C}) \simeq H^*_{\mathrm{orb}}(\overline{\mathcal{M}}, \mathbb{C}; \delta_L). \tag{3}$$

1.3. **Overview of Maulik and Shen's cohomological mirror symmetry.** Let us outline how the isomorphism (2) is proved in the cohomological setting of [50]. For simplicity, as in the main body of the paper [50], let us forget the Hodge structure and concentrate on the isomorphism of cohomology groups; the isomorphism of Hodge structures is then obtained by running the same argument using the theory of mixed Hodge modules.

First, for each $\gamma \in \Gamma$, one relates the γ-fixed locus $\mathcal{M}_\gamma$ with a relative moduli space for the associated cyclic cover $\pi := \pi_\gamma : C_\gamma \to C$. The π-relative SL-Higgs moduli space $\mathcal{M}_\pi$ is introduced in [35, 54] and defined as a fibre of a map between GL-Higgs moduli spaces on C_γ and C given by taking the determinant and trace of the pushfoward along π (see Definition 5.3). It comes equipped with a restricted Hitchin fibration $h_\pi : \mathcal{M}_\pi \to \mathcal{A}_\pi$ which is equivariant with respect to the action of $G_\pi := \mathrm{Gal}(C_\gamma/C)$ and has geometric quotient given by $h_\gamma : \mathcal{M}_\gamma \to \mathcal{A}_\gamma$. Consequently, the cohomology of $\mathcal{M}_\gamma$ is isomorphic to the G_π-equivariant cohomology of $\mathcal{M}_\pi$ and in fact this is true relative to $\mathcal{A}_\gamma$ (see [50, Lemma 1.7]).

In the second step, Maulik and Shen use a cohomological correspondence following work of Yun [66], which is based on ideas of Ngô [55] and the description of the generic fibres via Prym varieties, to construct a morphism

$$\beta_\gamma^{\mathrm{naive}} : (h_L)_*\mathbb{C}_\kappa \to i_{\gamma *}(h_{\gamma *}\mathbb{C})_\kappa\{-d_\gamma\} \in D^b_c(\mathcal{A}_L) \tag{4}$$

The morphism $\beta_\gamma^{\mathrm{naive}}$ is an isomorphism when restricted to a dense open of $\mathcal{A}_L$ by an explicit computation of Yun [66]. Note that Yun works in a much more general setting (Higgs bundles for an arbitrary reductive group G and its Langlands dual), interprets this isomorphism conceptually in terms of endoscopic groups of G, and already connects the picture to mirror symmetry. However, it is not clear whether $\beta_\gamma^{\mathrm{naive}}$ is an isomorphism over the full Hitchin base, and a key innovation of Maulik and Shen is to construct a variant they can study over the whole of $\mathcal{A}_L$.

For that purpose, they first consider versions of the above constructions for D-twisted Higgs bundles, where D is a divisor with $\deg(D) > 2g-2$ and $\deg(D)$ is even; we shall denote the corresponding moduli spaces and morphisms with a superscript D. This leads to a morphism

$$\beta_\gamma^D : ((h_L^D)_*\mathbb{C})_\kappa \to i^D_{\gamma *}(h^D_{\gamma *}\mathbb{C})_\kappa\{-d_\gamma^D\} \in D^b_c(\mathcal{A}^D_L). \tag{5}$$

The advantage of working with a divisor of larger even degree is that the geometry of the D-twisted Hitchin fibration $h^D : \mathcal{M}^D \to \mathcal{A}^D$ is simpler, in the sense that the supports of in the decomposition theorem for h^D are known by work of Chaudouard-Laumon [18]; however, $\dim \mathcal{M}^D \neq 2\dim \mathcal{A}^D$ and the D-twisted Higgs moduli space $\mathcal{M}^D$ is no longer algebraic symplectic. The morphism β^D_γ is again generically an isomorphism by [66]. Maulik and Shen use the decomposition theorem together with an analysis of the supports for the π-relative Hitchin maps $h^D_\pi : \mathcal{M}^D_\pi \to \mathcal{A}^D_\pi$ over the full Hitchin base, based on work of Chaudouard-Laumon [18] in the GL_n-case and of de Cataldo [22] in the SL_n-case, to prove that β^D_γ is actually an isomorphism.

Maulik and Shen then build on this to also treat the case of odd degree divisors with $\deg(D) > 2g-2$ as well as the original case of classical Higgs bundles $D = K_C$. The mechanism to extend the construction of β^D_γ to these cases is vanishing cycles. Let us concentrate on the case of odd degree divisors with $\deg(D) > 2g-2$; the case $D = K_C$ is obtained by iterating the construction twice starting from a divisor of degree $2g$. Fixing a point $p \in C$, the divisor $D+p$ has even degree $> 2g-2$, so we have an isomorphism β^{D+p}_γ. Maulik and Shen then construct a function $\mu_{\mathcal{A}} : \mathcal{A}^{D+p}_L \to \mathbb{A}^1$ with the property that $\mathcal{A}^D_L \subset \mathcal{A}^{D+p}_L$ is the critical locus of $\mu_{\mathcal{A}}$ (see [50, Theorem 4.5]). For the vanishing cycles functor $\phi_{\mu_{\mathcal{A}}} : D^b_c(\mathcal{A}^{D+p}_L) \to D^b_c(\mathcal{A}^D_L)$, they show that

$$\phi_{\mu_{\mathcal{A}}}((h^{D+p}_L)_*\mathbb{C}) \simeq (h^D_L)_*\mathbb{C}(a)[b]$$

for a certain Tate twist a and shift b whose precise values do not matter now; this requires working with vanishing cycles on certain simple Artin stacks. They then prove a similar statement relating the π-relative Hitchin fibrations h^{D+p}_π and h^D_π. Putting the two together then allows them to construct an isomorphism

$$\beta_\gamma^D : ((h_L^D)_*\mathbb{C})_\kappa \simeq i^D_{\gamma *}(h^D_{\gamma *}\mathbb{C})_\kappa\{-d_\gamma^D\} \in D^b_c(\mathcal{A}^D_L) \tag{6}$$

from β^{D+p}_γ and finish the proof of [50].

1.4. **Results and methods.** Our first main result is a motivic version of the isomorphism (2).

Theorem 1.1. *Let C be a smooth projective geometrically connected genus g curve over an algebraically closed field k of characteristic zero. Fix a rank n and coprime degree d and $L \in \mathrm{Pic}^d(C)$. For a divisor D on C with either $D = K_C$ or $\deg(D) > 2g-2$, the following statements hold in $\mathrm{DM}(k,\Lambda)$ where $\Lambda := \mathbb{Q}(\zeta_n)$.*

(i) *For each $\gamma \in \Gamma$ corresponding to $\kappa := \kappa(\gamma) \in \widehat{\Gamma}$, we have an isomorphism*

$$\nu^D_{\gamma,\mathrm{mot}} : M(\mathcal{M}^D_\gamma)_\kappa\{d_\gamma\} \xrightarrow{\sim} M(\mathcal{M}^D_L)_\kappa.$$

(ii) *There is a motivic mirror symmetry isomorphism*

$$M_{\mathrm{orb}}(\overline{\mathcal{M}}^D, \delta_L) \simeq M(\mathcal{M}^D_L),$$

where $M_{\mathrm{orb}}(\overline{\mathcal{M}}^D, \delta_L)$ is the orbifold motive with respect to the gerbe δ_L (see §5.3).

The second statement in this theorem is just a fancy way to record the sum of the first statement over all $\gamma \in \Gamma$; in particular, we *define* the left-hand side as the sum over κ of the left-hand sides of the isomorphisms $\nu^D_{\gamma,\mathrm{mot}}$, and leave aside the question of a more general theory of twisted orbifold motives. The theorem implies that the (orbifold) Chow groups of the SL-Higgs and PGL-Higgs moduli spaces are isomorphic (see Corollary 6.20, which includes some stronger consequences on motivic cohomology and algebraic K-theory). See §1.5 for how this relates to other results in the area.

The morphism $\nu^D_{\gamma,\mathrm{mot}}$ is obtained by pushing forward to $\mathrm{Spec}(k)$ a morphism

$$\beta^D_{\gamma,\mathrm{mot}} : (h^D_L)_*\mathbb{1} \to i^D_{\gamma*}(h^D_{\gamma*}\mathbb{1})(-d^D_\gamma)[-2d^D_\gamma] \in \mathrm{DM}(\mathcal{A}_L, \Lambda)$$

defined in §6.1 and then dualising. The construction of $\beta^D_{\gamma,\mathrm{mot}}$ is parallel to the one of β^D_γ above. For D with $\deg(D) > 2g-2$ and even, we define $\beta^D_{\gamma,\mathrm{mot}}$ as a motivic correspondence lifting the cohomological correspondence (4). For other divisors D and, in particular for $D = K_C$, the morphism $\beta^D_{\gamma,\mathrm{mot}}$ is constructed using motivic vanishing cycles [9]. This requires extending some constructions and results of Ayoub to Artin stacks as well as computing motivic vanishing cycles functors for homogeneous functions (see Appendix A).

We do not know if $\beta^D_{\gamma,\mathrm{mot}}$ is an isomorphism, and the method of [50], based on perverse sheaves and the decomposition theorem, is not at all available in $\mathrm{DM}(-,\Lambda)$. However, by construction and the main result of [50], the Betti realisations[2] of $\beta^D_{\gamma,\mathrm{mot}}$ and also of the induced morphism $\nu^D_{\gamma,\mathrm{mot}}$ are isomorphisms. The Betti realisation on the category $\mathrm{DM}_c(k,\Lambda)$ constructible motives with coefficients in a $\mathbb{Q}$-algebra is conjectured to be conservative, but this is a difficult open question in general [11]. However a result of Wildeshaus [64] ensures that this is true when restricting to the subcategory $\mathrm{DM}^{\mathrm{ab}}_c(k,\Lambda)$ of constructible *abelian* motives. Hence, to finish the proof of Theorem 1.1, it remains to show that both source and target of $\nu^D_{\gamma,\mathrm{mot}}$ are abelian.

In order to do this, we extend our previous work [39] to prove that, for a divisor D either with $D = K_C$ or $\deg(D) > 2g-2$, the motive of the moduli space of D-twisted GL-Higgs bundles of coprime rank and degree is generated by the motive of the curve (and in particular is abelian). We then consider SL-Higgs bundles. By [26, Proposition 5.7], we know that the motives of SL-Higgs moduli spaces are not in general generated by the motive of C when $k = \mathbb{C}$; we are nevertheless able to prove the motives of SL-Higgs moduli spaces are abelian in the coprime setting. Our second main result is the following theorem, which is proved in §3 and §4.

Theorem 1.2. *Let C be a smooth projective geometrically connected genus g curve over an arbitrary field k with $C(k) \neq \emptyset$ and let D be a divisor on C such that either $D = K_C$ or $\deg(D) > 2g-2$. Fix a rank n and coprime degree d and a line bundle $L \in \mathrm{Pic}^d(C)$. Then the following statements hold in $\mathrm{DM}(k,\mathbb{Q})$.*

(i) *The motive of the moduli space $\mathcal{M}^D := \mathcal{M}^D_{n,d}(C)$ of D-twisted* GL*-Higgs bundles is a direct factor of the motive of a large enough power of C. In particular, it is abelian.*

(ii) *The motive of the moduli space $\mathcal{M}^D_L := \mathcal{M}^D_{n,L}(C)$ of D-twisted* SL*-Higgs bundles is a direct factor of the motive of a product of étale covers of C. In particular, it is abelian.*

The strategy for SL-Higgs bundles follows the same lines as for GL-Higgs bundles in [39] which was based on the geometric ideas in [29]: one uses a motivic Białynicki–Birula decomposition associated to the $\mathbb{G}_m$-action on the Higgs moduli space given by scaling the Higgs field to relate to motives of moduli spaces of chains of vector bundle homomorphisms (of fixed total determinant in the case of SL-Higgs bundles). From this, we deduce the purity of $M(\mathcal{M}^D)$ and $M(\mathcal{M}^D_L)$ for Bondarko's weight structure and following [39, §6.3], it suffices to show that $M(\mathcal{M}^D)$ (resp. $M(\mathcal{M}^D_L)$) lies in the localising subcategory of $\mathrm{DM}(k,\mathbb{Q})$ generated by the motive of C (resp. of étale covers of C).

Next one uses variation of stability and Harder–Narasimhan recursions for moduli stacks of chains to reduce to describing the motive of the stack of injective chain homomorphisms

[2] Relative to some complex embedding of k; by the Lefschetz principle, one can always reduce to the case where k admits such an embedding, see §6.

(with fixed total determinant). At this point, the proof for SL-Higgs bundles is slightly more involved. One next considers a full flag version of the stack of injective chain homomorphisms which admits a forgetful map to the stack of injective chain homomorphisms following [36]. For GL-Higgs bundles, this map is small and a torsor under a product of symmetric groups on a dense open and so we can compute the motive of the stack of injective chain homomorphisms from that of the full flag version, which is an iterated projective bundle over the product of a power of C with a stack of vector bundles [36]; the motive of the latter we computed in [38].

For SL-Higgs bundles, we must first show the restriction of this small map to a closed substack is small. Furthermore, the full flag version of the stack of injective chain homomorphisms of fixed total determinant is an iterated projective bundle over a substack of the product of a power of C with a stack of vector bundles, which we identify with a stack of vector bundles with fixed determinant on a family of curves over a base scheme B, whose motive is abelian. To conclude, we extend our formula [38] for the motive of the stack of vector bundles to vector bundles with fixed determinant and also provide relative versions of these formulae (see Appendix B), which should be of independent interest.

1.5. **Related works and further directions.** Besides the works [35, 32, 48, 50, 66] which we have already mentioned and which inspired this paper, there have been other interesting results on topological mirror symmetry extending the original setup of Hausel-Thaddeus.

One can ask about similar statements for moduli of parabolic Higgs bundles (of fixed coprime rank and degree). A version of topological mirror symmetry in this case was established by Gothen and Oliveira [30] in ranks 2 and 3 and by Shiyu Shen [61] in general.

If n and d are not coprime and the corresponding moduli spaces are singular, one could ask if topological mirror symmetry (in the form of the endoscopic decomposition 2) holds for a suitable cohomology theory. One reasonable choice is intersection cohomology, and indeed a form of topological mirror symmetry for IH was conjectured by Mauri and established by him in rank 2 for $D = K_C$ [52] and by Maulik and Shen [51, Theorem 0.2] in arbitrary rank and degree but only for D-twisted Higgs bundles with $\deg(D) > 2g-2$. Maulik and Shen also raised the question of whether topological mirror symmetry holds for BPS cohomology, an invariant coming from mathematical physics which has been given a rigourous definition for Higgs bundles in [44] following ideas of Davison-Meinhardt and Toda. If $\deg(D) > 2g-2$, BPS cohomology coincides with intersection cohomology, but they differ when $D = K_C$. As discussed in [51, End of §3.6], topological mirror symmetry for BPS cohomology for $D = K_C$ should follow from results in [51, Theorem 0.2] and [45].

All results mentioned so far concern rational cohomology. One can also ask if topological mirror symmetry holds integrally. Groechenig and Shiyu Shen [31] observed that this cannot be the case for integral singular cohomology, because they showed that the integral cohomology on the SL_n-side is torsion-free, while the orbifold cohomology on the PGL_n-side has torsion. They go on to show that there is nevertheless an integral analogue, but at the level of (twisted) topological complex K-theory: assuming now again $(n,d) = 1$, there is an isomorphism of topological K-theory spectra

$$\mathrm{KU}(\mathcal{M}_L^{\mathrm{an}}) \simeq \mathrm{KU}(\overline{\mathcal{M}}^{\mathrm{an}}, \delta_L^{\mathrm{an}}).$$

Unlike the result of [50] and Theorem 1.1, this isomorphism is expected to be directly induced by the conjectural derived equivalence (1). The proof in [31] shares nevertheless some interesting similarities to [50] and to our arguments, which we would like to point out by sketching their strategy. They start with a correspondence over the elliptic locus of the Hitchin base (which in their case is a Fourier-Mukai kernel inspired by the one studied by Arinkin in the GL_n-case). They then use the decomposition theorem, results of de Cataldo on supports for the SL_n-Hitchin fibration [22] and vanishing cycles to get a statement about rational topological K-theory over the full Hitchin base. To be able to argue relatively to the Hitchin base, they use and refine ideas of Blanc, Moulinos and Brown about the topological K-theory of (sheaves of) dg-categories. To complete the proof, they upgrade this to a statement about integral topological K-theory by showing that the topological K-theory on both sides is torsion-free. In the case of $\mathcal{M}_L$, they reduce using the Atiyah-Hirzebruch spectral sequence to show that the singular cohomology of $\mathcal{M}_L$ is torsion-free. For this they use the chain wall-crossing techniques of [29, 28] similarly to our Section 4 (see Remark 4.7). We only became aware of this similarity in both proofs after our paper was completed.

Finally, we expect that Theorem 1.1 follows from a relative isomorphism of motivic sheaves over the Hitchin base, i.e. that the morphism $\beta^D_{\gamma,\mathrm{mot}}$ in Definition 6.15 is an isomorphism. Its Betti realisation is an isomorphism by [50] and Lemma 6.16, thus by conservativity on abelian motives and [12, Proposition 3.24] it would suffice to show that the motives of all fibres of the relevant Hitchin fibrations are abelian. It seems possible to prove this when the associated spectral curve is reduced, but the geometry of Hitchin fibres for non-reduced spectral curves is quite complicated.

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Motivic set-up. Let S be a finite type scheme over a field k and Λ be a $\mathbb{Q}$-algebra. We denote by $\mathrm{DM}(S,\Lambda)$ the triangulated category[3] of (Morel-Voevodsky) étale motivic sheaves over S with coefficients in Λ. This category can be defined in several equivalent ways (even more so because we work with rational coefficients), see [19, §16]. For concreteness we adopt the construction of [12, §3] where this category is denoted $\mathrm{DA}^{\text{ét}}(S,\Lambda)$. When $S = \mathrm{Spec}(k)$ is a perfect field, we write $\mathrm{DM}(k,\Lambda) := \mathrm{DM}(\mathrm{Spec}(k),\Lambda)$, and $\mathrm{DM}(k,\Lambda)$ is equivalently to the original definition of DM by Voevodsky using presheaves with transfers, see again [19, §16].

A central feature of $\mathrm{DM}(-,\Lambda)$ for the purpose of this paper is that it admits a full "six operation formalism", as well as a formalism of nearby and vanishing cycles. Almost everything that we need about this is contained in [8, 9, 12]. In Appendix A, we give a few complements about motivic sheaves and motivic vanishing cycles on algebraic stacks.

Let S be a finite type over a field k with structure morphism $\pi : S \to \mathrm{Spec}(k)$. In terms of the six operation formalism, S admits both an homological motive $M(S) := \pi_!\pi^!\mathbb{1}$ and a cohomological motive $M_{\mathrm{coh}}(S) := \pi_*\pi^*\mathbb{1}$, and the Verdier duality functor exchanges the two. Most of the paper is devoted to computing with (relative) cohomological motives, but we sometimes dualise the result to obtain statements about homological ones (which are in a sense more natural in Voevodsky's theory).

We denote by $\mathrm{DM}_c(S,\Lambda) \subset \mathrm{DM}(S,\Lambda)$ the subcategory of constructible objects [12, Definition 8.1] which in this context coincide with the compact objects in the triangulated sense [12, Proposition 8.3]. In our context, the six operations preserve constructible motives [12, Theoremes 8.10-12], so that almost all motivic sheaves appearing in this paper are constructible.

We denote by $\mathrm{DM}^{\mathrm{ab}}_c(k,\Lambda) \subset \mathrm{DM}_c(k,\Lambda)$ the subcategory of abelian motives, i.e. the thick tensor triangulated subcategory generated by the motives of all smooth projective curves over k (or equivalently the thick triangulated subcategory generated by the motives of all abelian varieties over k by [4, Proposition 4.5], hence the name).

Suppose we have a complex embedding[4] $\sigma : k \to \mathbb{C}$. If S is a finite type k-scheme, we denote by $S^{\mathrm{an}} := (S \times_{k,\sigma} \mathbb{C})^{\mathrm{an}}$ the associated complex analytic space. By [10], there is an associated Betti realisation functor, which is an exact functor

$$R_B : \mathrm{DM}(S,\Lambda) \to D(S^{\mathrm{an}},\Lambda)$$

to the triangulated category of sheaves of Λ-modules on S^{an}. This functor "commutes" with the six operations in the best possible sense (without restrictions for the left adjoint functors f^* and $f_!$, and when restricted to constructible objects for the right adjoint functors) [10, Theorem

[3]In Appendix A, where we also review how to extend DM and motivic vanishing cycles to Artin stacks; there it is essential for technical reasons to we consider the ∞-categorical enhancement of $\mathrm{DM}(S,\Lambda)$, but this is not necessary in the body of the paper.

[4]Such an embedding may not exist for every k for cardinality reasons, but an application of the Lefschetz principle immediately reduces the proof of the main theorem to the case where it does; see §6.5 for details.

3.19]. It also commutes with nearby cycles functors when restricted to constructible objects [10, Theorem 4.9] (including in the context of algebraic stacks, see A.9).

Notation. For $j \in \mathbb{Z}$ and $M \in \mathrm{DM}(S, \Lambda)$, we write $M\{j/2\} := M(\lfloor j/2 \rfloor)[j] \in \mathrm{DM}(S, \Lambda)$. Note that the "Tate twist" $\{j/2\}$ is pure if and only if j is even.

Let $M \in \mathrm{DM}(S, \Lambda)$ be a motive. We write $\langle M \rangle^{\otimes}$ (resp. $\langle\langle M \rangle\rangle^{\otimes}$) for the smallest thick tensor subcategory (resp. smallest localising tensor subcategory) of $\mathrm{DM}(S, \Lambda)$ containing M.

2. Background on Higgs moduli spaces

In this section, we introduce D-twisted Higgs bundles and their SL-counterparts.

2.1. D-twisted Higgs bundles.

For a divisor D on C with $D = K_C$ or $\deg(D) > 2g - 2$, we have the following notion of D-twisted Higgs bundles, which for $D = K_C$ gives the classical notion.

Definition 2.1. A D-twisted Higgs bundle is a pair $(E, \Phi : E \to E \otimes \mathcal{O}_C(D))$ consisting of a vector bundle E and an $\mathcal{O}_C$-linear homomorphism Φ.

Note that this notion only depends on D up to linear equivalence. In particular, if $\deg(D) > 2g - 2$, we can and will assume that D is effective, as in [50].

There is a notion of (semi)stability for D-twisted Higgs bundles involving verifying an inequality of slopes for all Φ-invariant subbundles of E. We let $\mathcal{M}^D := \mathcal{M}^D_{n,d}(C)$ denote the moduli space of semistable D-twisted Higgs bundles of rank n and degree d.

Generally speaking, whenever we introduce an object for D-Higgs bundles, we use a superscript D, which we drop when $D = K_C$.

2.2. SL-Higgs bundles.

In order to have analogues of moduli spaces of (D-twisted) SL-Higgs bundles in non-zero degrees d, we consider fibres of the morphism

$$(\det, \mathrm{tr}) : \mathcal{M}^D := \mathcal{M}^D_{n,d}(C) \to \mathcal{M}^D_{1,d}(C)$$

over $(L, 0)$. We write $\mathcal{M}^D_L = \mathcal{M}^D_{n,L}(C) := (\det, \mathrm{tr})^{-1}(L, 0)$ for the D-twisted SL-Higgs moduli space. If n and d are coprime, then $\mathcal{M}^D_L$ is smooth. We have $\dim(\mathcal{M}^D_L) = (n^2 - 1)\deg(D)$.

2.3. PGL-Higgs bundles.

The moduli of PGL_n-Higgs bundles does not really play a role in this paper, however it is central to the original motivation of Hausel-Thaddeus [35] so we include a brief overview. A PGL_n-Higgs bundle of degree d on C consists of a principal PGL_n-bundle $P \to C$ of degree d and a Higgs field $\Phi \in H^0(C, P \times^{\mathrm{PGL}_n} \mathfrak{pgl}_n)$.

The surjection $\mathrm{GL}_n \to \mathrm{PGL}_n$ induces a morphism between the stacks of GL_n and PGL_n-Higgs bundles preserving semistability. If the degree and rank are coprime, then the stack of semistable PGL_n-Higgs bundles is the quotient of the stack of semistable GL_n-Higgs bundles by the action of $T^* \mathrm{Jac}(C)$ (see [28, Section 3]). Equivalently, using the natural surjection $\mathrm{GL}_n \to \mathrm{PGL}_n$, the stack of semistable PGL_n-Higgs bundles is the quotient of the stack of semistable SL_n-Higgs bundles by the finite group $\Gamma := \mathrm{Jac}(C)[n]$. Consequently, the moduli space of semistable PGL_n-Higgs bundles of degree d can be defined as the following orbifold

$$\overline{\mathcal{M}} := [\mathcal{M}_L / \Gamma] \simeq [\mathcal{M} / T^* \mathrm{Jac}(C)]$$

which is smooth as a Deligne–Mumford stack, although its coarse moduli space $\mathcal{M}_L / \Gamma$ is singular. Note that because of that second presentation $\overline{\mathcal{M}}$ does not depend on the choice of line bundle L appearing in the SL-Higgs moduli space. There are also D-twisted variants of the PGL_n-Higgs moduli spaces given by

$$\overline{\mathcal{M}}^D := [\mathcal{M}^D_L / \Gamma] \simeq [\mathcal{M}^D / \mathrm{Jac}(C) \times H^0(C, \mathcal{O}(D))].$$

3. Motives of D-twisted Higgs moduli spaces

In this section, we generalise our previous result concerning motives of moduli spaces of Higgs bundles [39, Theorem 1.1] to moduli spaces of D-twisted Higgs bundles. Throughout this section, we work over an arbitrary field k and assume that $C(k) \neq \emptyset$. We fix a coprime rank n and degree d and a divisor D on C with $\deg(C) > 2g-2$.

3.1. **Hitchin's scaling action and moduli of chains.** The action of $\mathbb{G}_m$-action on $\mathcal{M}^D := \mathcal{M}^D_{n,d}(C)$ scaling the Higgs field has fixed points $[E,\Phi]$ where $E \cong \oplus_{i=0}^r E_i$ (with $r=0$ allowed) and $\Phi(E_i) \subset E_{i+1} \otimes \mathcal{O}_C(D)$ which determine a chain of vector bundle homomorphisms

$$F_0 \to F_1 \to F_2 \to \cdots \to F_r$$

where $F_i := E_i \otimes \mathcal{O}_C(iD)$.

For a brief overview of semistability for chains (which depends on a choice of stability parameter) and chain moduli spaces and stacks see [39, §2.3] and [3, 29]. Let $\mathcal{C}\mathrm{h}^{\alpha_D,ss}_{\underline{m},\underline{e}}$ denote the moduli space of chains $F_0 \to \cdots \to F_r$ with tuples of ranks $\underline{m}$ and degrees $\underline{e}$ which are semistable with respect to the chain stability parameter

$$\alpha_D := (r\deg(D), (r-1)\deg(D), \dots, \deg(D), 0) \in \mathbb{R}^{r+1}.$$

We refer to α_D as the D-twisted Higgs stability parameter. Since $\deg(D) > 2g-2$, this stability parameter lies in the cone $\Delta_r^\circ := \{(\alpha_0, \dots, \alpha_r) \in \mathbb{R}^{r+1} : \alpha_{i-1} - \alpha_i > 2g-2 \text{ for } 1 \leq i \leq r\}$ of stability parameters with well-understood deformation theory [3, Section 3]. Therefore, if α_D is non-critical for the invariants $\underline{m}$ and $\underline{e}$ (so α_D-semistability and α_D-stability coincide for chains with these invariants), then $\mathcal{C}\mathrm{h}^{\alpha_D,ss}_{\underline{m},\underline{e}}$ is a smooth projective variety by [3, Theorem 3.8 vi)].

Proposition 3.1. *The $\mathbb{G}_m$-action on $\mathcal{M}^D$ is semi-projective with fixed locus*

$$(\mathcal{M}^D)^{\mathbb{G}_m} = \bigsqcup_{(\underline{m},\underline{e}) \in \mathcal{I}} \mathcal{C}\mathrm{h}^{\alpha_D,ss}_{\underline{m},\underline{e}}$$

where the D-twisted Higgs stability parameter α_D is non-critical for all the invariants $\underline{m}$ and $\underline{e}$ appearing in this finite index set $\mathcal{I}$. Consequently, there is a motivic Białynicki-Birula decomposition

$$M(\mathcal{M}^D) \simeq \bigoplus_{(\underline{m},\underline{e}) \in \mathcal{I}} M(\mathcal{C}\mathrm{h}^{\alpha_D,ss}_{\underline{m},\underline{e}})\{c_{\underline{m},\underline{e}}\},$$

where $c_{\underline{m},\underline{e}}$ denotes the codimension of the corresponding Białynicki-Birula stratum in $\mathcal{M}^D$.

Proof. The fact that this $\mathbb{G}_m$-action is semi-projective follows by the same argument for K_C-Higgs bundles (see [39, Proposition 2.2]). As in [39, Proposition 2.5], one shows that semistability of the $\mathbb{G}_m$-fixed point $[E \cong \oplus_{i\geq 0} E_i, \Phi]$ in $\mathcal{M}^D$ corresponds to α_D-semistability of the corresponding chain $F_\bullet$ with $F_i := E_i \otimes \mathcal{O}_C(iD)$. Hence, the claimed description of the $\mathbb{G}_m$-fixed locus follows as in [39, Corollary 2.6]. Since the $\mathbb{G}_m$-action is semi-projective, one applies the motivic Białynicki-Birula decomposition [39, Theorem A.4] to finish the proof. □

3.2. **Stacks of chains, wall-crossings and Harder–Narasimhan recursions.** We let $\mathfrak{Ch}^{\alpha,ss}_{\underline{m},\underline{e}}$ denote the substack of α-semistable chains and let $\mathfrak{Ch}^{\alpha,\tau}_{\underline{m},\underline{e}}$ denote the substack of chains of α-HN type τ. Let us recall a few facts we will need (for details, see [39, §2.3] and [3, 29]). If $C(k) \neq \emptyset$ and α is non-critical for $\underline{m}$ and $\underline{e}$, then the stack of α-semistable chains is a trivial $\mathbb{G}_m$-gerbe over the moduli space $\mathcal{C}\mathrm{h}^{\alpha,ss}_{\underline{m},\underline{e}}$. Furthermore, if $\alpha \in \Delta_r^\circ$, then $\mathfrak{Ch}^{\alpha,ss}_{\underline{m},\underline{e}}$ is smooth and so is the stack $\mathfrak{Ch}^{\alpha,\tau}_{\underline{m},\underline{e}}$ of chains with α-HN type τ, as taking the associated graded for the α-HN filtration is an affine space fibration by [29, Lemma 4.6 and Proposition 4.8].

These HN strata appear in wall-crossings as we vary the stability parameter. We will employ the wall-crossing argument in [29] to end up in a chamber where the stack of semistable chains can be more readily described. More precisely, we will relate them to stacks of injective chain homomorphisms in the following sense.

Definition 3.2. Let $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ denote the substack of $\mathfrak{Ch}_{\underline{m},\underline{e}}$ consisting of chains

$$F_\bullet = (F_0 \overset{\phi_1}{\to} F_1 \to \cdots \to F_{r-1} \overset{\phi_r}{\to} F_r)$$

such that all the homomorphisms ϕ_i are injective.

If $\underline{m} = (m, \dots, m)$ is constant, $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ coincides with the stack of generically surjective chain maps (see [36]); in particular, this stack is smooth by [29, Lemma 4.9] and [3, Theorem 3.8 v)]. In [39], we computed the motive of $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ using Hecke modification maps and motivic descriptions of small maps; since we will later prove something similar for SL-Higgs bundles, we include a short proof.

Proposition 3.3. *If $\underline{m} = (m, \dots, m)$ is constant and for $\underline{e} = (e_0, \dots, e_r)$ we set $l_i := e_i - e_{i-1}$, then the motive of $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ in $\mathrm{DM}(k, \mathbb{Q})$ is given by*

$$M(\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}) \simeq M(\mathfrak{Bun}_{m,e_r}) \otimes \bigotimes_{i=1}^{r} M(\mathrm{Sym}^{l_i}(C \times \mathbb{P}^{m-1})).$$

In particular, the motive of $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ lies in $\langle\!\langle M(C) \rangle\!\rangle^{\otimes}$.

Proof. Let $\mathfrak{Coh}_{0,l}$ denote the stack of length l torsion sheaves and let $\widetilde{\mathfrak{Coh}}_{0,l}$ be the stack of length l torsion sheaves with a full flag; that is for a k-scheme S we have

$$\widetilde{\mathfrak{Coh}}_{0,l}(S) = \langle \mathcal{T}_1 \hookrightarrow \mathcal{T}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{T}_l \ : \ \mathcal{T}_i \in \mathfrak{Coh}_{0,i}(S) \rangle.$$

Then Laumon showed the forgetful map $\widetilde{\mathfrak{Coh}}_{0,l} \to \mathfrak{Coh}_{0,l}$ is a small map and a S_l-torsor over the dense open on which the support of the torsion sheaf consists of l distinct points [46, Theorem 3.3.1]. Similarly, Heinloth considered in [36, Proposition 11] a full flag version $\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ of $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ defined by

$$\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e}}(S) = \langle \mathcal{F}_0 = \mathcal{F}_1^0 \hookrightarrow \mathcal{F}_1^1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_1^{l_1} = \mathcal{F}_1 = \mathcal{F}_2^0 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_r^{r_l} = \mathcal{F}_r : \mathcal{F}_i^j \in \mathfrak{Bun}_{m,e_i+j}(S) \rangle,$$

which admits a forgetful map $f' \colon \widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e}} \to \mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ that is also small and a $(\prod_{i=1}^r S_{l_i})$-torsor on the dense open consists of chains $F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_r$ such that the support of F_i/F_{i-1} consists of l_i distinct points for each $1 \leq i \leq r$, as it is obtained by pulling back products of the small maps considered by Laumon under a smooth morphism. In fact, as in [36, Proposition 11] we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}} \arrow[r, "\mathrm{gr}"] & \prod_{i=1}^r \mathfrak{Coh}_{0,l_i} \times \mathfrak{Bun}_{m,e_r} \arrow[r, "\beta"] & \prod_{i=1}^r \mathrm{Sym}^{l_i}(C) \times \mathfrak{Bun}_{m,e_r} \\
\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e}} \arrow[u, "f'"] \arrow[r] \arrow[rr, bend right, "\mathrm{supp}"'] & \prod_{i=1}^r \widetilde{\mathfrak{Coh}}_{0,l_i} \times \mathfrak{Bun}_{m,e_r} \arrow[u, "f"] \arrow[r] & \prod_{i=1}^r C^{l_i} \times \mathfrak{Bun}_{m,e_r}, \arrow[u]
\end{tikzcd}$$

where $\mathrm{gr}(F_0 \hookrightarrow \cdots \hookrightarrow F_r) := (F_1/F_0, \dots, F_r/F_{r-1}, F_r)$ is a smooth morphism and

$$\mathrm{supp}(F_0 = F_1^0 \hookrightarrow \cdots \subset F_1^{l_1} = F_1 = F_2^0 \hookrightarrow \cdots \hookrightarrow F_r^{l_r} = F_r) := (\mathrm{supp}(F_i^j/F_i^{j-1})_{1 \leq i \leq r, 1 \leq j \leq l_i}, F_r)$$

is a $(\sum_{i=1}^r l_i)$-iterated $\mathbb{P}^{m-1}$-bundle, as we take $\sum_{i=1}^r l_i$ successive elementary Hecke modifications of a rank m bundle. Since supp is a $(\sum_{i=1}^r l_i)$-iterated $\mathbb{P}^{m-1}$-bundle, we have

$$M(\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e}}) \simeq M(\mathfrak{Bun}_{m,e_r}) \otimes \bigotimes_{i=1}^{r} M(C \times \mathbb{P}^{m-1})^{\otimes l_i}.$$

Since f' is a small map and a torsor under $G = \prod_{i=1}^r S_{l_i}$ on a dense open, by applying the motivic description of such small maps (see Theorem 3.4 below and [41, Theorem 2.1]) we obtain

$$M(\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}) \simeq M(\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e}})^G \simeq M(\mathfrak{Bun}_{m,e_r}) \otimes \bigotimes_{i=1}^{r} M(\mathrm{Sym}^{l_i}(C \times \mathbb{P}^{m-1})),$$

which completes the proof. □

To complete the proof, we state the relevant result on small maps from [41], which generalises [21] and [38, Theorem 2.13].

Theorem 3.4. [41, Theorem 2.1] *Let $\pi : \mathfrak{X} \to \mathcal{Y}$ be a small proper surjective morphism of smooth Artin stacks such that there exists a dense open $\mathcal{Y}^\circ \subset \mathcal{Y}$ with preimage $\mathfrak{X}^\circ$ such that $\pi : \mathfrak{X}^\circ \to \mathcal{Y}^\circ$ is a G-torsor. Then the G-action on $M(\mathfrak{X}^\circ)$ in $\mathrm{DM}(k, \mathbb{Q})$ extends to $M(X)$ and we have*

$$M(\mathfrak{X})^G \simeq M(\mathcal{Y}).$$

We can now employ the wall-crossing argument of [29] as in [39, §6] to conclude an analogous result to [39, Theorem 1.1].

Theorem 3.5. *Assume that $C(k) \neq \emptyset$. Let D be a divisor on C with either $D = K_C$ or $\deg(D) > 2g-2$. The motive of the D-twisted Higgs moduli space $\mathcal{M}^D := \mathcal{M}^D_{n,d}(C)$ for coprime rank and degree lies in the subcategory $\langle M(C)\rangle^{\otimes}$ of $\mathrm{DM}(k, \mathbb{Q})$ and is a direct factor of $M(C^m)$ for sufficiently high m.*

Proof. The case where $D = K_C$ is covered by [39, Theorem 1.1], so we assume $\deg(D) > 2g-2$. The final statement follows from the first using a purity argument as in [39, §6.3]. For the first, by Proposition 3.1, it suffices to show for each index $(\underline{m}, \underline{e}) \in \mathcal{I}$ that $M(\mathcal{C}\mathrm{h}^{\alpha_D,ss}_{\underline{m},\underline{e}}) \in \langle M(C)\rangle^{\otimes}$. As the map $\mathfrak{Ch}^{\alpha_D,ss}_{\underline{m},\underline{e}} \to \mathcal{C}\mathrm{h}^{\alpha_D,ss}_{\underline{m},\underline{e}}$ from the stack to its good moduli space is a trivial $\mathbb{G}_m$-gerbe, it suffices to show that $M(\mathcal{C}\mathrm{h}^{\alpha_D,ss}_{\underline{m},\underline{e}}) \in \langle\!\langle M(C)\rangle\!\rangle^{\otimes}$ by [39, Lemma 6.5]. For this, we use Proposition 3.3 and [39, Theorem 6.3] recursively, which employs the wall-crossing argument and HN recursion of [29]. The basic idea is that, as stated in [39, Proposition 2.9], there is a ray in the cone Δ_r° of stability parameters starting at α_D and ending at a stability parameter α_∞ such that either

- $\mathfrak{Ch}^{\alpha_\infty,ss}_{\underline{m},\underline{e}} = \emptyset$ if $\underline{m}$ is non-constant or
- $\mathfrak{Ch}^{\alpha_\infty,ss}_{\underline{m},\underline{e}} \subset \mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ and $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ is a union of α_∞-HN strata if $\underline{m}$ is constant.

The cone Δ_r° of stability parameters admits a wall and chamber decomposition and as this ray crosses each of the (finitely many) walls between α_D and α_∞, the semistable locus for the stability parameter on the wall is the union of the semistable loci on either side of the wall and finitely many HN-strata. Since by induction the motives of HN-strata can be related to motives of stacks of semistable for $\alpha \in \Delta_r^\circ$ (*cf.* [39, Lemma 6.2]), we see from looking at the Gysin triangles associated to each wall-crossing that it suffices to show by a HN-induction that $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ lies in $\langle\!\langle M(C)\rangle\!\rangle^{\otimes}$ for $\underline{m}$ constant, which is proven in Proposition 3.3. □

4. Motives of SL-Higgs moduli spaces

Throughout this section, we assume k is an arbitrary field and $C(k) \neq \emptyset$. We fix a rank n, a line bundle L on C of degree d coprime to n and a divisor D on C with either $D = K_C$ or $\deg(D) > 2g-2$. Our goal is to prove that the motive of the D-twisted SL-Higgs moduli space $\mathcal{M}^D_L := \mathcal{M}^D_{n,L}(C)$ is abelian.

4.1. **The scaling action on the SL-Higgs moduli space.** The scaling $\mathbb{G}_m$-action on $\mathcal{M}^D$ restricts to $\mathcal{M}^D_L$ and the fixed loci are chain moduli spaces with fixed total determinant

$$\mathcal{C}\mathrm{h}^{\alpha_D,ss}_{\underline{m},\underline{e},L\otimes\mathcal{O}_C(\sum_i im_iD)} := \left\{ F_0 \to \cdots \to F_r : \det\left(\bigoplus_{i=0}^{r} F_i\right) \cong L \otimes \mathcal{O}_C\left(\sum_i im_iD\right)\right\},$$

as if $E \simeq \oplus_{i=0}^r E_i$ has determinant L and $F_i := E_i \otimes \mathcal{O}_C(iD)$, then $\oplus_{i=0}^r F_i$ has determinant $L \otimes \mathcal{O}_C(\sum_i im_i D)$. Consequently, we have the following motivic Białynicki-Birula decomposition

$$M(\mathcal{M}_L^D) \simeq \bigoplus_{(\underline{m},\underline{e}) \in \mathcal{I}} M(\mathcal{C}\mathrm{h}^{\alpha_D,ss}_{\underline{m},\underline{e},L\otimes\mathcal{O}_C(\sum_i im_iD)})\{c_{\underline{m},\underline{e}}\}, \tag{7}$$

where $c_{\underline{m},\underline{e}}$ denotes the codimension of the corresponding Białynicki-Birula stratum in $\mathcal{M}_L^D$.

4.2. **The stack of injective chain homomorphisms of fixed total determinant.** For an arbitrary line bundle L on C (different in general from the L above, and in particular not assumed to be of degree d) and constant tuple of ranks $\underline{m}$, the aim of this section is to prove that the motive of the substack $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e},L} \hookrightarrow \mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ of injective chain homomorphisms with total determinant L

$$\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e},L} := \left\langle F_0 \hookrightarrow F_1 \hookrightarrow \cdots \hookrightarrow F_r \in \mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}} : \det\left(\bigoplus_{i=0}^r F_i\right) \cong L \right\rangle$$

is abelian. We will prove this by considering stacks of full flags $\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e},L} \hookrightarrow \widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e}}$ with total determinant L and proving that

i) the motive of $\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e},L}$ is abelian (Proposition 4.1 below) and
ii) the forgetful map $f'_L : \widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e},L} \to \mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e},L}$ is small and a torsor under a product of symmetric groups on a dense open (Proposition 4.3 below).

In both proofs it will be useful to consider a commutative diagram given by taking fibres of the commutative diagram in Proposition 3.3 over $L \in \mathrm{Pic}(C)$ under a certain weighted determinant map. As above, let us write $l_i := e_i - e_{i-1}$. First, similarly to [29, §5], we define a weighted determinant map

$$\begin{array}{rccc} \omega\colon & \prod_{i=1}^r C^{(l_i)} \times \mathfrak{Bun}_{m,e_r} & \to & \mathrm{Pic}(C) \\ & (D_1,\dots,D_r,F_r) & \mapsto & \det(F)^{\otimes r+1} \otimes \bigotimes_{i=1}^r \mathcal{O}_C(-iD_i) \end{array}$$

so that $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e},L}$ is the fibre over L of the composition $\omega \circ \beta \circ \mathrm{gr} : \mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}} \to \mathrm{Pic}(C)$. Let us write $\mathcal{T} := \prod_{i=1}^r \mathfrak{Coh}_{0,l_i}$ and $\widetilde{\mathcal{T}} := \prod_{i=1}^r \widetilde{\mathfrak{Coh}}_{0,l_i}$ and $C^{(\underline{l})} := \prod_{i=1}^r C^{(l_i)}$ and $C^{\underline{l}} := \prod_{i=1}^r C^{l_i}$. Let $\gamma := \omega \circ \beta$ and we denote the corresponding morphisms in the full flag setting by $\widetilde{\gamma} := \widetilde{\omega} \circ \widetilde{\beta}$. Then we have a commutative diagram

$$\begin{tikzcd}
\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}} \arrow[rr, "\mathrm{gr}"] & & \mathcal{T} \times \mathfrak{Bun}_{m,e_r} \arrow[rr, "\beta"] & & C^{(\underline{l})} \times \mathfrak{Bun}_{m,e_r} \arrow[ddrr, bend left, "\omega"] & & \\
 & \mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e},L} \arrow[ul, hook'] \arrow[rr, "\mathrm{gr}_L"] & & \gamma^{-1}(L) \arrow[ul, hook'] \arrow[rr] & & \omega^{-1}(L) \arrow[ul, hook'] & \\
 & & & & & & \mathrm{Pic}(C) \\
 & \widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e},L} \arrow[uu, "f'_L"] \arrow[rr] \arrow[dl, hook] & & \widetilde{\gamma}^{-1}(L) \arrow[uu, "f_L"] \arrow[rr] \arrow[dl, hook] & & \widetilde{\omega}^{-1}(L) \arrow[uu] \arrow[dl, hook] & \\
\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e}} \arrow[uuuu, "f'"] \arrow[rr, "\widetilde{\mathrm{gr}}"'] & & \widetilde{\mathcal{T}} \times \mathfrak{Bun}_{m,e_r} \arrow[uuuu, "f"] \arrow[rr, "\widetilde{\beta}"'] & & C^{\underline{l}} \times \mathfrak{Bun}_{m,e_r} \arrow[uuuu] \arrow[uurr, bend right, "\widetilde{\omega}"'] & &
\end{tikzcd}$$

where f and f' are small and $\prod_{i=1}^r S_{l_i}$-torsors on a dense open, gr and also its base change gr_L are smooth, and $\mathrm{supp} := \widetilde{\beta} \circ \widetilde{\mathrm{gr}}$ is an iterated projective bundle (see Proposition 3.3).

Proposition 4.1. *The motive of $\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e},L}$ is abelian.*

Proof. Since $\mathrm{supp} := \widetilde{\beta} \circ \widetilde{\mathrm{gr}} : \widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e}} \to C^{\underline{l}} \times \mathfrak{Bun}_{m,e_r}$ is a $(\sum_{i=1}^r l_i)$-iterated $\mathbb{P}^{m-1}$-bundle (see Proposition 3.3), its restriction $\mathrm{supp}_L : \widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e},L} \to \widetilde{\omega}^{-1}(L)$ is also an iterated projective bundle and so it suffices to prove that the motive of $\widetilde{\omega}^{-1}(L)$ is abelian.

For $\underline{c} := (c_{i,j})_{1 \leq i \leq r, 1 \leq j \leq l_i} \in C^{\underline{l}}$, let $\mathcal{O}_C(\underline{c}) := \mathcal{O}_C(-\sum_{i=1}^r \sum_{j=1}^{l_i} ic_{ij})$. Then by definition

$$\widetilde{\omega}^{-1}(L) = \left\langle (\underline{c}, F) \in C^{\underline{l}} \times \mathfrak{Bun}_{m,e_r} : \det(F)^{\otimes r+1} \cong L \otimes \mathcal{O}_C(\underline{c}) \right\rangle.$$

Note that as $\det(F)^{\otimes r+1}$ has degree $(r+1)e_r$, so does $L \otimes \mathcal{O}_C(\underline{c})$. Consider the fibre product B

$$\begin{tikzcd} B \arrow[r] \arrow[d] & C^{\underline{l}} \arrow[d] \\ \mathrm{Pic}^{e_r}(C) \arrow[r, "\cdot(r+1)"] & \mathrm{Pic}^{(r+1)e_r}(C) \end{tikzcd}$$

where the bottom arrow is the map taking $(r+1)$-powers and the right arrow is given by $\underline{c} \mapsto L \otimes \mathcal{O}_C(\underline{c})$. Hence, the scheme B parametrises pairs $(N, \underline{c}) \in \mathrm{Pic}^{e_r}(C) \times C^{\underline{l}}$ such that $N^{\otimes r+1} \cong L \otimes \mathcal{O}_C(\underline{c})$. Let $\mathcal{N} \in \mathrm{Pic}^{e_r}(C \times B)$ denote the pullback of the Poincaré line bundle on $C \times \mathrm{Pic}^{e_r}(C)$; then

$$\widetilde{\omega}^{-1}(L) \cong \mathfrak{Bun}_{C \times B/B, m, \mathcal{N}}$$

is isomorphic to the stack of rank m vector bundles on $C \times B/B$ with determinant $\mathcal{N}$.

To show that the motive of $\mathfrak{Bun}_{C \times B/B,m,\mathcal{N}}$ is abelian, it suffices to show that $M(B)$ is abelian by Theorem B.4. To prove that the motive of B is abelian, we let $C' \to C$ denote the finite étale cover of C obtained by base change under the multiplication by $r+1$ map on $\mathrm{Jac}(C)$, where we fix $x_0 \in C$ to determine an Abel–Jacobi map $C \to \mathrm{Jac}(C)$ given by $c \mapsto \mathcal{O}_C(x_0 - c)$. We shall view elements in C' as pairs (c, M) where $c \in C$ and M is a line bundle with $M^{\otimes r+1} \cong \mathcal{O}(x_0 - c)$. We claim that there is a surjective morphism $(C')^{\underline{l}} \to B$. To define this map, we use the universal property of the fibre product B: we take the natural morphism $(C')^{\underline{l}} \to C^{\underline{l}}$ and the morphism $(C')^{\underline{l}} \to \mathrm{Pic}^{e_r}(C)$ given by

$$(c_{ij}, M_{ij})_{1 \leq i \leq r, 1 \leq j \leq l_i} \mapsto L' \otimes \bigotimes_{i=1}^{r} \bigotimes_{j=1}^{l_i} M_{ij}^{\otimes i},$$

where $L' \in \mathrm{Pic}^{e_r}(C)$ is a fixed $(r+1)$-root of $\mathcal{O}_C(\underline{x_0}) := L \otimes \mathcal{O}_C(-\sum_{i=1}^r \sum_{j=1}^{l_i} ix_0)$. Since

$$\left(L' \otimes \bigotimes_{i=1}^{r} \bigotimes_{j=1}^{l_i} M_{ij}^{\otimes i} \right)^{\otimes r+1} \simeq L \otimes \mathcal{O}_C \left(-\sum_{i=1}^{r} \sum_{j=1}^{l_i} ix_0 \right) \otimes \bigotimes_{i=1}^{r} \bigotimes_{j=1}^{l_i} \mathcal{O}(i(x_0 - c_{ij})) \simeq L \otimes \mathcal{O}_C(\underline{c})$$

the corresponding compositions from $(C')^{\underline{l}}$ to $\mathrm{Pic}^{(r+1)e_r}(C)$ commute and consequently there is a morphism $(C')^{\underline{l}} \to B$, which is clearly surjective. By Lemma 4.2 below, we conclude that B is abelian and thus also $\widetilde{\omega}^{-1}(L)$ is abelian. □

To complete the proof, we use the following result, which is well-known to experts but for which we did not find a suitable reference. Recall that there is a fully faithful embedding of the category of Chow motives with rational coefficients over k into $\mathrm{DM}(k, \mathbb{Q})$, so that it is enough to show that $M(B)$ is abelian as a Chow motive in the sense of [65, Definition 1.1]. Recall as well that abelian Chow motives in this sense are Kimura finite-dimensional [65, Proposition 1.8].

Lemma 4.2. *Let $X \to Y$ be a surjective morphism of smooth projective k-varieties. If the Chow motive $M(X)$ is Kimura finite-dimensional (in particular, if $M(X)$ is abelian), then $M(Y)$ is a direct factor of $M(X)$. In particular, if $M(X)$ is abelian, then $M(Y)$ is abelian.*

Proof. The surjective morphism $X \to Y$ induces a surjective morphism of Chow motives in the sense of [43, Definition 6.5] by [43, Remark 6.6]. By [43, Lemma 6.8], this implies there is a morphism $\eta : M(Y) \to M(X)$ of Chow motives such that $M(f) \circ \eta = \mathrm{id}_{M(Y)}$. By [43, Proposition 6.9], the fact that $M(X)$ is Kimura finite-dimensional then also implies that $M(Y)$ is Kimura finite-dimensional.

Now consider the category $\mathcal{M}_{\mathrm{rat}}^{\mathrm{fd}}(k,\mathbb{Q})$ of Kimura finite-dimensional Chow motives over k with $\mathbb{Q}$-coefficients and the category $\mathcal{M}_{\mathrm{num}}(k,\mathbb{Q})$ of pure motives over k with rational coefficients with respect to numerical equivalence. Denote by $(-)^{\mathrm{num}} : \mathcal{M}_{\mathrm{rat}}^{\mathrm{fd}}(k,\mathbb{Q}) \to \mathcal{M}_{\mathrm{num}}(k,\mathbb{Q})$ the natural functor. This functor is conservative and lifts idempotents by [5, Corollaires 3.15-16].

The category $\mathcal{M}_{\mathrm{num}}(k,\mathbb{Q})$ is abelian semi-simple by Jannsen's theorem, so the existence of η implies that $M(f)^{\mathrm{num}} : M(Y)^{\mathrm{num}} \to M(X)^{\mathrm{num}}$ makes $M(Y)^{\mathrm{num}}$ into a direct factor of $M(X)^{\mathrm{num}}$. Let $p : M(X) \to M(X)$ be a projector such that p^{num} has image $M(Y)^{\mathrm{num}}$. In other words, the composition $M(Y)^{\mathrm{num}} \xrightarrow{M(f)^{\mathrm{num}}} M(X)^{\mathrm{num}} \xrightarrow{p^{\mathrm{num}}} \mathrm{Im}(p^{\mathrm{num}})$ is an isomorphism. Since $(-)^{\mathrm{num}}$ is conservative, this implies that $M(Y) \simeq \mathrm{Im}(p)$ is a direct factor of $M(X)$. □

It is not obvious that f'_L is small, as it is the base change of (the small map) f' under the non-flat morphism $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e},L} \hookrightarrow \mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e}}$. To prove that f'_L is small we will instead prove that f_L is small.

Proposition 4.3. *The forgetful map $f'_L : \widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e},L} \to \mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e},L}$ is small and a $\prod_{i=1}^r S_{l_i}$-torsor on a dense open.*

Proof. It suffices to prove that f_L is small and a $\prod_{i=1}^r S_{l_i}$-torsor on a dense open, as f'_L is the pullback of f_L under the smooth map gr_L. Recall that the forgetful map

$$\widetilde{\mathcal{T}} := \prod_{i=1}^{r} \widetilde{\mathfrak{Coh}}_{0,l_i} \to \mathcal{T} := \prod_{i=1}^{r} \mathfrak{Coh}_{0,l_i}$$

is small and a $\prod_{i=1}^r S_{l_i}$-torsor on a dense open.

Let $\mathcal{L} \to C \times \mathcal{T}$ be the line bundle with $\mathcal{L}_t := L \otimes \bigotimes_{i=1}^r \mathcal{O}_C(i \operatorname{supp} T_i)$ over $t = (T_1, \ldots, T_r) \in \mathcal{T}$. Then by definition

$$\gamma^{-1}(L) = \left\langle (t,F) \in \mathcal{T} \times \mathfrak{Bun}_{m,e_r} : \det(F)^{\otimes r+1} \cong \mathcal{L}_t \right\rangle =: \mathfrak{Bun}_{C\times\mathcal{T}/\mathcal{T},m,\det(\,)^{\otimes r+1}\simeq\mathcal{L}}.$$

Let $\mathcal{T}'$ be the fibre product

$$\begin{tikzcd} \mathcal{T}' \arrow[r] \arrow[d] & \mathcal{T} \arrow[d, "\mathcal{L}"] \\ \mathrm{Pic}^{e_r}(C) \arrow[r, "\cdot(r+1)"] & \mathrm{Pic}^{(r+1)e_r}(C) \end{tikzcd}$$

where the bottom morphism is multiplication by $(r+1)$ and the right morphism is given by $\mathcal{L}$. Let $\mathcal{L}' \in \mathrm{Pic}^{(r+1)e_r}_{C\times\mathcal{T}'/\mathcal{T}'}(\mathcal{T}')$ denote the pullback of $\mathcal{L}$. Then by construction all $(r+1)$-roots of $\mathcal{L}'$ exist and

$$\mathfrak{Bun}_{C\times\mathcal{T}'/\mathcal{T}',m,\det(\,)^{\otimes r+1}\simeq\mathcal{L}'} = \bigsqcup_{\substack{(r+1)-\text{root} \\ \mathcal{N} \text{ of } \mathcal{L}'}} \mathfrak{Bun}_{C\times\mathcal{T}'/\mathcal{T}',m,\mathcal{N}}.$$

Since each $\mathfrak{Bun}_{C\times\mathcal{T}'/\mathcal{T}',m,\mathcal{N}}$ is smooth[5] over $\mathcal{T}'$, we have that $\mathfrak{Bun}_{C\times\mathcal{T}'/\mathcal{T}',m,\det(\,)^{\otimes r+1}\simeq\mathcal{L}'}$ is smooth over $\mathcal{T}'$ is smooth. Consequently

$$\gamma^{-1}(L) \simeq \mathfrak{Bun}_{C\times\mathcal{T}/\mathcal{T},m,\det(\,)^{\otimes r+1}\simeq\mathcal{L}} \to \mathcal{T}$$

is smooth by étale descent under the finite étale cover $\mathcal{T}' \to \mathcal{T}$. Since f_L is the base change of $\widetilde{\mathcal{T}} \to \mathcal{T}$ under the smooth map $\gamma^{-1}(L) \to \mathcal{T}$, we conclude that the morphism f_L is also small and a $\prod_{i=1}^r S_{l_i}$-torsor on a dense open. □

We can now conclude that the stack of injective chain homomorphisms with fixed total determinant has abelian motive.

Corollary 4.4. *For a constant tuple of ranks $\underline{m}$, the motive of $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e},L}$ is abelian.*

[5]Here $\mathcal{T}'$ is a stack, but we can base change to an atlas of $\mathcal{T}'$ to prove $\mathfrak{Bun}_{C\times\mathcal{T}'/\mathcal{T}',m,\mathcal{M}}$ is smooth via descent.

Proof. This follows from Propositions 4.1 and 4.3, as by Theorem 3.4, the motive of $\mathfrak{Ch}^{\mathrm{inj}}_{\underline{m},\underline{e},L}$ is a direct factor of the motive of $\widetilde{\mathfrak{Ch}}^{\mathrm{inj}}_{\underline{m},\underline{e},L}$ and thus is also abelian. □

4.3. **The SL-Higgs moduli space has abelian motive.** We are now able to prove the main result of this section.

Theorem 4.5. *Assume that $C(k) \neq \emptyset$. Let D be a divisor on C with either $D = K_C$ or $\deg(D) > 2g-2$. The motive in $\mathrm{DM}(k,\mathbb{Q})$ of the D-twisted* SL*-Higgs moduli space $\mathcal{M}_L^D := \mathcal{M}_{n,L}^D(C)$ for a line bundle L of coprime degree to n is abelian.*

Proof. The proof follows by adapting the argument of Theorem 3.5 but using the motivic Białynicki-Birula decomposition (7) in place of Proposition 3.1, and Corollary 4.4 in place of Proposition 3.3. □

Remark 4.6. From the proof of Proposition 4.1, one sees that

$$M(\mathcal{M}_L^D) \in \langle M(C_r) : 1 \leq r \leq n \rangle^{\otimes},$$

where $C_r \to C$ is the r^{2g}-étale cover obtained by pullback along the multiplication by r map on $\mathrm{Jac}(C)$. In fact, one could ask if this category is genuinely larger than the category generated by $M(C)$. In [26, Proposition 5.7], we show with L. Fu that if C is a general complex curve then $M(\mathcal{M}_L) \notin \langle M(C) \rangle^{\otimes}$ and so this category really is larger in this case.

We can now prove Theorem 1.2.

Proof of Theorem 1.2. This follows from Theorems 3.5 and 4.5, as well as Remark 4.6 combined with the argument in [39, §6.3] to show the further claim that the motives are direct factors of products of curves. □

Remark 4.7. As mentioned in the introduction, in upcoming work of Groechenig and Shiyu Shen [31], they also adapt the techniques of [29, 28] to study the class of the SL-Higgs moduli space in the Grothendieck ring of varieties in order to deduce its Betti cohomology is torsion-free. However, in [31], they describe the class of the stack of injective chain homomorphisms of fixed total determinant using étale covers of products of symmetric powers of curves, whereas our proof is simplified by working with the flag version of the stack and using properties of small maps similarly to the approach for GL-Higgs moduli spaces in [36] .

5. Motivic isotypical decompositions and orbifold motives

In this section, we assume that k is an algebraically closed field of characteristic zero and consider motives with coefficients in $\Lambda = \mathbb{Q}(\zeta_n)$, where ζ_n is a primitive nth root of unity.

Fix a rank n and line bundle $L \in \mathrm{Pic}^d(C)$ of coprime degree to n. Recall that $\mathrm{Jac}(C)$ acts on $\mathcal{M}^D := \mathcal{M}_{n,d}^D(C)$ and also $\Gamma := \mathrm{Jac}(C)[n]$ acts on $\mathcal{M}_L^D := \mathcal{M}_{n,L}^D(C)$ by tensoring. Associated to the Γ-action on the motive $M(\mathcal{M}_L^D)$ of the D-twisted SL-Higgs moduli space, we have the following isotypical decomposition in $\mathrm{DM}(k,\Lambda)$

$$M(\mathcal{M}_L^D) \simeq \bigoplus_{\kappa \in \widehat{\Gamma}} M(\mathcal{M}_L^D)_\kappa \simeq M(\mathcal{M}_L^D/\Gamma) \oplus \bigoplus_{\kappa \neq 0 \in \widehat{\Gamma}} M(\mathcal{M}_L^D)_\kappa,$$

where the Γ-invariant piece $M(\mathcal{M}_L^D)^\Gamma \simeq M(\mathcal{M}_L^D/\Gamma) \simeq M(\overline{\mathcal{M}}^D)$ is the motive of the D-twisted PGL_n-Higgs moduli space. Note that the piece indexed by non-trivial characters κ is non-zero, as Γ acts non-trivially on $M(\mathcal{M}_L^D)$. Indeed this was already observed on the level of cohomology in rank $n = 2$ by Hitchin [37] and the above decomposition on the level of cohomology was recently described by Maulik and Shen [50]. In this section, we introduce the set-up and notation of [50] in order to be able to describe the motivic isotypical pieces above.

5.1. **Weil pairing and cyclic covers.** The n-torsion in the Jacobian $\Gamma := \mathrm{Jac}(C)[n]$ has a natural non-degenerate Weil pairing

$$\langle -,-\rangle : \Gamma \times \Gamma \to \mu_n$$

which allows us to identify $\Gamma \cong \widehat{\Gamma}$. Alternatively, we can use the Abel-Jacobi map to identify Γ

$$\Gamma \cong H^1(C, \mathbb{Z}/n\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(C), \mathbb{Z}/n\mathbb{Z})$$

with cyclic covers of C of degree dividing n. The Weil pairing corresponds to the intersection pairing on $H^1(C, \mathbb{Z}/n\mathbb{Z})$.

Notation 5.1. For $\gamma \in \Gamma$, we denote the corresponding character in $\widehat{\Gamma}$ by $\kappa := \kappa(\gamma) = \langle \gamma, -\rangle$ and the corresponding cyclic cover by $\pi := \pi_\gamma : C_\gamma \to C$ and write $m_\gamma := \mathrm{ord}(\gamma) = \deg(\pi)$, which divides n and so we write $n_\gamma := n/m_\gamma$, and write $G_\pi := \mathrm{Gal}(C_\gamma/C) \cong \mathbb{Z}/m_\gamma\mathbb{Z}$.

Note that $g_{C_\gamma} = 1 + m_\gamma(g_C - 1)$ by Riemann–Hurwitz. More concretely, if $\gamma \in \Gamma$ corresponds to $L_\gamma \in \mathrm{Jac}(C)$ of degree m_γ dividing n, then C_γ is constructed as a closed subscheme of the total space of L_γ given by taking fibrewise the m_γ-roots of unity.

5.2. **Fixed loci and relative Higgs moduli spaces for cyclic covers.** In this section, we fix $\gamma \in \Gamma$. Our goal is to interpret the motive of the γ-fixed locus in the D-twisted SL-Higgs moduli space in terms of a direct summand of a motive of a relative Higgs moduli space for the associated cyclic cover $\pi : C_\gamma \to C$. In fact, a description of the γ-fixed locus in the moduli spaces of vector bundles in terms of a relative moduli space for π was given by Narasimhan and Ramanan [54], and as observed in [35, Section 7], the arguments also extend to Higgs bundles. The compatibility of this description with the corresponding Hitchin fibrations was described in [50, Section 1.5].

Definition 5.2. The fixed locus in $\mathcal{M}_L^D$ of an element $\gamma \in \Gamma$ is denoted $\mathcal{M}_\gamma^D := (\mathcal{M}_L^D)^\gamma$. We let $\mathcal{A}_\gamma^D$ denote the image of $\mathcal{M}_\gamma^D$ under the D-twisted SL-Hitchin map $h_L^D\colon \mathcal{M}_L \to \mathcal{A}_L^D$ and write $h_\gamma^D\colon \mathcal{M}_\gamma^D \to \mathcal{A}_\gamma^D$ for the restricted Hitchin map and define

$$d_\gamma^D := \mathrm{codim}(i_\gamma^D : \mathcal{A}_\gamma^D \hookrightarrow \mathcal{A}_L^D).$$

The fixed locus $\mathcal{M}_\gamma^D$ is the coarse moduli space of the corresponding fixed locus at the level of moduli stacks, which is a smooth Deligne-Mumford stack.

There is an induced Γ-action on $\mathcal{M}_\gamma^D$, which also has an associated motivic isotypical decomposition in $\mathrm{DM}(k, \Lambda)$. Note that there is no Γ-action on the D-twisted SL-Hitchin base $\mathcal{A}_L^D$ and thus $\mathcal{A}_\gamma^D$ is not a fixed locus, but rather the image of a fixed locus under the SL-Hitchin map.

Now consider the cyclic cover $\pi : C_\gamma \to C$ associated to γ of degree $m_\gamma := \mathrm{ord}(\gamma) = \deg(\pi_\gamma)$ with $n_\gamma := n/m_\gamma$ and $G_\pi := \mathrm{Gal}(C_\gamma/C) \cong \mathbb{Z}/m_\gamma\mathbb{Z}$.

Definition 5.3. We let $D_\gamma := \pi^* D$ and define a map

$$\begin{array}{rccc} Q_\gamma\colon & \mathcal{M}_{n_\gamma,d}^{D_\gamma}(C_\gamma) & \to & \mathcal{M}_{1,d}^D(C) \\ & [E,\Phi] & \mapsto & [\det(\pi_*(E)), \mathrm{tr}(\pi_*(\Phi))] \end{array}$$

and we define the π-relative D-twisted SL-Higgs moduli space to be $\mathcal{M}_\pi^D := Q_\gamma^{-1}([L,0])$. The D-twisted GL-Hitchin fibration $h_{n_\gamma,d}^{D_\gamma}(C_\gamma) : \mathcal{M}_{n_\gamma,d}^{D_\gamma}(C_\gamma) \to \mathcal{A}_{n_\gamma,d}^{D_\gamma}(C_\gamma)$ restricts to a Hitchin fibration

$$h_\pi^D : \mathcal{M}_\pi^D \twoheadrightarrow \mathcal{A}_\pi^D.$$

As explained in [50, §1.2], the morphism Q_γ is smooth, as it is the composition of two smooth maps. Hence the π-relative D-twisted SL-Higgs moduli space is smooth, but is not connected (see [50, Proposition 1.1]). We summarise the geometric properties from [50] which we need.

Proposition 5.4. [50, §1.5] *The morphism $\mathcal{M}_\pi^D \to \mathcal{M}_L^D$ given by pushforward along π has image $\mathcal{M}_\gamma^D$. Furthermore, there is a commutative diagram*

$$\begin{tikzcd} \mathcal{M}_\pi^D \arrow[r, "p_\gamma^D"] \arrow[d, "h_\pi^D"'] & \mathcal{M}_\gamma^D \arrow[d, "h_\gamma^D"] \\ \mathcal{A}_\pi^D \arrow[r, "q_\gamma^D"] & \mathcal{A}_\gamma^D \end{tikzcd}$$

where p_γ^D and q_γ^D are geometric G_π-quotients and p_γ^D is Γ-equivariant. The action of G_π on $\mathcal{M}_\pi^D$ is free and permutes the connected components (however, the action of G_π on $\mathcal{A}_L$ is not free). The quotient $\mathcal{M}_\gamma^D \simeq \mathcal{M}_\pi^D/G_\pi$ is connected.

Consequently, we can make some dimension computations which are used (sometimes implicitly) in [50]. We also collect various dimension formulae in Appendix C.

Lemma 5.5. *The following dimension formulae hold.*

i) $\dim \mathcal{M}_\pi^D = \dim \mathcal{M}_\gamma^D = (nn_\gamma - 1)\deg(D)$.
ii) $\dim \mathcal{A}_\pi^D = \dim \mathcal{A}_\gamma^D = \frac{n(n_\gamma+1)\deg(D)}{2} - (n-1)(g-1) - \deg(D)$.
iii) $\operatorname{codim}_{\mathcal{M}_L^D}(\mathcal{M}_\gamma^D) = n(n-n_\gamma)\deg(D)$.
iv) $d_\gamma^D = \frac{n(n-n_\gamma)\deg(D)}{2}$.

Proof. The last two formulae follow from the first two together with the formulae [22, Eq.(78)] for $\dim(\mathcal{M}_L^D)$ and $\dim(\mathcal{A}_L^D)$.

By Proposition 5.4, we have $\mathcal{M}_\gamma^D \simeq \mathcal{M}_\pi/G_\gamma$ hence $\dim \mathcal{M}_\pi^D = \dim \mathcal{M}_\gamma^D$ and the same argument implies $\dim \mathcal{A}_\pi^D = \dim \mathcal{A}_\gamma^D$. If $\gamma = \mathrm{id}$, then $\mathcal{M}_\pi^D = \mathcal{M}_L^D$ and $\mathcal{A}_\pi^D = \mathcal{A}_L^D$, in which case we also have $n_\gamma = n$ and we are done by [22, Eq.(78)]. We thus assume $\gamma \neq \mathrm{id}$.

Since $\mathcal{M}_\pi^D$ is a fibre of the fibration Q_γ, we have

$$\dim \mathcal{M}_\pi^D = \dim \mathcal{M}_{n_\gamma,d}^{D_\gamma}(C_\gamma) - \dim \mathcal{M}_{1,d}^D(C).$$

Since $\gamma \neq \mathrm{id}$, we have $m_\gamma \neq 1$ and $\deg(D_\gamma) = m_\gamma \deg(D) > 2g-2$ (even if $D = K_C$). By the formula in [57, Proposition 7.1.(c)] and Riemann-Roch, we deduce that

$$\dim \mathcal{M}_\pi^D = n_\gamma^2(m_\gamma \deg(D)) + 1 - (g + (\deg(D) + 1 - g)) = (nn_\gamma - 1)\deg(D)$$

as claimed.

By [50, Eq.(18)], [22, Eq.(77)] and Riemann-Roch, we have

$$\begin{aligned} \dim \mathcal{A}_\pi^D &= \dim \mathcal{A}_{n_\gamma}^{D_\gamma}(C_\gamma) - \dim H^0(C_\gamma, D_\gamma)^{G_\gamma} \\ &= \frac{n_\gamma(n_\gamma+1)}{2} m_\gamma \deg(D) - n_\gamma m_\gamma (g-1) - \dim H^0(C, D) \\ &= \frac{n(n_\gamma+1)}{2} \deg(D) - n(g-1) - n(g-1) - (\deg(D) + 1 - g) \\ &= \frac{n(n_\gamma+1)\deg(D)}{2} - (n-1)(g-1) - \deg(D) \end{aligned}$$

which finishes the proof. □

Note that, in the case $D = K_C$, we have

$$d_\gamma = \frac{1}{2}\operatorname{codim}_{\mathcal{M}_L}(\mathcal{M}_\gamma)$$

(reflecting the symplectic geometry of that case) but this does not hold otherwise.

Based on this, we want to relate the motives of the fixed loci of the Γ-action and the relative Higgs moduli spaces. We need the following standard result.

Lemma 5.6. *Here k can be an arbitrary field. Let G be a finite group acting on a k-variety Y such that the action admits a geometric quotient $f : Y \to X$. Let $M \in \mathrm{DM}(X, \Lambda)$. Then there is an induced G-action on p_*p^*M and the unit map $M \to p_*p^*M$ factor through an isomorphism $M \simeq (p_*p^*M)^G$.*

Proof. This is a special case of [8, Corollaire 2.1.166] which applies as, in the terminology of *loc. cit.*, $\mathrm{DM}(-, \Lambda)$ is $\mathbb{Q}$-linear and separated (note that in characteristic 0 this last condition follows from étale descent by [8, Proposition 2.1.162], but in fact holds for any k [12, Theoreme 3.9]). □

Corollary 5.7. *The object $q^D_{\gamma *} h^D_{\pi *} \mathbb{1}$ admits a G_π-action such that there is an isomorphism*

$$(q^D_{\gamma *}(h^D_{\pi *}\mathbb{1}))^{G_\pi} \simeq h^D_{\gamma *}\mathbb{1}$$

in $\mathrm{DM}(\mathcal{A}^D_\gamma, \Lambda)$. Consequently, we have the following isomorphism in $\mathrm{DM}(k, \Lambda)$

$$M(\mathcal{M}^D_\pi)^{G_\pi} \simeq M(\mathcal{M}^D_\gamma).$$

Moreover, this isomorphism is Γ-equivariant.

Proof. The first statement follows from Proposition 5.4 and Lemma 5.6 applied to p^D_γ, as well as the fact that h^D_π is G_π-equivariant:

$$(q_{\gamma *}(h^D_{\pi *}\mathbb{1}))^{G_\pi} \simeq (h^D_{\gamma *} p_{\gamma *}\mathbb{1})^{G_\pi} \simeq h^D_{\gamma *}(p_{\gamma *}\mathbb{1})^{G_\pi} \simeq h^D_{\gamma *}\mathbb{1}.$$

The second follows by pushforward to $\mathrm{Spec}(k)$. The Γ and G_π-actions commute and the constructions are all Γ-equivariant, so the resulting isomorphism is Γ-equivariant. □

5.3. **The orbifold motive of the PGL-Higgs moduli space.** We consider the D-twisted PGL_n-Higgs moduli space as the orbifold quotient

$$\overline{\mathcal{M}}^D = [\mathcal{M}^D_L/\Gamma]$$

which has a natural gerbe δ_L obtained as by descending a Γ-equivariant μ_n-gerbe on $\mathcal{M}^D_L$ (see [35, Section 3]).

The following definition is just the natural extension of Hausel and Thaddeus's description [35] of the stringy E-polynomial of $\overline{\mathcal{M}}^D$ to the motivic context (see also [48]).

Definition 5.8. The orbifold motive of the D-twisted PGL-Higgs moduli space $\overline{\mathcal{M}}^D$ with respect to the gerbe δ_L is defined in $\mathrm{DM}(k, \Lambda)$ as follows

$$M_{\mathrm{orb}}(\overline{\mathcal{M}}^D, \delta_L) := \bigoplus_{\gamma \in \Gamma} M(\mathcal{M}^D_\gamma)_{\kappa(\gamma)}\{d_\gamma\} = M(\overline{\mathcal{M}}^D) \oplus \bigoplus_{0 \neq \gamma \in \Gamma} M(\mathcal{M}^D_\gamma)_{\kappa(\gamma)}\{d_\gamma\},$$

where $\mathcal{M}^D_\gamma$ is the γ-fixed locus in the D-twisted SL-Higgs moduli space and d_γ is the codimension appearing in Definition 5.2.

6. Proof of motivic mirror symmetry

In this section, we assume that k is an algebraically closed field of characteristic zero and consider motives with coefficients in $\Lambda = \mathbb{Q}(\zeta_n)$. We fix $\gamma \in \Gamma$ corresponding to $\kappa = \kappa(\gamma) \in \widehat{\Gamma}$ and cyclic cover $\pi = \pi_\gamma : C_\gamma \to C$ of degree m_γ with $n = n_\gamma m_\gamma$ and Galois group G_π as in §5.1.

Let us outline the structure of this section. In §6.1-6.3, we will construct the morphism $\beta^D_{\gamma,\mathrm{mot}} \in \mathrm{DM}(\mathcal{A}^D_L, \Lambda)$ whose Betti realisation is the map β^D_γ of (6), which Maulik and Shen show is an isomorphism. Since the cohomological construction of β^D_γ uses cohomological correspondences and vanishing cycles, we will use motivic correspondences and motivic vanishing cycles, which are discussed in §6.1 and Appendix A respectively. In §6.2, we use motivic correspondences to construct $\beta^D_{\gamma,\mathrm{mot}}$ in the case when $\deg(D) > 2g - 2$ and even. In §6.3, we use motivic vanishing cycles to construct $\beta^D_{\gamma,\mathrm{mot}}$ in general by passing from $D + p$ to D. To complete the proof, we need to show the pushfoward $\nu^D_{\gamma,\mathrm{mot}}$ of $\beta^D_{\gamma,\mathrm{mot}}$ to k is an isomorphism. In §6.4, we prove that the motives appearing as the source and target of $\nu^D_{\gamma,\mathrm{mot}}$ are abelian in order to conclude that $\nu^D_{\gamma,\mathrm{mot}}$ is an isomorphism using a conservativity argument in §6.5.

6.1. **Motivic correspondences.** In this section, we discuss motivic correspondences, which are lifts to motivic sheaves of the cohomological correspondences of [2, Exposé III §3]. We follow the presentation of [67, Appendix A], since this is the source used by [50] and we wish to lift their results to motives; however, we use a covariant convention like in [2] instead of the contravariant one in [67] which seems unnecessarily confusing.

Note that the results of this subsection apply more generally when k is any field and Λ is any $\mathbb{Q}$-algebra (and indeed much more generally to other types of motivic sheaves). We sometimes denote $\mathrm{DM}(X, \Lambda)$ by $\mathrm{DM}(X)$.

Definition 6.1. For a commutative (but not necessarily cartesian) diagram of finite type separated k-schemes[6]

(8)
$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

such that the induced morphism $r := (p, q) : Z \to X \times_S Y$ is proper, we define a *motivic correspondence supported on* Z from $M \in \mathrm{DM}(X, \Lambda)$ to $N \in \mathrm{DM}(Y, \Lambda)$ to be a morphism

$$\zeta : p^*M \to q^!N$$

in $\mathrm{DM}(Z, \Lambda)$.

Motivic correspondences can be pushed forward to morphisms in $\mathrm{DM}(S, \Lambda)$ as follows.

Construction 6.2. Given a motivic correspondence $\zeta : p{*}M \to q^!N$ as above, we will construct a morphism

$$\zeta_\sharp : f_!M \to g_*N$$

in $\mathrm{DM}(S, \Lambda)$. For this, we first associate to the motivic correspondence ζ supported on Z a motivic correspondence $\zeta_{X\times_S Y}$ supported on $X \times_S Y$ via the following map of morphism groups

$$\begin{aligned} \mathrm{DM}(Z)(p^*M, q^!N) \xrightarrow{r_*} \mathrm{DM}(X \times_S Y)(r_*p^*M, r_*q^!N) \simeq \mathrm{DM}(X \times_S Y)(r_*r^*\pi_X^*M, r_!r^!\pi_Y^!N) \\ \to \mathrm{DM}(X \times_S Y)(\pi_X^*M, \pi_Y^!N) \end{aligned}$$

where π_X and π_Y denote the projections from $X \times_S Y$ to X and Y. Here the first map comes from the functoriality of r_*, the next isomorphism relies on the properness of r and the final map is given by pre- and post-composition with the unit and counit for the adjunctions $r^* \dashv r_*$ and $r_! \dashv r^!$. Then we define $\zeta_\sharp$ to be the image of the motivic correspondence $\zeta_{X\times_S Y}$ supported on $X \times_S Y$ under the following isomorphisms

$$\mathrm{DM}(X \times_S Y)(\pi_X^*M, \pi_Y^!N) \simeq \mathrm{DM}(X)(M, \pi_{X*}\pi_Y^!N) \simeq \mathrm{DM}(X)(M, f^!g_*N) \simeq \mathrm{DM}(S)(f_!M, g_*N)$$

where the first and last isomorphism are adjunctions and the middle one is base change.

The construction above is natural in Z and ζ in various ways. We only need the following lemma on compatibility with a finite group action, which is a straightforward consequence of the naturality of the construction with respect to isomorphisms.

Lemma 6.3. *In the situation of Diagram* (8)*, assume that there exists a finite group H acting on the whole diagram (i.e. on the individual schemes and such that the morphisms are equivariant). Let M and N be also be H-equivariant objects in* $\mathrm{DM}(X, \Lambda)$ *and* $\mathrm{DM}(Y, \Lambda)$ *respectively; this induces H-equivariant structures on p^*M, $q^!N$, $f_!M$ and g_*N. If the motivic correspondence*

$$\zeta : p^*M \to q^!N$$

is H-equivariant, then the induced morphism

$$\zeta_\sharp : f_!M \to g_*N$$

[6] For us it suffices to work with finite type separated schemes, but the construction works in greater generality.

is also H-equivariant.

We note that Construction 6.2 commutes with Betti realisation.

Lemma 6.4. *Let $\sigma : k \to \mathbb{C}$ be a complex embedding and suppose $\zeta : p^*M \to q^!N$ is a motivic correspondence supported on Z as above between constructible motives M and N. Since the Betti realisation functor R_B commutes with p^* and $q^!$ on constructible objects, we have an induced cohomological correspondence*

$$R_B(\zeta) : p^* R_B M \to q^! R_B N$$

and similarly an induced morphism

$$R_B(\zeta_\sharp) : f_! R_B M \to g_* R_B N.$$

Then we have an equality

$$R_B(\zeta_\sharp) = (R_B\zeta)_\sharp$$

where this right side should be understood as the construction in [67, §A.1].

Moreover, if ζ is equivariant in the sense of Lemma 6.3, then the induced equivariant structures on $R_B(\zeta_\sharp) = (R_B\zeta)_\sharp$ coincide.

Proof. This follows directly from the fact that the Betti realisation commutes with the six operations on constructible motives [10, Theorem 3.19]. ☐

Let us explain how, in certain situations, the fundamental class of Z in its rational Chow group (i.e. Borel-Moore rational motivic homology) provides a natural motivic correspondence.

Definition 6.5. In the situation of Diagram (8), suppose that Y is smooth of dimension e over k and that the morphism q is equidimensional of dimension d. Then Z is a (usually singular) equidimensional variety of dimension $d + e$, so it has a fundamental class

$$[Z] \in \mathrm{CH}_{d+e}(Z) \otimes \Lambda.$$

We have isomorphisms

$$\begin{aligned}\mathrm{CH}_{d+e}(Z) \otimes \Lambda &\simeq \mathrm{Hom}_{\mathrm{DM}(k)}(\Lambda\{d+e\}, (p_Y \circ q)_*(p_Y \circ q)^!\Lambda) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(Z)}(\Lambda_Z\{d+e\}, q^!\Lambda_Y\{e\}) \\ &\simeq \mathrm{Hom}_{\mathrm{DM}(Z)}(\Lambda_Z\{d\}, q^!\Lambda_Y)\end{aligned}$$

where the second isomorphism follows from relative purity for the smooth structure morphism $p_Y : Y \to \mathrm{Spec}(k)$. Through these isomorphisms, the class $[Z]$ induces a motivic correspondence supported on Z from $M = \Lambda_X\{d\}$ to $N = \Lambda_Y$

$$[Z] : \Lambda_Z\{d\} \to q^!\Lambda_Y$$

and corresponding morphism $[Z]_\sharp : f_!\Lambda_X\{d\} \to g_*\Lambda_Y$ in $\mathrm{DM}(Z, \Lambda)$.

6.2. **A motivic endoscopic correspondence for Higgs bundles.** In §5.2 for $\gamma \in \Gamma$, we constructed various Higgs moduli spaces and Hitchin maps fitting into a commutative diagram

$$\begin{tikzcd} \mathcal{M}^D_\pi \arrow[r, "p^D_\gamma"] \arrow[d, "h^D_\pi"'] & \mathcal{M}^D_\gamma \arrow[r, hook] \arrow[d, "h^D_\gamma"] & \mathcal{M}^D_L \arrow[d, "h^D_L"] \\ \mathcal{A}^D_\pi \arrow[r, "q^D_\gamma"] & \mathcal{A}^D_\gamma \arrow[r, hook, "\iota^D_\gamma"] & \mathcal{A}^D_L. \end{tikzcd}$$

In this section, we assume that $\deg(D)$ is even with $\deg(D) > 2g-2$, and construct a morphism

$$\beta^D_{\gamma,\mathrm{mot}} : (h^D_L)_*\mathbb{1} \to (i^D_\gamma)_*(h^D_\gamma)_*\mathbb{1})\{-d^D_\gamma\} \in \mathrm{DM}(\mathcal{A}^D_L, \Lambda)$$

as an application of the formalism of motivic correspondences. This construction mimics the cohomological construction of Maulik and Shen [50] which builds on the work of Yun [67].

For the construction of this correspondence, Maulik and Shen first construct a line bundle $L' \in \mathrm{Pic}(C)$ from L which satisfies

$$\deg(L') \equiv \deg(L) \pmod{n} \tag{9}$$

and in particular $\gcd(\deg(L'), n) = \gcd(\deg(L'), n_\gamma) = 1$. Equation (9) ensures we can tensor by a line bundle to relate moduli spaces with determinant L and L'. Consider the SL-Higgs moduli space $\mathcal{M}^D_{L'} := \mathcal{M}^D_{n,L'}$ with determinant L' which has Hitchin map $h^D_{L'} : \mathcal{M}^D_{L'} \to \mathcal{A}^D_{L'}$. We can also consider the Hitchin maps γ-fixed locus and π-relative moduli spaces for L' which we denote by adding a subscript L' (e.g. $h^D_{\gamma,L'} : \mathcal{M}^D_{\gamma,L'} \to \mathcal{A}^D_{\gamma,L'}$). All the Hitchin bases are independent of the choice of L and so we drop this additional subscript for the Hitchin base.

To define the correspondence, Maulik and Shen introduce a (singular) variety Σ in [50, §3.3], which fits into a commutative diagram

(10)
$$\begin{tikzcd}
& \Sigma \arrow[dl] \arrow[dr] & \\
\mathcal{M}^D_L \times_{\mathcal{A}^D_L} \mathcal{A}^D_\pi \arrow[dr, "h^D_L \times \mathrm{id}_{\mathcal{A}^D_\pi}"'] & & \mathcal{M}^D_{\pi,L'} \arrow[dl, "h^D_{\pi,L'}"] \\
& \mathcal{A}^D_\pi = \mathcal{A}^D_{\pi,L'} &
\end{tikzcd}$$

where all the morphisms are proper and both G_π and Γ act on Σ in such a way that the diagram is $(G_\pi \times \Gamma)$-equivariant.

We do not need to know anything more about Σ for this paper. Let us nevertheless indicate the idea of the construction. The variety Σ is the Zariski closure of the graph of a morphism

$$g_u : \mathcal{M}^{D,\mathrm{reg}}_L \times_{\mathcal{A}^D_L} \mathcal{A}^{D,\heartsuit}_\pi \to \mathcal{M}^{D,\mathrm{reg}}_{\pi,L'} \times_{\mathcal{A}^D_\pi} \mathcal{A}^{D,\heartsuit}_\pi$$

where the additional decorations on the moduli spaces and Hitchin bases denote suitable open subsets. The restriction to these opens allows, as a special case of the BNR correspondence [15], to parametrise Higgs bundles via line bundles on spectral curves; moreover, the generic SL_n-spectral curve on the locus $\mathcal{A}^D_\gamma \subset \mathcal{A}^D_L$ is nodal and the generic spectral curve for $\mathcal{M}^{D,\mathrm{reg}}_{\pi,L'}$ is its smooth normalisation. The morphism g_u is then given by pullback of line bundles from the nodal curve to its normalisation. The introduction of the line bundle L' is necessary to match up the determinants of the associated Higgs bundles. We refer to [50, §3.1-3] for details (which, again, are immaterial to our argument).

By Definition 6.5, Σ induces a $(G_\pi \times \Gamma)$-equivariant morphism in $\mathrm{DM}(\mathcal{A}^D_\pi, \Lambda)$

$$[\Sigma]_\sharp : (q^D_\gamma)^*(i^D_\gamma)^*(h^D_L)_*\mathbb{1} \simeq (h^D_L \times \mathrm{id}_{\mathcal{A}^D_\pi})_*\mathbb{1} \to (h^D_{\pi,L'})_*\mathbb{1}\{-d^D_\gamma,\}$$

where the first isomorphism follows from proper base change. Since we are working with coefficients in Λ, we can thus take the κ-isotypical part for any $\kappa \in \widehat{\Gamma}$ and get a morphism

$$[\Sigma]_{\sharp,\kappa} : (q^D_\gamma)^*((i^D_\gamma)^*(h^D_L)_*\mathbb{1})_\kappa \to ((h^D_{\pi,L'})_*\mathbb{1})_\kappa\{-d^D_\gamma\}.$$

We will first pushforward along $q^D_\gamma : \mathcal{A}^D_\pi \to \mathcal{A}^D_\gamma$ and take G_π-invariants and then we will pushfoward along $i^D_\gamma : \mathcal{A}^D_\gamma \to \mathcal{A}^D_L$. In the first step, we note that in $\mathrm{DM}(\mathcal{A}^D_\gamma, \Lambda)$, we have

$$\big((q^D_\gamma)_*(q^D_\gamma)^*((i^D_\gamma)^*(h^D_L)_*\mathbb{1})_\kappa\big)^{G_\pi} \simeq ((i^D_\gamma)^*(h^D_L)_*\mathbb{1})_\kappa$$

by Lemma 5.6 and we have

$$\big((q^D_\gamma)_*((h^D_{\pi,L'})_*\mathbb{1})_\kappa\big)^{G_\pi} \simeq ((h^D_{\gamma,L'})_*\mathbb{1})_\kappa$$

by Corollary 5.7. Pushing forward these last two isomorphisms along $i^D_\gamma : \mathcal{A}^D_\gamma \to \mathcal{A}^D_L$ and combining with $[\Sigma]_{\sharp,\kappa}$ as well as an adjunction, we obtain a morphism in $\mathrm{DM}(\mathcal{A}^D_L, \Lambda)$

$$((h^D_L)_*\mathbb{1})_\kappa \longrightarrow (i^D_\gamma)_*(i^D_\gamma)^*((h^D_L)_*\mathbb{1})_\kappa \xrightarrow{(i^D_\gamma)_*((q^D_\gamma)_*[\Sigma]_{\sharp,\kappa})^{G_\pi}} i^D_{\gamma*}((h^D_{\gamma,L'})_*\mathbb{1})_\kappa\{-d^D_\gamma\}. \tag{11}$$

The final step is to pass from L' back to L following [50, §3.4]. Since k is algebraically closed, Equation (9) implies the existence of a line bundle N such that $L' = L \otimes N^{\otimes n}$ and tensoring with N produces a $(G_\pi \times \Gamma)$-equivariant isomorphism

$$\mathcal{M}^D_\pi := \mathcal{M}^D_{\pi,L} \simeq \mathcal{M}^D_{\pi,L'}$$

of $\mathcal{A}^D_\pi$-schemes. This directly provides a G_π-equivariant isomorphism

$$(h^D_\pi)_*\mathbb{1} = (h^D_{\pi,L})_*\mathbb{1} \simeq (h^D_{\pi,L'})_*\mathbb{1}$$

which by Corollary 5.7 gives an isomorphism in $\mathrm{DM}(\mathcal{A}_\gamma, \Lambda)$

$$(h^D_\gamma)_*\mathbb{1} = (h^D_{\gamma,L})_*\mathbb{1} \simeq (h^D_{\gamma,L'})_*\mathbb{1} \tag{12}$$

Definition 6.6. For D of even degree with $\deg(D) > 2g-2$, by combining the morphism (11) with the isomorphism (12) above, we obtain a morphism in $\mathrm{DM}(\mathcal{A}^D_L, \Lambda)$

$$\beta^D_{\gamma,\mathrm{mot}} : ((h^D_L)_*\mathbb{1})_\kappa \to (i^D_\gamma)_*((h^D_\gamma)_*\mathbb{1})_\kappa\{-d^D_\gamma\}.$$

Lemma 6.7. *Assume D has even degree with $\deg(D) > 2g-2$ and fix a complex embedding $\sigma : k \to \mathbb{C}$. The Betti realisation of $\beta^D_{\gamma,\mathrm{mot}}$ is the morphism β^D_γ in* (5)*, which is precisely the isomorphism c^D_κ of* [50, Theorem 3.2].

Proof. This follows from Lemma 6.4 and further applications of [10, Theorem 3.19] on the compatibility of the Betti realisation with the six operations on constructible motives. ☐

6.3. **Passing from $D+p$ to D with motivic vanishing cycles.** Our goal in this section is to extend the construction of $\beta^D_{\gamma,\mathrm{mot}}$ to the case where $\deg(D)$ is odd and the case $D = K_C$ by using vanishing cycles to pass from $D+p$ to D. In this section, we assume that D is either K_C or of degree $> 2g-2$, and we fix an additional point $p \in C(k)$. We start by reviewing the relevant geometric constructions from [50, §4.2-3].

We need to work with moduli stacks of Higgs bundles rather than moduli spaces[7], as in Diagram (13) below, the map of stacks ev_p is smooth and we use smooth base change for vanishing cycles (Proposition A.6). We write $\mathfrak{M}^D_L$ (resp. $\mathfrak{M}^D_\pi$) for the stack of stable D-twisted (resp. π-relative) Higgs bundles of rank n and determinant L. These are smooth Deligne-Mumford stacks; moreover, the natural morphisms $\delta^D_L : \mathfrak{M}^D_L \to \mathcal{M}^D_L$ and $\delta^D_\gamma : \mathfrak{M}^D_\pi \to \mathcal{M}_\pi$ are μ_n-gerbes (the gerbe δ^D_L was used in §5.3).

To pass from $D+p$ to D, we restrict to Higgs bundles on p. Since SL_n-Higgs bundles on a point up to isomorphism correspond to trace-free matrices up to conjugation, the stack of SL_n-Higgs bundles on p is $\mathfrak{M}_{\mathrm{SL}_n}(p) \simeq [\mathfrak{sl}_n/\mathrm{SL}_n]$ with good moduli space $\mathcal{M}_{\mathrm{SL}_n}(p) \simeq \mathfrak{sl}_n /\!\!/ \mathrm{SL}_n$ which is isomorphic to the Hitchin base $\mathcal{A}_{\mathrm{SL}_n}(p) \simeq \mathfrak{t}_n /\!\!/ S_n$, where $\mathfrak{t}_n \subset \mathfrak{sl}_n$ is a Cartan subalgebra.

For $\mathfrak{M}^D_L$, Maulik and Shen construct a commutative diagram

$$\begin{tikzcd}
\mathfrak{M}^D_L \arrow[r, hook, "\iota_{\mathfrak{M}}"] \arrow[d, "\delta^D_L"'] & \mathfrak{M}^{D+p}_L \arrow[r, "\mathrm{ev}_p"] \arrow[d, "\delta^{D+p}_L"] \arrow[rr, bend left, "\mu_{\mathfrak{M}}"] & \mathfrak{M}_{\mathrm{SL}_n}(p) \simeq [\mathfrak{sl}_n/\mathrm{SL}_n] \arrow[d] \arrow[rd, "\widehat{\mu}"] & \\
\mathcal{M}^D_L \arrow[r, hook, "\iota_{\mathcal{M}}"] \arrow[d, "h^D_L"'] & \mathcal{M}^{D+p}_L \arrow[r] \arrow[d, "h^{D+p}_L"] \arrow[rr, bend right, "\mu_{\mathcal{M}}"'] & \mathcal{M}_{\mathrm{SL}_n}(p) \simeq \mathfrak{sl}_n /\!\!/ \mathrm{SL}_n \arrow[r, "\mu"] \arrow[d, "\wr"] & \mathbb{A}^1 \\
\mathcal{A}^D_L \arrow[r, hook, "\iota_{\mathcal{A}}"] & \mathcal{A}^{D+p}_L \arrow[r] \arrow[rru, bend right, "\mu_{\mathcal{A}}"'] & \mathcal{A}_{\mathrm{SL}_n}(p) \simeq \mathfrak{t}_n /\!\!/ S_n &
\end{tikzcd} \tag{13}$$

which roughly speaking relates the difference between D-twisted and $(D+p)$-twisted Higgs bundles with the restriction map to Higgs bundles on the point p. We do not review the full construction, but record the following properties which are used below.

(i) The morphism ev_p is smooth [50, Proposition 4.1].

[7] In [50, §4] the authors do not distinguish between the moduli stacks and the moduli spaces, but we do in order to spell out some arguments precisely. Moreover, their D (resp. $D-p$) is what we call $D+p$ (resp. D).

(ii) The function μ is induced by the SL_n-equivariant quadratic form

$$\mathfrak{sl}_n \to \mathbb{A}^1, \quad g \mapsto \mathrm{Tr}(g^2)$$

(see [50, Equation (98)]).

(iii) The closed embedding $\iota_{\mathfrak{M}}$ is the critical locus of the function $\mu_{\mathfrak{M}}$ by [50, Theorem 4.5(a)].

(iv) The closed embedding $\iota_{\mathcal{A}}$ is the critical locus of the function $\mu_{\mathcal{A}}$ by [50, Lemma 4.3].

(v) The codimension c of $\iota_{\mathfrak{M}}$ is equal to $\dim \mathfrak{sl}_n = n^2 - 1$ by [22, Eq.(78) in §6.1].

For the π-relative moduli stack $\mathfrak{M}_\pi^D$, we let H_π be the appropriate subgroup of SL_n which parametrises automorphisms of π-relative Higgs bundles over the point p (see [50, §4.2]) and $\mathfrak{h}_\pi$ denote its Lie algebra. Then Maulik and Shen show there is a similar diagram

$$\begin{tikzcd}
\mathfrak{M}_\pi^D \arrow[r, hook, "\iota_{\mathfrak{M},\pi}"] \arrow[d, "\delta_\pi^D"'] & \mathfrak{M}_\pi^{D+p} \arrow[r, "\mathrm{ev}_{p,\pi}"] \arrow[d, "\delta_\pi^{D+p}"] \arrow[rrd, bend left, "\mu_{\mathfrak{M},\pi}"] & \mathfrak{M}_\pi(p) \simeq [\mathfrak{h}_\pi/H_\pi] \arrow[d] \arrow[rd, "\widehat{\mu}_\pi"] & \\
\mathcal{M}_\pi^D \arrow[r, hook, "\iota_{\mathcal{M},\pi}"] \arrow[d, "h_\pi^D"'] & \mathcal{M}_\pi^{D+p} \arrow[r] \arrow[d, "h_\pi^{D+p}"] \arrow[rr, bend right=15, "\mu_{\mathcal{M},\pi}"'] & \mathcal{M}_\pi(p) \simeq \mathfrak{h}_\pi /\!\!/ H_\pi \arrow[r, "\mu_\pi"] \arrow[d, "\wr"] & \mathbb{A}^1 \\
\mathcal{A}_\pi^D \arrow[r, hook, "\iota_{\mathcal{A},\pi}"] & \mathcal{A}_\pi^{D+p} \arrow[r] \arrow[rru, bend right, "\mu_{\mathcal{A},\pi}"'] & \mathcal{A}_{\mathrm{SL}_n}(p) \simeq \mathfrak{t}_\pi /\!\!/ W_\pi &
\end{tikzcd} \tag{14}$$

with the following properties.

(i′) The morphism $\mathrm{ev}_{p,\pi}$ is smooth by [50, Proposition 4.1].

(ii′) The function $\mu_{\mathfrak{M},\pi}$ is induced by the SL_n-equivariant quadratic form

$$\mathfrak{h}_\pi \to \mathbb{A}^1, \quad g \mapsto \mathrm{Tr}(g^2)$$

(see [50, Equation (98)]).

(iii′) The closed embedding $\iota_{\mathfrak{M},\pi}$ is the critical locus of the function $\mu_{\mathfrak{M},\pi}$ [50, Theorem 4.5(a)].

(iv′) The closed embedding $\iota_{\mathcal{A},\pi}$ is the critical locus of the function $\mu_{\mathcal{A},\pi}$ by [50, Lemma 4.3].

(v′) The codimension c_π of $\iota_{\mathfrak{M},\pi}$ is equal to $\dim \mathfrak{h}_\pi = n_\gamma n - 1$ by Lemma 5.5.

Remark 6.8. Diagram (13) is a special case of Diagram (14) and Properties (i)-(v) are special cases of (i′)-(v′). We present both separately, as this special case is going to be used in combination with the general case in the next section.

Now we use the formalism of motivic nearby cycles of [9, Chapter 3], extended to motivic vanishing cycles on Artin stacks as in Definition A.3 in Appendix A. Recall for a half integer $r \in \frac{1}{2}\mathbb{Z}$, we defined Tate twists $\{r\} := (\lfloor r \rfloor)[2r]$, which are pure if and only if $r \in \mathbb{Z}$.

Remark 6.9. In [50], the vanishing cycle functors are shifted by $[-1]$ in order to use the fact that $\phi_f[-1]$ preserves perverse sheaves. We prefer to stick to the conventions in [9, Chapter 3] and not shift by $[-1]$.

Theorem 6.10. *We have the following isomorphisms.*

(i) $\widetilde{\phi}_{\mu_{\mathfrak{M}}} \mathbb{1} \simeq \iota_{\mathfrak{M}*}\mathbb{1}\{-(c-1)/2\}$ *in* $\mathrm{DM}(\mathfrak{M}_L^{D+p}, \Lambda)$ *as* Γ*-equivariant objects,*

(ii) $\widetilde{\phi}_{\mu_{\mathfrak{M},\pi}} \mathbb{1} \simeq \iota_{\mathfrak{M},\pi*}\mathbb{1}\{-(c_\pi-1)/2\}$ *in* $\mathrm{DM}(\mathfrak{M}_\pi^{D+p}, \Lambda)$ *as* $(\Gamma \times G_\pi)$*-equivariant objects,*

(iii) $\widetilde{\phi}_{\mu_{\mathcal{A}}}((h_L^{D+p})_*\mathbb{1})_\kappa \simeq \iota_{\mathcal{A}*}((h_L^D)_*\mathbb{1})_\kappa\{-(c-1)/2\}$ *in* $\mathrm{DM}(\mathcal{A}_L^{D+p}, \Lambda)$*,*

(iv) $\widetilde{\phi}_{\mu_{\mathcal{A},\pi}}((h_\pi^{D+p})_*\mathbb{1})_\kappa \simeq \iota_{\mathcal{A},\pi_*}((h_\pi^D)_*\mathbb{1})_\kappa\{-(c_\pi-1)/2\}$ *in* $\mathrm{DM}(\mathcal{A}_\pi^{D+p}, \Lambda)$ *as* G_π*-equivariant object,*

Proof. It suffices to prove Statements (ii) and (iv), as (i) and (iii) are special cases of these.

For Statement (ii), by Theorem A.10 we can compute the motivic vanishing cycles functor for the quadratic form $q : \mathfrak{h}_\pi \to \mathbb{A}^1$ given by $g \mapsto \mathrm{Tr}(g^2)$: we have $\widetilde{\phi}_q(\mathbb{1}) \simeq \iota_{0_*}\mathbb{1}\{-(c_\pi - 1)/2\}$, where $c_\pi = \dim \mathfrak{h}_\pi$ and $\iota_0 : \{0\} = \mathrm{crit}(q) \hookrightarrow \mathfrak{h}_\pi$. Since everything is H_π-equivariant, on the stack quotient $[\mathfrak{h}_\pi/H_\pi]$ we also have

$$\widetilde{\phi}_{\hat{\mu}_\pi}(\mathbb{1}) \simeq \hat{\iota}_{0_*}\mathbb{1}\{-(c_\pi - 1)/2\}, \tag{15}$$

where $\hat{\iota}_0 : [\{0\}/H_\pi] \hookrightarrow [\mathfrak{h}_\pi/H_\pi]$. Finally, we have isomorphisms

$$\widetilde{\phi}_{\mu_{\mathfrak{M}},\pi}\mathbb{1} \simeq \widetilde{\phi}_{\mu_{\mathfrak{M}},\pi}\mathrm{ev}_{p,\pi}^*\mathbb{1} \simeq \mathrm{ev}_{p,\pi}^*\widetilde{\phi}_{\hat{u}_\pi}(\mathbb{1}) \simeq \mathrm{ev}_{p,\pi}^*\hat{\iota}_{0*}\mathbb{1}\{-(c_\pi-1)/2\} \simeq \iota_{\mathfrak{M},\pi*}\mathbb{1}\{-(c_\pi-1)/2\},$$

where the second isomorphism is smooth base change for vanishing cycles (see Proposition A.6) for the smooth map $\mathrm{ev}_{p,\pi}$, the third isomorphism is Equation (15) and the final isomorphism is smooth base change for stacks (see Theorem A.1 (iv)).

To prove Statement (iv), we first take the pushforward of the isomorphism (ii) along the morphism δ_π^{D+p}, which is proper since it is a gerbe for a finite group and is also representable by Deligne–Mumford stacks. By proper base change for vanishing cycles (see Proposition A.5) and the fact that the top-left square of Diagram (14) commutes and is Γ-equivariant, we obtain a Γ-equivariant isomorphism

$$\widetilde{\phi}_{\mu_{\mathcal{M}},\pi}(\delta_\pi^{D+p})_*\mathbb{1} \simeq \iota_{\mathcal{M},\pi*}(\delta_\pi^{D})_*\mathbb{1}\{-(c_\pi-1)/2\}.$$

By pushing forward this isomorphism along the proper morphism h_π^{D+p} and using proper base change for vanishing cycles, we obtain isomorphisms

$$\begin{aligned}\widetilde{\phi}_{\mu_{\mathcal{A}},\pi}(h_\pi^{D+p})_*(\delta_\pi^{D+p})_*\mathbb{1} \simeq (h_\pi^{D+p})_*\widetilde{\phi}_{\mu_{\mathcal{M}},\pi}(\delta_\pi^{D+p})_*\mathbb{1} &\simeq (h_\pi^{D+p})_*\iota_{\mathcal{M},\pi*}(\delta_\pi^{D})_*\mathbb{1}\{-(c_\pi-1)/2\}\\ &\simeq \iota_{\mathcal{A},\pi_*}(h_\pi^{D})_*(\delta_\pi^{D})_*\mathbb{1}\{-(c_\pi-1)/2\}\end{aligned}$$

where the last isomorphism follows from the commutativity of Diagram (14). By Lemma 6.11 below, we have $(\delta_\pi^D)_*\mathbb{1} \simeq \mathbb{1}$ and similarly for δ_π^{D+p}; therefore, we obtain

$$\widetilde{\phi}_{\mu_{\mathcal{A},\pi}}(h_\pi^{D+p})_*\mathbb{1} \simeq \iota_{\mathcal{A},\pi_*}(h_\pi^{D})_*\mathbb{1}\{-(c_\pi-1)/2\}.$$

The whole bottom-left square of Diagram (14) as well as the maps $\mu_{\mathcal{M},\pi}$ and $\mu_{\mathcal{A},\pi}$ are Γ-equivariant, and this implies that all the isomorphisms above commute with Γ-actions. We can then take the κ-isotypical component to obtain

$$(\widetilde{\phi}_{\mu_{\mathcal{A},\pi}}(h_\pi^{D+p})_*\mathbb{1})_\kappa \simeq (\iota_{\mathcal{A},\pi_*}(h_\pi^{D})_*\mathbb{1})_\kappa\{-(c_\pi-1)/2\},$$

which we can rewrite as (iv) as $\widetilde{\phi}_{\mu_{\mathcal{A},\pi}}$ and $\iota_{\mathcal{A},\pi_*}$ commute with Γ-actions. The whole construction is G_π-equivariant and we obtain that the resulting isomorphisms are G_π-equivariant. □

To complete the proof, we need the following lemma, which in our setting reflects the classical fact that BG has trivial cohomology with rational coefficients when G is a finite group.

Lemma 6.11. *Let G be a finite group.*

(i) *For a morphism of Artin stacks $f : \mathcal{Y} \to \mathfrak{X}$ which is an (étale) G-torsor, the unit natural transformation* $\mathrm{id} \to f_*f^*$ *induces a natural isomorphism* $\mathrm{id} \simeq (f_*f^*-)^G$.

(ii) *For a morphism of Artin stacks $\delta : \mathfrak{X} \to \mathcal{Z}$ which is an (étale) G-gerbe, the unit natural transformation* $\mathrm{id} \to \delta_*\delta^*$ *induces an isomorphism* $\mathbb{1} \simeq \delta_*\mathbb{1}$.

Proof. If $g : \mathcal{W} \to \mathfrak{X}$ is a étale surjective morphism, then $g^* : \mathrm{DM}(\mathfrak{X}, \Lambda) \to \mathrm{DM}(\mathcal{W}, \Lambda)$ is conservative; this follows by étale descent in exactly the same way as in the scheme case, because g then admits a section locally for the étale topology. By assumption on f, there exists such a g for which $\mathcal{Y} \times_{\mathfrak{X}} \mathcal{W} \simeq \mathcal{W} \times G \to \mathcal{W}$ is a trivial G-torsor. By conservativity and proper base change, this reduces the proof of (i) to the case of a trivial G-torsor, which is immediate.

For (ii), by the same conservativity argument and proper base change argument (using that δ is proper and Deligne-Mumford representable, see Theorem A.1 (vi)), we can reduce to the case where $\delta : \mathcal{Z} \times_k BG \to \mathcal{Z}$ is a trivial G-gerbe. In other words, we can consider $\mathcal{Z} \times_k BG$ as the quotient of the trivial G-action on $\mathcal{Z}$, and this gives us a (non-trivial) G-torsor $f : \mathcal{Z} \to \mathcal{Z} \times_k BG$. By applying (i) to f, we have an isomorphism

$$\delta_*\mathbb{1} \simeq \delta_*(f_*\mathbb{1})^G.$$

By definition, the object $\delta_*(f_*\mathbb{1})^G$ is a direct factor of $\delta_*f_*\mathbb{1} \simeq \mathbb{1}$ in $\mathrm{DM}(\mathcal{Z}, \Lambda)$ so we have an induced morphism $\theta : \delta_*\mathbb{1} \to \mathbb{1}$. Moreover, it is easy to see using the naturality of adjunctions that the composition $\mathbb{1} \xrightarrow{\eta} \delta_*\mathbb{1} \xrightarrow{\theta} \mathbb{1}$ is the identity. This shows that the direct factor $\delta_*(f_*\mathbb{1})^G$ of $\mathbb{1}$ is isomorphic to $\mathbb{1}$, and completes the proof. □

Lemma 6.12. *We have the following dimension formulae*

(i) $c - c_\pi = 2(d_\gamma^{D+p} - d_\gamma^D)$,

(ii) $\lfloor (c-1)/2 \rfloor - \lfloor (c_\pi - 1)/2 \rfloor = d_\gamma^{D+p} - d_\gamma^D$.

Proof. By Properties (v) and (v′) from the lists following Diagrams (13) and (14), we have $c - c_\pi = n^2 - n_\gamma^2 m = n(n - n_\gamma)$ and then (i) follows from Lemma 5.5. If n is odd, then so is its divisor n_γ; this shows that $c - c_\pi$ is always even. The second statement then follows, as if $a, b \in \mathbb{Z}$ and $a - b$ is even, then $\lfloor a/2 \rfloor - \lfloor b/2 \rfloor = (a-b)/2$. □

We now obtain a corollary analogous to [50, Corollary 4.6]. For this, note that the morphism $\mu_{\mathcal{A},\pi} : \mathcal{A}_\pi^{D+p} \to \mathbb{A}^1$ is invariant under $G_\pi = \mathrm{Gal}(\pi : C_\gamma \to C)$ and so induces $\mu_{\mathcal{A},\gamma} : \mathcal{A}_\gamma^{D+p} \to \mathbb{A}^1$ which coincides with the restriction of $\mu_\mathcal{A} : \mathcal{A}_L^{D+p} \to \mathbb{A}^1$ to $i_\gamma^{D+p} : \mathcal{A}_\gamma^{D+p} \hookrightarrow \mathcal{A}_L^{D+p}$.

Corollary 6.13. *There is an isomorphism* $\widetilde{\phi}_{\mu_{\mathcal{A},\gamma}}((h_\gamma^{D+p})_* \mathbb{1})_\kappa \simeq (\iota_{\mathcal{A},\gamma})_*((h_\gamma^D)_* \mathbb{1})_\kappa\{-(c_\pi - 1)/2\}$.

Proof. This follows from Theorem 6.10 (including the G_π-equivariance) and 5.7 exactly in [50, Corollary 4.6]. □

Before we define $\beta_{\gamma,\mathrm{mot}}^D$ in the missing cases, we need one final lemma.

Lemma 6.14. *There is an isomorphism*

$$\iota_\mathcal{A}^* \widetilde{\phi}_{\mu_\mathcal{A}} (i_\gamma^{D+p})_*((h_\gamma^{D+p})_* \mathbb{1})_\kappa\{(c-1)/2 - d_\gamma^{D+p}\} \simeq (i_\gamma^D)_*((h_\gamma^D)_* \mathbb{1})_\kappa\{-d_\gamma^D\}.$$

Proof. This follows from the chain of isomorphisms

$$\begin{aligned}
&\iota_\mathcal{A}^* \widetilde{\phi}_{\mu_\mathcal{A}} (i_\gamma^{D+p})_*((h_\gamma^{D+p})_* \mathbb{1})_\kappa\{(c-1)/2 - d_\gamma^{D+p}\} \\
&\simeq \iota_\mathcal{A}^* (i_\gamma^{D+p})_* \widetilde{\phi}_{\mu_{\mathcal{A},\gamma}} ((h_\gamma^{D+p})_* \mathbb{1})_\kappa\{(c-1)/2 - d_\gamma^{D+p}\} \\
&\simeq \iota_\mathcal{A}^* (i_\gamma^{D+p})_* (\iota_{\mathcal{A},\gamma})_* ((h_\gamma^D)_* \mathbb{1})_\kappa(\lfloor (c-1)/2 \rfloor - \lfloor (c_\pi - 1)/2 \rfloor - d_\gamma^{D+p})[c - c_\pi - 2d_\gamma^{D+p}] \\
&\simeq (i_\gamma^D)_*((h_\gamma^D)_* \mathbb{1})_\kappa\{-d_\gamma^D\},
\end{aligned}$$

where the first isomorphism is proper base change for vanishing cycles (Proposition A.5) as $\mu_\mathcal{A} \circ i_\gamma^{D+p} = \mu_{\mathcal{A},\gamma}$, the second isomorphism follows from Corollary 6.13 and the final isomorphism follows from Lemma 6.12 and proper base change $\iota_\mathcal{A}^*(i_\gamma^{D+p})_* \simeq (i_\gamma^D)_*(\iota_{\mathcal{A},\gamma})^*$ for the cartesian square

$$\begin{tikzcd}
\mathcal{A}_\gamma^D \arrow[r, "\iota_{\mathcal{A},\gamma}"] \arrow[d, "i_\gamma^D"'] & \mathcal{A}_\gamma^{D+p} \arrow[d, "i_\gamma^{D+p}"] \\
\mathcal{A}_L^D \arrow[r, "\iota_\mathcal{A}"] & \mathcal{A}_L^{D+p}
\end{tikzcd}$$

together with the isomorphism $(\iota_{\mathcal{A},\gamma})^*(\iota_{\mathcal{A},\gamma})_* \simeq \mathrm{id}$. □

Definition 6.15 (Construction of $\beta_{\gamma,\mathrm{mot}}^D$)**.**

(1) For D of even degree, we have defined in §6.2 the morphism

$$\beta_{\gamma,\mathrm{mot}}^D : ((h_L^D)_* \mathbb{1})_\kappa \to (i_\gamma^D)_*((h_\gamma^D)_* \mathbb{1})_\kappa\{-d_\gamma^D\} \in \mathrm{DM}(\mathcal{A}_L^D, \Lambda).$$

(2) For D of odd degree greater than $2g-2$, we define $\beta_{\gamma,\mathrm{mot}}^D \in \mathrm{DM}(\mathcal{A}_L^D, \Lambda)$ as

$$\begin{aligned}
\beta_{\gamma,\mathrm{mot}}^D : ((h_L^D)_* \mathbb{1})_\kappa \xleftarrow{\sim} \iota_\mathcal{A}^* \iota_{\mathcal{A}*} ((h_L^D)_* \mathbb{1})_\kappa &\simeq \iota_\mathcal{A}^* \widetilde{\phi}_{\mu_\mathcal{A}} ((h_L^{D+p})_* \mathbb{1})_\kappa\{(c-1)/2\} \\
&\xrightarrow{(\star)} \iota_\mathcal{A}^* \widetilde{\phi}_{\mu_\mathcal{A}} (i_\gamma^{D+p})_*((h_\gamma^{D+p})_* \mathbb{1})_\kappa\{(c-1)/2 - d_\gamma^{D+p}\} \\
&\simeq (i_\gamma^D)_*((h_\gamma^D)_* \mathbb{1})_\kappa\{-d_\gamma^D\},
\end{aligned}$$

where the first map comes from adjunction, the next isomorphism comes from Theorem 6.10 (iii), the morphism $(\star)$ is $\iota_\mathcal{A}^* \widetilde{\phi}_{\mu_\mathcal{A}} \beta_{\gamma,\mathrm{mot}}^{D+p}$ for the map $\beta_{\gamma,\mathrm{mot}}^{D+p}$ constructed in (1), and the final isomorphism is given by Lemma 6.14.

(3) For $D = K_C$, we define

$$\beta_{\gamma,\mathrm{mot}} := \beta^{K_C}_{\gamma,\mathrm{mot}} : ((h_L)_*\mathbb{1})_\kappa \to (i_\gamma)_*((h_\gamma)_*\mathbb{1})_\kappa\{-d_\gamma\} \text{ in } \mathrm{DM}(\mathcal{A}_L,\Lambda).$$

by taking a point p on C so that $K_C + p$ is of odd degree greater than $2g-2$ and we have defined $\beta^{K_C+p}_{\gamma,\mathrm{mot}}$ in (2) and we then construct $\beta^{K_C}_{\gamma,\mathrm{mot}}$ from $\beta^{K_C+p}_{\gamma,\mathrm{mot}}$ in exactly the same way that $\beta^{D}_{\gamma,\mathrm{mot}}$ is constructed from $\beta^{D+p}_{\gamma,\mathrm{mot}}$ in (2).

Lemma 6.16. *For D with $D = K_C$ or $\deg(D) > 2g-2$ and $k \hookrightarrow \mathbb{C}$, the Betti realisation of $\beta^D_{\gamma,\mathrm{mot}}$ is the morphism β^D_γ in* (6)*, which is the isomorphism c^D_κ of* [50, Theorem 3.2].

Proof. Our construction is parallel to the one of [50], so this follows from combining 6.7 and Theorem A.9, noting that all the motives on stacks we encounter in the proof satisfy the condition of (iii) in Theorem A.9 by smooth base change. □

Remark 6.17. The Betti realisation functor $R_B : \mathrm{DM}_c(S,\Lambda) \to D(S^{\mathrm{an}},\Lambda)$ is conjectured to be conservative [11], so we expect that $\beta^D_{\gamma,\mathrm{mot}}$ is an isomorphism. This is however not necessary for the proof of the main theorem.

6.4. **Higgs moduli spaces with abelian motives.** We have already seen that $M(\mathcal{M}^D_L)$ is abelian (Theorem 4.5) and to apply a conservativity argument, we also need the following result.

Proposition 6.18. *For each $\gamma \in \Gamma$ with corresponding $\kappa = \kappa(\gamma) \in \widehat{\Gamma}$, the κ-isotypical piece of the motive of the γ-fixed locus $M(\mathcal{M}^D_\gamma)_\kappa \in \mathrm{DM}(k,\Lambda)$ is abelian.*

Proof. The idea is to relate this motive to the motive of the π-relative moduli space $\mathcal{M}^D_\pi$. We first apply Corollary 5.7 to get an isomorphism

$$M(\mathcal{M}^D_\gamma)_\kappa \simeq M(\mathcal{M}^D_\pi)^{G_\pi}_\kappa. \tag{16}$$

In [50, Proposition 2.10], Maulik and Shen construct an isomorphism

$$R_B(((h^D_\pi)_*\mathbb{1})_\kappa) \simeq R_B(((h^D_\pi)_*\mathbb{1})^\Gamma)$$

in $D((\mathcal{A}^D_\pi)^{\mathrm{an}},\Lambda)$. Their construction uses the interaction of the action of Γ with the connected components of $\mathcal{M}^D_\pi$ described in [34] and is entirely motivic (it relies on averaging and taking isotypical components for actions of some finite abelian groups), and so lifts to an isomorphism

$$((h^D_\pi)_*\mathbb{1})_\kappa \simeq ((h^D_\pi)_*\mathbb{1})^\Gamma.$$

By pushing this isomorphism forward to $\mathrm{Spec}(k)$ and dualising, we get an isomorphism

$$M(\mathcal{M}^D_\pi)_\kappa \simeq M(\mathcal{M}^D_\pi)^\Gamma.$$

Combining this with the isomorphism (16) above, we get an isomorphism

$$M(\mathcal{M}^D_\gamma)_\kappa \simeq M(\mathcal{M}^D_\pi)^{G_\pi\times\Gamma}.$$

We now claim that $M(\mathcal{M}^D_\pi)^\Gamma$ is a direct factor of the motive of $\mathcal{M}^{D_\gamma}_{n_\gamma,d}(C_\gamma)$. The cohomological counterpart of this is explained in [50, §5.3, Equations (118) and (119)] and we just adapt the argument (note that our notation is slightly different). Consider the algebraic group

$$\mathcal{M}^D_{1,0} \simeq \mathrm{Pic}^0(C) \times H^0(C,\mathcal{O}_C(D)).$$

There is a morphism

$$\mathcal{M}^D_{1,0} \times \mathcal{M}^D_\pi \to \mathcal{M}^{D_\gamma}_{n_\gamma,d}(C_\gamma), \quad ((\mathcal{E}_1,\theta_1),(\mathcal{F},\theta)) \mapsto (\pi^*\mathcal{E}_1 \otimes \mathcal{F}, \pi^*\theta_1 + \theta).$$

As explained in [50] after Equation (118), this morphism factors through the (free) diagonal action of Γ on the LHS and gives rise to an isomorphism

$$(\mathcal{M}^D_{1,0} \times \mathcal{M}^D_\pi)/\Gamma \simeq \mathcal{M}^{D_\gamma}_{n_\gamma,d}(C_\gamma).$$

Moreover, we have $M(\mathcal{M}^D_{1,0}) \simeq M(\mathrm{Pic}^0(C))$ by $\mathbb{A}^1$-homotopy invariance and Γ acts trivially on the motive $M(\mathrm{Pic}^0(C))$ by [25, Proof of Theorem 4.2]. Combining this with the Künneth formula, we get an isomorphism

$$M(\mathcal{M}^{D_\gamma}_{n_\gamma,d}(C_\gamma)) \simeq M(\mathrm{Pic}^0(C)) \otimes M(\mathcal{M}^D_\pi)^\Gamma.$$

The claim follows as $\mathbb{1}$ is a canonical direct factor of $M(\mathrm{Pic}^0(C))$.

By Theorem 3.5, the motive of $\mathcal{M}^{D_\gamma}_{n_\gamma,d}(C_\gamma)$ is abelian and thus we deduce that the direct factor $M(\mathcal{M}^D_\gamma)_\kappa$ is also abelian. □

Remark 6.19. We do not know whether the motives of $\mathcal{M}^D_\gamma$ or $\mathcal{M}^D_\pi$ are abelian (although we suspect they are), but we do not need to know this for the conservativity argument we employ in the proof of Theorem 1.1.

6.5. **Completing the proof.** By pushing-forward the morphism $\beta_{\gamma,\mathrm{mot}}$ constructed in Definitions 6.6 and 6.15 to k and dualising, one obtains a morphism in $\mathrm{DM}(k,\Lambda)$ which we denote by

$$\nu^D_{\gamma,\mathrm{mot}} : M(\mathcal{M}^D_\gamma)_\kappa\{d_\gamma\}) \to M(\mathcal{M}^D_L)_\kappa.$$

We are now able to prove Theorem 1.1, which states that $\nu^D_{\gamma,\mathrm{mot}}$ is an isomorphism.

Proof of Theorem 1.1. We first note that the motives in the source and target of $\nu^D_{\gamma,\mathrm{mot}}$ are abelian: $M(\mathcal{M}^D_\gamma)_\kappa$ is abelian by Proposition 6.18 above and $M(\mathcal{M}^D_L)_\kappa$ is abelian as it is a direct factor of $M(\mathcal{M}^D_L)$, which is abelian by Theorem 4.5.

In order to consider the Betti realisation and apply a conservativity argument, we can assume without loss of generality that k admits an embedding $\sigma : k \hookrightarrow \mathbb{C}$ by an application of the Lefschetz principle. More precisely, let k_0 be a subfield of k, finitely generated over $\mathbb{Q}$, such that C is the base change of a smooth projective curve C_0 over k_0, and let $\bar{k}_0$ be the algebraic closure of k_0 in k. Then $\bar{k}_0$ is algebraically closed, admits a complex embedding and the morphism $\nu^D_{\gamma,\mathrm{mot}}$ for C is the image of the analoguous morphism $\nu^D_{\gamma,\mathrm{mot}}(C_0 \times_{k_0} \bar{k}_0)$ in $\mathrm{DM}_c(\bar{k}_0,\Lambda)$ via the base change functor $\mathrm{DM}(\bar{k}_0,\Lambda) \to \mathrm{DM}(k,\Lambda)$ (because all the constructions we have done commute with this base change functor), so it suffices to show $\nu^D_{\gamma,\mathrm{mot}}(C_0 \times_{k_0} \bar{k}_0)$ is an isomorphism.

Since the Betti realisation associated to σ commutes with the six operations (hence sends motivic correspondences to the corresponding cohomological correspondence) and commutes with vanishing cycles [10], the Betti realisation of $\nu^D_{\gamma,\mathrm{mot}}$ is the pushforward to k of the morphism β^D_γ constructed by Maulik and Shen (see Lemmas 6.7 and 6.16 for details); in particular, Maulik and Shen showed that this Betti realisation is an isomorphism [50, Theorem 0.3]. Since the motives appearing are abelian, the conservativity result of Wildeshaus [64, Theorem 1.12] implies that $\nu^D_{\gamma,\mathrm{mot}}$ is an isomorphism. □

We end by giving some more concrete consequences of Theorem 1.1.

Corollary 6.20. *We adopt the notation of Theorem 1.1 and also fix a smooth k-variety X.*

(i) *Let $[X] \in K_0(\mathcal{M}_{\mathrm{rat}}(k,\mathbb{Q}))$ denote the virtual motivic class of a k-variety in the Grothendieck ring of Chow motives, and write $\mathbb{L} := [\mathbb{P}^1] - [\mathrm{Spec}(k)] \in K_0(\mathcal{M}_{\mathrm{rat}}(k,\mathbb{Q}))$ for the Lefschetz motive. Define the orbifold twisted virtual motive $[\overline{\mathcal{M}}^D]^{\mathrm{orb}}_{\delta_L} \in K_0(\mathcal{M}_{\mathrm{rat}}(k,\mathbb{Q}))$ of $\overline{\mathcal{M}}^D$ as*

$$[\overline{\mathcal{M}}^D]^{\mathrm{orb}}_{\delta_L} := \sum_{\gamma\in\Gamma} [\mathcal{M}^D_\gamma]_{\kappa(\gamma)} \mathbb{L}^{d_\gamma}.$$

Then

$$[\mathcal{M}^D_L] = [\overline{\mathcal{M}}^D]^{\mathrm{orb}}_{\delta_L}.$$

(ii) *Let $\gamma \in \Gamma$ corresponding to $\kappa := \kappa(\gamma) \in \widehat{\Gamma}$. For $a, b \in \mathbb{Z}$, there is an isomorphism of motivic cohomology groups (i.e. higher Chow groups)*

$$H^a_{\mathrm{mot}}(\mathcal{M}^D_L \times_k X, \Lambda(b))_\kappa \simeq H^{a-2d_\gamma}_{\mathrm{mot}}(\mathcal{M}^D_\gamma \times_k X, \Lambda(b-d_\gamma))_\kappa.$$

In particular, taking $a = 2b = i \geq 0$, we have an isomorphism

$$\mathrm{CH}^i(\mathcal{M}_L^D \times_k X, \Lambda)_\kappa \simeq \mathrm{CH}^{i-d_\gamma}(\mathcal{M}_\gamma^D \times_k X, \Lambda)_\kappa.$$

These isomorphisms are 'functorial in $M(X)$', so that if X and Y are smooth projective, they are natural with respect to the action of correspondences in $\mathrm{CH}^*(X \times_k Y)$.

(iii) *For $a, b \in \mathbb{Z}$, there is an isomorphism*

$$H^a_{\mathrm{mot}}(\mathcal{M}_L^D \times_k X, \Lambda(b)) \simeq H^{a-2d_\gamma}_{\mathrm{mot,orb}}(\overline{\mathcal{M}}^D \times_k X, \delta_L; \Lambda(b - d_\gamma))_\kappa$$

where $H^*_{\mathrm{mot,orb}}(-, \delta_L; \Lambda(-))$ denotes orbifold twisted motivic cohomology for Deligne-Mumford stacks, with the same properties as in (i)*.*

(iv) *For $m \geq 0$, there is an isomorphism*

$$K_m(\mathcal{M}_L^D \times_k X, \Lambda) \simeq K_m(\overline{\mathcal{M}}^D \times_k X, \delta_L; \Lambda)$$

of (twisted) algebraic K-theory groups with coefficients in Λ. If X and Y are smooth projective varieties, this isomorphism is natural with respect to K-theoretic correspondences in $K_0(X \times_k Y)$.

Proof. Part (i) follows from Theorem 1.1 together with the fact that Voevodssky's embedding of $\mathcal{M}_{\mathrm{rat}}(k, \mathbb{Q})$ into $\mathrm{DM}_c(k, \mathbb{Q})$ induces an isomorphism on Grothendieck rings which sends the Lefschetz motive $\mathbb{L}$ onto the Tate motive $\mathbb{Q}\{1\}$ [17, Corollary 6.4.3].

Part (ii) follows from the representability of motivic cohomology/higher Chow groups for smooth varieties in $\mathrm{DM}(k, \Lambda)$ [53, Theorem 19.1]. Part (iii) is simply obtained by summing the first statement over all κ (because this is how we define orbifold twisted motivic cohomology).

For part (iv), recall that if X is a smooth k-variety, then there is a Chern character isomorphism

$$\mathrm{ch} : K_m(X, \Lambda) \simeq \bigoplus_{i \in \mathbb{Z}} H^{2i-m}_{\mathrm{mot}}(X, \Lambda(i))$$

see e.g. [19, Corollary 16.2.21]. Moreover, if Y is a smooth k-variety with an action of a finite abelian n-torsion group G, then [63, Theorem 1] provides an isomorphism

$$v : K_*([Y/G], \Lambda) \simeq \bigoplus_{g \in G} K_*(Y^g, \Lambda)^G.$$

To obtain this precise form of the formula from Vistoli's theorem, observe that the formula simplifies when G is abelian and one tensors with a large enough cyclotomic field so that all characters of G become defined. We claim that if δ is a gerbe on the quotient stack $[Y/G]$, there is a similar isomorphism

$$\tilde{v} : K_*([Y/G], \delta; \Lambda) \simeq \bigoplus_{g \in G} K_*(Y^g, \Lambda)_{\kappa(g)}.$$

where $\kappa(g) \in \widehat{G}$ is the character of G provided by δ. Unfortunately we do not have a reference for this claim, but it seems likely that Vistoli's argument can be adapted to the twisted case. The analoguous formula for twisted topological K-theory is established in [47, Theorem 3.11].

We thus have a sequence of isomorphisms

$$\begin{aligned}
K_m(\mathcal{M}_L^D \times_k X, \Lambda) &\overset{\mathrm{ch}}{\simeq} \bigoplus_{i\in\mathbb{Z}} H^{2i-m}_{\mathrm{mot}}(\mathcal{M}_L^D \times_k X, \Lambda(i)) \\
&\simeq \bigoplus_{i\in\mathbb{Z}} \bigoplus_{\kappa\in\widehat{\Gamma}} H^{2i-m}_{\mathrm{mot}}(\mathcal{M}_L^D \times_k X, \Lambda(i))_\kappa \\
&\simeq \bigoplus_{i\in\mathbb{Z}} \bigoplus_{\kappa\in\widehat{\Gamma}} H^{2(i-d_\gamma)-m}_{\mathrm{mot}}(\mathcal{M}_\gamma^D \times_k X, \Lambda(i-d_\gamma))_{\kappa(\gamma)} \\
&\simeq \bigoplus_{\kappa\in\widehat{\Gamma}} \bigoplus_{i\in\mathbb{Z}} H^{2(i-d_\gamma)-m}_{\mathrm{mot}}(\mathcal{M}_\gamma^D \times_k X, \Lambda(i-d_\gamma))_{\kappa(\gamma)} \\
&\overset{\mathrm{ch}^{-1}}{\simeq} \bigoplus_{\gamma\in\Gamma} (K_m(\mathcal{M}_\gamma^D \times_k X, \Lambda))_{\kappa(\gamma)} \\
&\overset{\tilde{v}^{-1}}{\simeq} K_m(\overline{\mathcal{M}}^D, \delta_L; \Lambda),
\end{aligned}$$

where the third isomorphism comes from Part (ii). □

Remark 6.21. The isomorphism of rational algebraic K-theory of part (iv) is not intended to be induced by the conjectural derived equivalence (1), and it seems unlikely that it extends to an isomorphism of integral algebraic K-theory.

Appendix A. Motivic sheaves and motivic vanishing cycles for stacks

In this appendix, Λ denotes an arbitrary $\mathbb{Q}$-algebra. We first summarise how to extend DM to Artin stacks following the approach of Khan [42, Appendix A] and then we extend the construction of motivic nearby and vanishing cycles functors to Artin stacks by following the approach for schemes of Ayoub [9, Chapitre 3].

A.1. **Extending DM to Artin stacks.** In this section, we review how to extend $\mathrm{DM}(-,\Lambda)$ to Artin stacks by étale descent. For this, we follow the Khan's approach of [42, Appendix A] to extending étale motivic homotopy categories $\mathrm{SH}_{\text{ét}}(-)$ to derived Artin stacks using an ∞-categorical approach. His construction and results apply just as well to $\mathrm{DM}(-,\Lambda)$: the key inputs are the six operations for schemes and the étale descent property, which are satisfied in both cases. Khan's construction works in two steps, first he uses Nisnevich descent to extend to (derived) algebraic spaces and then étale descent to extend to (derived) Artin stacks. For us, all stacks we are interested in are non-derived Artin stacks, and so we will state everything for non-derived Artin stacks. Since we only work with Artin stacks with an atlas given by a scheme, we could strictly speaking bypass the first step.

To do this extension, it is necessary to use the formalism of ∞-categories. In particular, when we invoke categorical notions such as functor, (co)limits, adjunctions, etc. these should be interpreted as ∞-categorical. The following theorem summarises the main results of Khan's construction in the setting of étale motivic sheaves.

Theorem A.1 (Khan)**.** *The formalism of six operations on* $\mathrm{DM}(-,\Lambda)$ *extends to algebraic spaces. Moreover, the presheaf of* ∞*-categories*

$$X \to \mathrm{DM}(X,\Lambda), f \to f^*$$

on the site of algebraic spaces admits a right Kan extension to the site of Artin stacks with the following properties.

(i) [42, Eq. (A.4)] *Let* $\mathfrak{X}$ *be an Artin stack. There is an* ∞*- (in fact* $(2,1)$*-)category* $\mathrm{Lis}_{\mathfrak{X}}$ *of smooth morphisms* $X \to \mathfrak{X}$ *with* X *a scheme, and we have*

$$\mathrm{DM}(\mathfrak{X},\Lambda) \simeq \lim_{X\in\mathrm{Lis}_{\mathfrak{X}}} \mathrm{DM}(X,\Lambda).$$

In particular, the collection of functors $(u^* : \mathrm{DM}(\mathfrak{X},\Lambda) \to \mathrm{DM}(X,\Lambda))_{(u:X\to\mathfrak{X})\in\mathrm{Lis}_{\mathfrak{X}}}$ *is jointly conservative.*

(ii) [42, Eq. (A.3)] *More precisely, but less canonically, one has the following description in terms of a fixed atlas. If $p : X \to \mathfrak{X}$ is a smooth surjection from an algebraic space X to an Artin stack $\mathfrak{X}$, then*

$$\mathrm{DM}(\mathfrak{X},\Lambda) = \lim \Big(\mathrm{DM}(X,\Lambda) \rightrightarrows \mathrm{DM}(X \times_{\mathfrak{X}} X, \Lambda) \substack{\rightarrow\\ \rightarrow\\ \rightarrow} \mathrm{DM}(X \times_{\mathfrak{X}} X \times_{\mathfrak{X}} X, \Lambda) \cdots \Big).$$

(iii) [42, Theorem A.5 (i)] *For every Artin stack $\mathfrak{X}$, there is a closed symmetric monoidal structure on* $\mathrm{DM}(\mathfrak{X},\Lambda)$ *and a pair of adjoint bifunctors* $(\otimes, \underline{\mathrm{Hom}})$.

(iv) (Adjunctions, projection and base change formulae, [42, Theorem A.5 (ii-iv)]) *For any locally of finite type morphism $f : \mathfrak{X} \to \mathcal{Y}$ between Artin stacks, there are adjunctions*

$$f^* : \mathrm{DM}(\mathcal{Y},\Lambda) \rightleftarrows \mathrm{DM}(\mathfrak{X},\Lambda) : f_*$$

and

$$f_! : \mathrm{DM}(\mathfrak{X},\Lambda) \rightleftarrows \mathrm{DM}(\mathcal{Y},\Lambda) : f^!$$

which satisfy the projection formula $f_!(\mathcal{F}) \otimes \mathcal{G} \simeq f_!(\mathcal{F} \otimes f^*(\mathcal{G}))$ and base change formulae:

$$g^* f_! \xrightarrow{\sim} \tilde{f}_! \tilde{g}^* \qquad \textit{and} \qquad \tilde{g}_* \tilde{f}^! \xrightarrow{\sim} f^! g_*$$

for any cartesian square

$$\begin{tikzcd} \mathcal{W} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] & \mathcal{Z} \arrow[d, "g"] \\ \mathfrak{X} \arrow[r, "f"] & \mathcal{Y}. \end{tikzcd}$$

(v) (Purity isomorphism and smooth base change, [42, Theorem A.13]) *For any smooth morphism $f : \mathfrak{X} \to \mathcal{Y}$ of pure relative dimension d, there is a purity isomorphism $f^! \simeq f^*\{-d\}$ and smooth base change $f^* g_* \simeq \tilde{g}_* \tilde{f}^*$ for a cartesian square as above.*

(vi) (Proper base change, [42, Theorems A.5 (iv) and A.7]) *If $f : \mathfrak{X} \to \mathcal{Y}$ is Deligne–Mumford-representable (i.e. represented by Deligne–Mumford stacks), there is a natural transformation $f_! \to f_*$, which is an isomorphism if f is proper. For f proper and Deligne–Mumford-representable, there is a proper base change $g^* f_* \simeq \tilde{f}_* \tilde{g}^*$ for a cartesian square as above.*

Example A.2. For a finite group scheme G/S, the morphism $\delta : BG \to S$ is not representable, but is Deligne–Mumford-representable and is proper; thus the natural transformation $\delta_! \to \delta_*$ is an isomorphism. More generally, the same is true for any G-gerbe δ.

The Betti realisation also extends readily to the context of Artin stacks. First, we need to extend its target. Let X be a finite type scheme over $\mathbb{C}$. We denote by $D(X^{\mathrm{an}}, \Lambda)$ the ∞-category of sheaves of complexes of Λ-modules on the topological space X^{an}. Then the assignment $X \mapsto D(X^{\mathrm{an}}, \Lambda)$ has the same basic functoriality as $X \mapsto \mathrm{DM}(X,\Lambda)$, and one can follow the approach of [42, Appendix A] to define a category $D(\mathfrak{X}^{\mathrm{an}}, \Lambda)$ for every finite type Artin stack $\mathfrak{X}$ over $\mathbb{C}$. Let $\sigma : k \to \mathbb{C}$ be a complex embedding. There is a Betti realisation functor

$$R_B : \mathrm{DM}(X,\Lambda) \to D(X^{\mathrm{an}}, \Lambda)$$

defined at the triangulated level in [10] but which can be easily refined to an ∞-functor, see e.g. [7, Definition 1.21].

The Betti realisation functor for motives of schemes commutes with pullback by arbitrary morphisms [10, Theoreme 3.19 A], so in particular by smooth morphisms. By Theorem A.1 (i), this implies that we can extend the Betti realisation functor to a functor

$$R_B : \mathrm{DM}(\mathfrak{X},\Lambda) \to D(\mathfrak{X}_\sigma, \Lambda).$$

A.2. **Motivic nearby and vanishing cycles functors for stacks.** Motivic nearby cycles for the categories $\mathrm{DM}(-,\Lambda)$ of étale motivic sheaves on schemes have been introduced in [9, Chapitre 3] and studied further in [10, 12]. The closely related functor of motivic vanishing cycles was not defined in those references, but is not too difficult to construct once we have motivic nearby cycles. Our goal is to extend this to Artin stacks. We could employ Theorem A.1 (i) combined with the smooth base change properties for nearby cycles of schemes, but we prefer a slightly different approach with a concrete formula.

Throughout this subsection, we let S be a Noetherian finite dimensional base scheme of characteristic zero; for this paper we only need $S = \mathrm{Spec}(k)$, but the general case is exactly the same. Since we are in characteristic 0, we only need the "tame" version of these constructions, which is the only one considered in [9, Chapitre 3]. Furthermore, as we work with coefficients in a $\mathbb{Q}$-algebra, we can also use the alternative "logarithmic" description of motivic nearby cycles considered in [9, Section 3.6]. For this, let $\mathcal{L}\mathrm{og}^\vee \in \mathrm{DM}(\mathbb{G}_{m,S},\Lambda)$ be the (dual) logarithm motive constructed in [9, Definition 3.6.29], which comes with a morphism $\mathbb{1}_{\mathbb{G}_{m,S}} \to \mathcal{L}\mathrm{og}^\vee$.

Definition A.3. For an Artin stack $\mathfrak{X}$ and regular function $f : \mathfrak{X} \to \mathbb{A}^1_S$, construct the commutative diagram with cartesian squares:

$$\begin{tikzcd}
\mathfrak{X}_\eta \arrow[r, "j_{\mathfrak{X}}"] \arrow[d, "f_\eta"'] & \mathfrak{X} \arrow[d, "f"'] & \mathfrak{X}_0 \arrow[l, "i_{\mathfrak{X}}"'] \arrow[d, "f_0"] \\
\mathbb{G}_{m,S} \arrow[r, "j"] & \mathbb{A}^1_S & S \arrow[l, "i"']
\end{tikzcd}$$

We define the *unipotent nearby cycles functor* of f as

$$\psi_f^{\mathrm{uni}} : \mathrm{DM}(\mathfrak{X}_\eta,\Lambda) \to \mathrm{DM}(\mathfrak{X}_0,\Lambda), \quad M \mapsto i_{\mathfrak{X}}^*(j_{\mathfrak{X}})_*(M \otimes f_\eta^*\mathcal{L}\mathrm{og}^\vee).$$

There is a natural transformation $i_{\mathfrak{X}}^* \to i_{\mathfrak{X}}^* j_{\mathfrak{X}*} j_{\mathfrak{X}}^* \to i_{\mathfrak{X}}^* j_{\mathfrak{X}*}(j_{\mathfrak{X}}^*(-) \otimes f_\eta^*\mathcal{L}\mathrm{og}^\vee) = \psi_f^{\mathrm{uni}} j_{\mathfrak{X}}^*$ induced by adjunction and the morphism $\mathbb{1}_{\mathbb{G}_{m,S}} \to \mathcal{L}\mathrm{og}^\vee$ and we define the *unipotent vanishing cycles functor* of f as the cofibre of this natural transformation

$$\phi_f^{\mathrm{uni}} : \mathrm{DM}(\mathfrak{X},\Lambda) \to \mathrm{DM}(\mathfrak{X}_0,\Lambda), \quad M \mapsto \mathrm{cofib}\left(i_{\mathfrak{X}}^*(M) \to \psi_f^{\mathrm{uni}} j_{\mathfrak{X}}^*(M)\right).$$

For convenience, we also let $\widetilde{\phi}_f^{\mathrm{uni}} := (i_{\mathfrak{X}})_* \circ \phi_f^{\mathrm{uni}} : \mathrm{DM}(\mathfrak{X},\Lambda) \to \mathrm{DM}(\mathfrak{X},\Lambda)$.

To construct the nearby and vanishing cycles functors, for $n > 0$, let $p_n : \mathbb{A}^1_S \to \mathbb{A}^1_S$ denote the nth power map and let $\mathfrak{X}_n := \mathfrak{X} \times_{f,\mathbb{A}^1_S,p_n} \mathbb{A}^1_S$ denote the base change with projection maps $f_n : \mathfrak{X}_n \to \mathbb{A}^1_S$ and $e_n : \mathfrak{X}_n \to \mathfrak{X}$. We similarly construct $f_{\eta,n} : \mathfrak{X}_{\eta,n} \to \mathbb{G}_{m,S}$ and $f_{0,n} : \mathfrak{X}_{0,n} \to S$ by base-change along restrictions of p_n and we have an open immersion $j_n : \mathfrak{X}_{\eta,n} \to \mathfrak{X}_n$ and closed immersion $i_n : \mathfrak{X}_{0,n} \to \mathfrak{X}_n$.

The induced map $\mathfrak{X}_{0,n} \to \mathfrak{X}_0$ is a closed immersion which induces an isomorphism of reduced schemes, so we have compatible equivalences $\mathrm{DM}(\mathfrak{X}_{0,n},\Lambda) \simeq \mathrm{DM}(\mathfrak{X}_0,\Lambda)$ which are used implicitely in the following. We define the *nearby cycles functor* of f as

$$\psi_f : \mathrm{DM}(\mathfrak{X}_\eta,\Lambda) \to \mathrm{DM}(\mathfrak{X}_0,\Lambda), \quad M \mapsto \operatorname*{colim}_{n\in(\mathbb{N}^*,|)} \left(i_n^*(j_n)_*(e_{\eta,n}^* M \otimes f_{\eta,n}^*\mathcal{L}\mathrm{og}^\vee)\right).$$

Using the same construction as in the unipotent case, we have a natural transformation $i_{\mathfrak{X}}^* \to \psi_f j_{\mathfrak{X}}^*$ and we define the *vanishing cycles functor* of f as

$$\phi_f : \mathrm{DM}(\mathfrak{X},\Lambda) \to \mathrm{DM}(\mathfrak{X}_0,\Lambda), \quad M \mapsto \mathrm{cofib}\left(i_{\mathfrak{X}}^*(M) \to \psi_f j_{\mathfrak{X}}^*(M)\right).$$

The functors involved in the definition of ϕ_f commute with colimits, so that we also have

$$\phi_f(M) \simeq \operatorname*{colim}_{n\in\mathbb{N}^*} \left(\mathrm{cofib}\left(i_n^*(M) \to i_n^*(j_n)_*(e_{\eta,n}^*(j_{\mathfrak{X}}^* M) \otimes f_{\eta,n}^*\mathcal{L}\mathrm{og}^\vee)\right)\right).$$

For convenience, we also write $\widetilde{\phi}_f := (i_{\mathfrak{X}})_* \circ \phi_f : \mathrm{DM}(\mathfrak{X},\Lambda) \to \mathrm{DM}(\mathfrak{X},\Lambda)$.

Remark A.4. One advantage of this definition, compared to [9, Definition 3.5.6], is that it does not require introducing motives over diagrams of schemes. One can make sense of categories of motives over a diagram of schemes ∞-categorically, but we prefer to avoid this additional complication and to use the formalism of [42] to directly define nearby and vanishing cycles (both for schemes and stacks) at the level of ∞-categories.

Proposition A.5 (Proper base change for vanishing cycles of stacks)**.** *Let $g : \mathcal{Y} \to \mathfrak{X}$ be a proper Deligne–Mumford-representable morphism of stacks over S and $f : \mathfrak{X} \to \mathbb{A}^1_S$ be a regular function; consider the commutative diagram with cartesian squares*

(17)
$$\begin{tikzcd}
\mathcal{Y}_\eta \arrow[r, "j_{\mathcal{Y}}"] \arrow[d, "g_\eta"'] & \mathcal{Y} \arrow[d, "g"'] & \mathcal{Y}_0 \arrow[l, "i_{\mathcal{Y}}"'] \arrow[d, "g_0"] \\
\mathfrak{X}_\eta \arrow[r, "j_{\mathfrak{X}}"] \arrow[d, "f_\eta"'] & \mathfrak{X} \arrow[d, "f"'] & \mathfrak{X}_0 \arrow[l, "i_{\mathfrak{X}}"'] \arrow[d, "f_0"] \\
\mathbb{G}_{m,S} \arrow[r, "j"] & \mathbb{A}^1_S & S. \arrow[l, "i"']
\end{tikzcd}$$

Then there is a natural isomorphism

$$\phi_f g_* \simeq (g_0)_* \phi_{f\circ g} : \mathrm{DM}(\mathcal{Y}, \Lambda) \to \mathrm{DM}(\mathfrak{X}_0, \Lambda).$$

Proof. Since g is a proper Deligne–Mumford-representable morphism, the natural transformations $g_! \to g_*$ and $(g_\eta)_! \to (g_\eta)_*$ and $(g_0)_! \to (g_0)_*$ are isomorphisms by Theorem A.1 (vi).

Let us first show the corresponding statement for the unipotent vanishing cycles functor. Since the unipotent vanishing cycles functor is defined as a cofibre of the natural transformation $i^*_{\mathfrak{X}} \to \psi^{\mathrm{uni}}_f j^*_{\mathfrak{X}}$, it suffices to show there is a natural isomorphism

$$\psi^{\mathrm{uni}}_f (g_\eta)_* \simeq (g_0)_* \psi^{\mathrm{uni}}_{f\circ g} : \mathrm{DM}(\mathcal{Y}_\eta) \to \mathrm{DM}(\mathfrak{X}_0, \Lambda).$$

This natural isomorphism is obtained from the following chain of isomorphisms which are natural in $M \in \mathrm{DM}(\mathcal{Y}_\eta)$:

$$\begin{aligned}
\psi^{\mathrm{uni}}_f (g_\eta)_*(M) &:= i^*_{\mathfrak{X}} (j_{\mathfrak{X}})_*((g_\eta)_*(M) \otimes f^*_\eta \mathcal{L}\mathrm{og}^\vee) \\
&\simeq i^*_{\mathfrak{X}} (j_{\mathfrak{X}})_*(g_\eta)_*(M \otimes g^*_\eta f^*_\eta \mathcal{L}\mathrm{og}^\vee) \\
&\simeq i^*_{\mathfrak{X}} g_* (j_{\mathcal{Y}})_*(M \otimes (f\circ g)^*_\eta \mathcal{L}\mathrm{og}^\vee) \\
&\simeq (g_0)_*(i^*_{\mathcal{Y}} (j_{\mathcal{Y}})_*(M \otimes (f\circ g)^*_\eta \mathcal{L}\mathrm{og}^\vee) =: (g_0)_* \psi^{\mathrm{uni}}_{f\circ g}(M).
\end{aligned}$$

Here the first isomorphism comes from the projection formula, the second comes from the commutativity of the upper left square in (17) and the final isomorphism comes from proper base change for the proper Deligne–Mumford-representable map g (Theorem A.1 (vi)).

The corresponding statement for the full vanishing cycles functor and nearby cycles functor then follows as colimits commute with both $(g_\eta)_* \simeq (g_\eta)_!$ and $(g_0)_* \simeq (g_0)_!$ because they are left adjoints. □

We also need the next result about pulling back vanishing cycles along smooth morphisms.

Proposition A.6 (Smooth base change for vanishing cycles of stacks)**.** *Let $g : \mathcal{Y} \to \mathfrak{X}$ be a smooth morphism of stacks over S and $f : \mathfrak{X} \to \mathbb{A}^1_S$ be a regular function; then using the notation in Diagram* (17)*, there is a natural isomorphism*

$$g^*_0 \phi_f \simeq \phi_{f\circ g} g^* : \mathrm{DM}(\mathfrak{X}, \Lambda) \to \mathrm{DM}(\mathcal{Y}_0, \Lambda).$$

Proof. As in the proof of Proposition A.5, this boils down to constructing an isomorphism

$$g^*_0 \psi^{\mathrm{uni}}_f \simeq \psi^{\mathrm{uni}}_{f\circ g} g^*_\eta : \mathrm{DM}(\mathfrak{X}_\eta) \to \mathrm{DM}(\mathcal{Y}_0, \Lambda).$$

This isomorphism follows from the chain of isomorphisms which are natural in $M \in \mathrm{DM}(\mathfrak{X}_\eta)$:

$$\begin{aligned}
g^*_0 \psi^{\mathrm{uni}}_f(M) &:= g^*_0 i^*_{\mathfrak{X}} (j_{\mathfrak{X}})_*(M \otimes f^*_\eta \mathcal{L}\mathrm{og}^\vee) \\
&\simeq i^*_{\mathcal{Y}} g^* (j_{\mathfrak{X}})_*(M \otimes f^*_\eta \mathcal{L}\mathrm{og}^\vee) \\
&\simeq i^*_{\mathcal{Y}} (j_{\mathcal{Y}})_* g^*_\eta (M \otimes f^*_\eta \mathcal{L}\mathrm{og}^\vee) \\
&\simeq i^*_{\mathcal{Y}} (j_{\mathcal{Y}})_*(g^*_\eta(M) \otimes (f\circ g)^*_\eta \mathcal{L}\mathrm{og}^\vee) =: \psi^{\mathrm{uni}}_{f\circ g} g^*_\eta(M),
\end{aligned}$$

where the middle isomorphism follows from smooth base change (Theorem A.1 (v)). $\square$

Remark A.7. It follows from this smooth base change property that the functors ψ_f and ϕ_f defined above are canonically equivalent to the functors obtained by extending ψ_f and ϕ_f for schemes using Theorem A.1 (i).

Remark A.8. For a morphism of stacks $g : \mathcal{Y} \to \mathfrak{X}$ and regular function $f : \mathfrak{X} \to \mathbb{A}^1_S$, to avoid the notation g_0 in the above results, we can work with $\widetilde{\phi}_f := i_* \circ \phi_f : \mathrm{DM}(\mathfrak{X}, \Lambda) \to \mathrm{DM}(\mathfrak{X}, \Lambda)$. Then the above two results translate into the following statements.

(1) By Proposition A.5, for g proper and representable by Deligne–Mumford stacks, we have $\widetilde{\phi}_f g_* \simeq g_* \widetilde{\phi}_{f\circ g} : \mathrm{DM}(\mathcal{Y}, \Lambda) \to \mathrm{DM}(\mathfrak{X}, \Lambda)$.
(2) By Proposition A.6, for g smooth, we have $g^* \widetilde{\phi}_f \simeq \widetilde{\phi}_{f\circ g} g^* : \mathrm{DM}(\mathfrak{X}, \Lambda) \to \mathrm{DM}(\mathcal{Y}, \Lambda)$.

The theory of motivic vanishing cycles is strongly inspired by the theory of nearby and vanishing cycle functors for sheaves of Λ-modules on complex algebraic varieties [1, Exposé XIV]. Let X be a finite type $\mathbb{C}$-scheme and $f : X \to \mathbb{A}^1_{\mathbb{C}}$ be a regular function. For concreteness, we use the formulation in [10, Paragraph before Proposition 4.8] to define

$$\psi_f^{\mathrm{an}} : D(X_\eta^{\mathrm{an}}, \Lambda) \to D(X_0^{\mathrm{an}}, \Lambda)$$

and we define $\phi_f^{\mathrm{an}} : D(X^{\mathrm{an}}, \Lambda) \to D(X_0^{\mathrm{an}}, \Lambda)$ as a cofibre of the natural map $i^* \to \psi_f^{\mathrm{an}} \circ j^*$. By [1, Exposé XIV], these functors satisfy smooth and proper base change properties. Moreover, the definition is in terms of the six operations and so makes sense at the level of ∞-categories of sheaves.

We claim that these nearby and vanishing cycles functors in the sheaf setting can be readily extended to Artin stacks. Let $\mathfrak{X}$ be a finite type Artin stack over $\mathbb{C}$ and $f : \mathfrak{X} \to \mathbb{A}^1_{\mathbb{C}}$ be a regular function. As when defining the Betti realisation, we define $D(\mathfrak{X}^{\mathrm{an}}, \Lambda)$ as a limit over $\mathrm{Lis}_{\mathfrak{X}}$. Moreover, ϕ_f^{an} also satisfies smooth base change, so we can extend directly to a functor

$$\phi_f^{\mathrm{an}} : D(\mathfrak{X}^{\mathrm{an}}, \Lambda) \to D(\mathfrak{X}_0^{\mathrm{an}}, \Lambda)$$

Moreover, these motivic and sheaf vanishing cycles functors commute with Betti realisation in the following sense.

Theorem A.9. *Let $\sigma : k \to \mathbb{C}$ be a complex embedding. Let $\mathfrak{X}$ be a finite type Artin stack over k and $f : \mathfrak{X} \to \mathbb{A}^1_k$ be a regular function.*

(i) *There is a natural transformation*

$$\omega_f : R_B \circ \phi_f \to \phi_f^{\mathrm{an}} \circ R_B$$

of functors $\mathrm{DM}(\mathfrak{X}, \Lambda) \to \mathrm{DM}(\mathfrak{X}_0, \Lambda)$.
(ii) *The construction of ω_f commutes with base change by smooth morphisms (modulo smooth base change for vanishing cycles).*
(iii) *Let $M \in \mathrm{DM}(\mathfrak{X}, \Lambda)$ be such that for every finite type k-scheme X and every smooth morphism $u : X \to \mathfrak{X}$, the motive u^*M is constructible in* $\mathrm{DM}(X, \Lambda)$. *Then $\omega_f(M)$ is an isomorphism.*

Proof. Both sides of ω_f are obtained by passing to the limit over $\mathrm{Lis}_{\mathfrak{X}}$, so to show (i) and (ii) it suffices to construct ω_f for schemes and to proves that its construction commutes with base change by smooth morphisms.

Such a natural transformation ω_f is constructed for nearby cycles in the context of triangulated categories of motives over schemes by Ayoub in [10, Proposition 4.8]. He also proves that his construction commutes with base change by smooth morphisms. Unfortunately, Ayoub uses in *loc. cit.* a slightly different definition of nearby cycles involving diagram of schemes. We claim that with our logarithmic definition, the construction of ω_f is even simpler than in [10, Proposition 4.8].

Let us explain the construction of the analogue $\omega_f^{\mathrm{uni}} : R_B \circ \psi_f^{\mathrm{uni}} \to \psi_{f^{\mathrm{an}}} \circ R_B$; the general case then follows from passing to the limit over the n-th power maps. The key point is that by construction of $\mathcal{L}\mathrm{og}^\vee$ (see [9, Definition 3.6.29]), the object $R_B \mathcal{L}\mathrm{og}^\vee \in D(\mathbb{G}_{m,\mathbb{C}}^{\mathrm{an}}, \Lambda)$ is an ind-(unipotent local system) on $\mathbb{C}^\times$ and it becomes canonically trivialised when pulling back to the universal cover $\exp : \mathbb{C} \to \mathbb{C}^\times$. Looking at the definition of ψ_f^{an} in [10, Paragraph 4.8], we see that this immediately provides a natural transformation $\omega_f^{\mathrm{uni}} : R_B \circ \psi_f^{\mathrm{uni}} \to \psi_{f^{\mathrm{an}}} \circ R_B$. The fact that the construction of the resulting natural transformation ω_f satisfies base change by smooth morphisms is then an easy consequence of smooth base change for j_*. This finishes the proof of (i) and (ii).

It remains to show (iii). By (ii) and the fact that the collection of functors $(u_X^* : \mathrm{DM}(\mathfrak{X}, \Lambda) \to \mathrm{DM}(X, \Lambda))_{(u:X\to\mathfrak{X})\in \mathrm{Lis}_{\mathfrak{X}}}$ is jointly conservative, it suffices to show (iii) when $\mathfrak{X} = X$ is a scheme and M is constructible. Since it is a matter of checking that a certain morphism is an isomorphism, we can work at the level of the (triangulated) homotopy categories. In [10, Theoreme 4.9], Ayoub proves precisely this result for his definition of ω_f (and nearby cycles, but passing to vanishing cycles is then easy). We claim that the arguments in [10, Theoreme 4.9] apply just as well to our definition of ω_f. The proof of [10, Theoreme 4.9] proceeds by reducing, using smooth and proper base change for nearby cycles (and the resulting machinery of "specialisation systems"), to a very simple situation. The same reduction applies in our case, and the computation in the simple situation is then also easy to do. □

A.3. **Motivic vanishing cycles for homogeneous functions.** Our goal in this section is to prove the following theorem about motivic vanishing cycles functors for quadratic forms, which is a special case of a more general result about vanishing cycles of homogeneous functions (see Theorem A.11). Recall for a half integer $r \in \frac{1}{2}\mathbb{Z}$, we defined Tate twists $\{r\} := (\lfloor r \rfloor)[2r]$, which are pure if and only if $r \in \mathbb{Z}$.

Theorem A.10. *Let V be a vector space of dimension $d > 0$ over an algebraically closed field k of characteristic zero and $q : V \to \mathbb{A}^1$ be a non-degenerate quadratic form. Then in* $\mathrm{DM}(V_0, \Lambda)$*, we have*

$$\phi_q(\mathbb{1}_V) \simeq (i_0)_* \mathbb{1}\{-(d-1)/2\},$$

where $i_0 : \mathrm{Spec}(k) \to V_0 := q^{-1}(0)$ denotes the inclusion of the origin.

This result is well-known in the étale setting as part of the Picard–Lefschetz theory in SGA7 [1, Exposé XV 2.2.5 D E]. In fact, for most of the proof we can work with a non-degenerate homogeneous regular function $f : V \to \mathbb{A}^1$ and we will describe the vanishing cycles functor as the reduced cohomological motive of the fibre $V_1 := f^{-1}(1)$ as stated in the next theorem. Versions of the following theorem were already well-known in other contexts and particularly in Donaldson–Thomas theory. More precisely, the virtual motivic nearby cycles functor of a homogeneous form (or more generally a certain torus equivariant regular function) is described in [16, Theorem B.1] and in a weighted homogeneous setting in [56, Theorem 4.1.1], confirming a conjecture of Davison and Meinhardt [20].

Theorem A.11. *Let V be a vector space over a field k of characteristic zero and $f : V \to \mathbb{A}^1$ be a non-degenerate homogeneous function. Then in* $\mathrm{DM}(V_0, \Lambda)$*, we have*

$$\phi_f(\mathbb{1}_V) \simeq (i_0)_* \overline{M}_{\mathrm{coh}}(V_1),$$

where $i_0 : \mathrm{Spec}(k) \to V_0 := f^{-1}(0)$ denotes the inclusion of the origin.

Theorem A.10 directly follows from Theorem A.11 and Proposition A.12 below, which computes the (cohomological) motive of a (split) affine quadric and is based on Rost's computation in the projective setting [60] and Bachmann's computation in the affine setting [13].

Proposition A.12. *Let Q_n be a smooth affine quadric of dimension n over an algebraically closed field k; then in $DM(k, \Lambda)$ we have an isomorphism*

$$M_{\mathrm{coh}}(Q_n) \simeq \mathbb{1} \oplus \mathbb{1}\{-n/2\}.$$

Proof. We prove the proposition by induction on n. For $n = 0$, the quadric Q_0 consists of two points and thus $M_{\mathrm{coh}}(Q_0) \simeq \Lambda \oplus \Lambda$. For $n = 1$, the quadric Q_1 is isomorphic to $\mathbb{G}_m$ and thus $M_{\mathrm{coh}}(Q_1) \simeq \mathbb{1} \oplus \mathbb{1}(-1)[-1]$.

For the inductive step, we relate the reduced motives of Q_n and Q_{n-2} by using [13, Lemma 34] and [60, Proposition 1] to conclude

$$\overline{M}(Q_n) \simeq \overline{M}(Q_{n-2})\{1\}.$$

For example, this isomorphism appears in [14, Eq (1)]. By dualising, we conclude $\overline{M}_{\mathrm{coh}}(Q_n) \simeq \overline{M}_{\mathrm{coh}}(Q_{n-2})\{-1\}$ which completes the inductive proof. □

The goal for the rest of this section is to prove Theorem A.11. Throughout this section, we write $f_0 : V_0 \to \{0\}$ and $f_\eta : V_\eta \to \mathbb{G}_m$ as in the previous subsection.

First of all, as the critical locus of the non-degenerate homogeneous function $f : V \to \mathbb{A}^1$ is the origin, the vanishing cycles functor is concentrated at the origin in the following sense.

Lemma A.13. *Let X be a finite type k scheme and $f : X \to \mathbb{A}^1$ be a regular function with a single isolated critical point $x_0 \in X(k)$ over 0 with $i_0 : \mathrm{Spec}(k) \to X_0$ the corresponding closed immersion. We have*

$$\phi_f(\mathbb{1}_X) \simeq (i_0)_*(i_0)^*\phi_f(\mathbb{1}_X) \simeq (i_0)_*(f_0)_*\phi_f(\mathbb{1}_X).$$

Proof. Since the restriction $f^\times : X^\times \to \mathbb{A}^1$ of f to $X^\times := X \setminus \{x_0\}$ is smooth, we have $\psi_{f^\times}(\mathbb{1}_{X_\eta}) \simeq \mathbb{1}_{X_0}$ and $\phi_{f^\times}(\mathbb{1}_X) \simeq 0$. Let $X_0^\times := X_0 \setminus \{x_0\}$ and $j_0 : X_0^\times \hookrightarrow X_0$ denote the open immersion; then by smooth base change for vanishing cycles, we have $\phi_{f^\times}(\mathbb{1}_{X^\times}) \simeq j_0^*\phi_f(\mathbb{1}_X)$. By considering the localisation triangle for the pair $(j_0 : X_0^\times \hookrightarrow X_0,\ i_0 : \{x_0\} \hookrightarrow X_0)$

$$(j_0)_!(j_0)^*\phi_f(\mathbb{1}_X) \to \phi_f(\mathbb{1}_X) \to (i_0)_*(i_0)^*\phi_f(\mathbb{1}_X) \xrightarrow{+}$$

we obtain the first claimed isomorphism, as the left term is zero. The second claimed isomorphism then follows as i_0 is a section of f_0 and again using the localisation sequence for (i_0, j_0) together with the fact that $(j_0)^*\phi_f(\mathbb{1}_X) \simeq 0$. □

The next step is to relate the vanishing cycles functor for f with the vanishing cycles functor for $\mathrm{Id} : \mathbb{A}^1 \to \mathbb{A}^1$.

Proposition A.14. *For a homogeneous non-degenerate function $f : V \to \mathbb{A}^1$ on a vector space V over a field k, we have $\phi_{\mathrm{Id}}(f_*\mathbb{1}_V) \simeq (f_0)_*\phi_f(\mathbb{1}_V)$.*

Proof. Ideally we would like to prove this using proper base change, but the morphism f is not proper. To remedy this, we construct a fibrewise projectivisation $\widetilde{f} : \widetilde{V} \to \mathbb{A}^1$ with smooth boundary divisor $D := \widetilde{V} \setminus V$ which is a constant family over $\mathbb{A}^1$ and use proper base change for $\widetilde{f}$. More precisely, we define $\widetilde{V}$ to be the relative projective spectrum over $\mathbb{A}^1 = \mathrm{Spec}\, k[t]$ of the graded ring $k[V, z, t]/(f - tz^r)$ where r denotes the degree of the homogeneous function f and the coordinate t has weight 0, the coordinate z has weight 1 and the coordinates on V have weight 1. By construction, the fibre $\widetilde{V}_\lambda$ over $\lambda \in \mathbb{A}^1$ is the projective variety defined by the vanishing locus of the equation $f(v) = \lambda z^r$ in $\mathbb{P}(V \oplus k)$ with coordinates $[v : z]$ and it contains V_λ as the intersection with $V \cong \{z \neq 0\}$ and has boundary D_λ given by the projective variety $f(v) = 0$ in $\mathbb{P}(V) \cong \{z = 0\}$.

Let $f_D : D \to \mathbb{A}^1$ denote the restriction of $\widetilde{f}$ to D, so that we have a commutative diagram

$$\begin{tikzcd} V \arrow[r, "\nu"] \arrow[dr, "f"'] & \widetilde{V} \arrow[d, "\widetilde{f}"] & D \arrow[l, "\iota"'] \arrow[dl, "f_D"] \\ & \mathbb{A}^1 & \end{tikzcd}$$

with ν an open immersion and ι a closed immersion. Since $D \cong D_0 \times \mathbb{A}^1$ is a smooth constant family, we have

$$\phi_{f_D}(\Lambda_D) \simeq 0. \tag{18}$$

Furthermore, as $\widetilde{f}$ and f_D are both proper, by Proposition A.5, we have

$$\phi_{\mathrm{Id}}\widetilde{f}_* \simeq (\widetilde{f}_0)_*\phi_{\widetilde{f}} \quad \text{and} \quad \phi_{\mathrm{Id}}f_{D*} \simeq (f_{D,0})_*\phi_{f_D}. \tag{19}$$

By applying $\widetilde{f}_*$ to the localisation sequence for $(\nu : V \hookrightarrow \widetilde{V}, \iota : D \hookrightarrow \widetilde{V})$, we obtain

$$\widetilde{f}_*\iota_*\iota^!\mathbb{1}_{\widetilde{V}} \to \widetilde{f}_*\mathbb{1}_{\widetilde{V}} \to \widetilde{f}_*\nu_*\nu^*\mathbb{1}_{\widetilde{V}} \xrightarrow{+}$$

and we have isomorphisms $\widetilde{f}_*\nu_*\nu^*\mathbb{1}_{\widetilde{V}} \simeq f_*\mathbb{1}_V$ and $\widetilde{f}_*\iota_*\iota^!\mathbb{1}_{\widetilde{V}} \simeq (f_D)_*\iota^!\mathbb{1}_{\widetilde{V}} \simeq (f_D)_*\mathbb{1}_D\{-1\}$ using the purity isomorphism for the regular map ι. Next we apply the vanishing cycles functor for $\mathrm{Id} : \mathbb{A}^1 \to \mathbb{A}^1$ to this sequence to obtain

$$\phi_{\mathrm{Id}}((f_D)_*\mathbb{1}_D\{-1\}) \to \phi_{\mathrm{Id}}(\widetilde{f}_*\mathbb{1}_{\widetilde{V}}) \to \phi_{\mathrm{Id}}(f_*\mathbb{1}_V) \xrightarrow{+}$$

where the left term is zero and the middle term is isomorphic to $(\widetilde{f}_0)_*\phi_{\widetilde{f}}(\mathbb{1}_{\widetilde{V}})$ by Equations (18) and (19). Therefore, to prove that $\phi_{\mathrm{Id}}(f_*\mathbb{1}_V) \simeq (f_0)_*\phi_f(\mathbb{1}_V)$, it suffices to show that there is an isomorphism $(\widetilde{f}_0)_*\phi_{\widetilde{f}}(\mathbb{1}_{\widetilde{V}}) \to (f_0)_*\phi_f(\mathbb{1}_V)$. This final claim is proved by using Lemma A.13 for both f and $\widetilde{f}$ together with the isomorphism $\widetilde{i_0}^*\phi_{\widetilde{f}} \simeq i_0^*\phi_f$, where i_0 (resp. $\widetilde{i_0}$) denote the inclusion of the origin in V_0 (resp. $\widetilde{V}_0$), which follows from smooth base change for vanishing cycles applied to the open immersion ν. □

We are now ready to prove Theorem A.11.

Proof of Theorem A.11. Let r denote the degree of the homogeneous function $f : V \to \mathbb{A}^1$. We let $p_r : \mathbb{A}^1 \to \mathbb{A}^1$ denote the rth power map and write $f_{(r)} : V_{(r)} \to \mathbb{A}^1$ for the base change of f along p_r. Then, because f is homogeneous of degree r, the generic fibre $f_{(r),\eta} : V_{(r),\eta} \to \mathbb{G}_m$ is the constant family: $V_{(r),\eta} \cong V_{(r),1} \times \mathbb{G}_m \cong V_1 \times \mathbb{G}_m$.

By [9, Lemme 3.5.8 and following paragraph], the nearby cycles functor for $\mathrm{Id} : \mathbb{A}^1 \to \mathbb{A}^1$ is unchanged when applying $p_{r,\eta}^*$; that is, $\psi_{\mathrm{Id}} \simeq \psi_{\mathrm{id}} p_{r,\eta}^*$. Therefore, we have isomorphisms

$$\psi_{\mathrm{Id}}((f_\eta)_*\mathbb{1}_{V_\eta}) \simeq \psi_{\mathrm{Id}}(p_{r,\eta}^*(f_\eta)_*\mathbb{1}_{V_\eta}) \simeq \psi_{\mathrm{Id}}(f_{(r),\eta})_*\mathbb{1}_{V_{(r),\eta}} \simeq (f_1)_*\mathbb{1}_{V_1}$$

where the first isomorphism is the result of Ayoub, the middle isomorphism is smooth base change for p_r and the right isomorphism follows as $f_{(r),\eta}$ is constant. Here $f_1 : V_1 \to \{1\}$ is the restriction of f to the fibre over 1.

Since V_0 is an affine cone over 0, we have $(f_0)_*\mathbb{1}_{V_0} \simeq \mathbb{1}_k$ by rescaling. Even though f is not proper, we have $i^*f_*\mathbb{1}_V \simeq (f_0)_*\mathbb{1}_{V_0}$ as follows from localisation and proper base change for the compactification $\tilde{f}$ as in the proof of Proposition A.14. Therefore, we have

$$\phi_{\mathrm{Id}}(f_*\mathbb{1}_V) = \mathrm{cofib}\left(i^*f_*\mathbb{1}_V \to \phi_{\mathrm{Id}}((f_\eta)_*\mathbb{1}_{V_\eta})\right) \simeq \mathrm{cofib}\left(\mathbb{1}_k \to (f_1)_*\mathbb{1}_{V_1}\right) \simeq \overline{M}_{\mathrm{coh}}(V_1). \tag{20}$$

To complete the proof, we have isomorphisms

$$\phi_f(\mathbb{1}_V) \simeq (i_0)_*(f_0)_*\phi_f(\mathbb{1}_V) \simeq (i_0)_*\phi_{\mathrm{Id}}(f_*\mathbb{1}_V) \simeq (i_0)_*\overline{M}_{\mathrm{coh}}(V_1)$$

coming from Lemma A.13, Proposition A.14 and Equation (20). □

Remark A.15. We expect that Theorem A.11 can be generalised to the setting of a non-degenerate weighted homogeneous function $f : V \to \mathbb{A}^1$.

Appendix B. Motives of stacks of vector bundles with fixed determinant

Throughout this section, we assume that k is an arbitrary field and $C(k) \neq \emptyset$. We will compute motives of stacks of vector bundles over C (or families $\mathcal{C}/T$ of curves) with fixed determinant in $\mathrm{DM}(k, \mathbb{Q})$ by extending results in [40, 38].

B.1. **A formula for the motive of the stack of bundles with fixed determinant.** Fix a line bundle $L \to C$ and consider the stack $\mathfrak{Bun}_{n,L}$ of rank n vector bundles with determinant isomorphic to L. If $d = \deg(L)$, then $\mathfrak{Bun}_{n,L}$ is a smooth codimension g substack of $\mathfrak{Bun}_{n,d}$. In this section, we will prove the following explicit formula.

Theorem B.1. *Assume $C(k) \neq \emptyset$. Then in $\mathrm{DM}(k, \mathbb{Q})$, we have*

$$M(\mathfrak{Bun}_{n,L}) \simeq M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, \mathbb{Q}\{i\}).$$

Proof. Fix $x \in C(k)$ and for $l \in \mathbb{N}$, we consider as in [40, §4.3], the scheme

$$\mathrm{Div}_{n,L}(l) := \{E \subset \mathcal{O}_C(lx)^{\oplus n} : \mathrm{rk}(E) = n, \det(E) \cong L\}$$

which is a smooth codimension g closed subvariety of the Quot scheme $\mathrm{Div}_{n,d}(l)$ of length $nl-d$ torsion quotients of $\mathcal{O}_C(lx)^{\oplus n}$, where $d = \deg(L)$. Moreover, one has

$$M(\mathfrak{Bun}_{n,L}) \simeq \operatorname*{hocolim}_l M(\mathrm{Div}_{n,L}(l))$$

as in the proof of the first paragraph of [40, Theorem 4.6].

We also consider the smooth closed subscheme $\mathrm{FDiv}_{n,L}(l)$ in the full-flag Quot scheme $\mathrm{FDiv}_{n,d}(l)$ given by

$$\mathrm{FDiv}_{n,L}(l) := \left\{F_0 \subset \cdots \subset F_{nl-d} = \mathcal{O}_C(lx)^{\oplus n} : \mathrm{rk}(F_i) = n, \deg(F_i) = d+i, \det(F_0) \simeq L\right\}.$$

The support map $\mathrm{supp} : \mathrm{FDiv}_{n,d} \to C^{nl-d}$ given by $(F_0 \subset \cdots \subset F_{nl-d}) \mapsto \mathrm{supp}(F_i/F_{i-1})_{1\leq i\leq nl-d}$ is an $(nl-d)$-iterated projective bundle. Let $(C^{nl-d})_L := \mathrm{supp}(\mathrm{FDiv}_{n,L}(l))$; then by the projective bundle formula

$$M(\mathrm{FDiv}_{n,L}(l)) \simeq M((C^{nl-d})_L) \otimes M(\mathbb{P}^{n-1})^{\otimes(nl-d)}.$$

We will now complete the proof by adapting the argument in [38, §4.3]. Using the decompositions $M(\mathbb{P}^{n-1}) \simeq \oplus_{i=0}^{n-1}\mathbb{Q}\{i\}$ as in [38, Remark 4.6], we can write

$$M(\mathrm{FDiv}_{n,L}(l)) \simeq \bigoplus_{I\in\mathcal{I}_l} M((C^{nl-d})_L) \otimes \mathbb{Q}\{|I|\},$$

where $\mathcal{I}_l = \{0,\dots,n-1\}^{\times nl-d}$ and for $I = (i_1,\dots,i_{nl-d}) \in \mathcal{I}_l$, we write $|I| := \sum_{j=1}^{nl-d} i_j$. Let $\mathcal{B}_l$ denote the set of $\underline{m} = (m_0,\dots,m_{n-1}) \in \mathbb{N}^n$ with $\sum_{i=0}^{n-1} m_i = nl-d$. Then as in [38, Lemma 4.7], we can conclude that

$$M(\mathrm{Div}_{n,L}(l)) \simeq \bigoplus_{\underline{m}\in\mathcal{B}_l} M(C_L^{(\underline{m})})\{c_{\underline{m}}\},$$

where $c_{\underline{m}} := \sum_{i=0}^{n-1} im_i$ and $C_L^{(\underline{m})} \subset C^{(\underline{m})} := C^{(m_0)}\times\cdots\times C^{(m_{n-1})}$ is the image of $(C^{nl-d})_L$ under the quotient $C^{nl-d} \to C^{(m_0)} \times \cdots \times C^{(m_{n-1})}$ by the product of symmetric groups $\prod_{i=0}^{n-1} S_{m_i}$. Furthermore, as in [38, Lemma 4.7 (ii)], the transition map $M(\mathrm{Div}_{n,L}(l)) \to M(\mathrm{Div}_{n,L}(l+1))$ is induced by direct sums over $\underline{m} \in \mathcal{B}_l$ and $\underline{m'} \in \mathcal{B}_{l+1}$ of the maps $M(\kappa_{\underline{m},\underline{m'},L})\{c_{\underline{m}}\}$, where

$$\kappa_{\underline{m},\underline{m'},L} : C_L^{(\underline{m})} \to C_L^{(\underline{m'})}$$

is zero unless $\underline{m'} = \underline{m} + (n,0,\dots,0)$ (and thus $c_{\underline{m}} = c_{\underline{m'}}$) and in this case, $\kappa_{\underline{m},\underline{m'},L}$ is the restriction of the map $\kappa_{\underline{m},\underline{m'}} : C^{(\underline{m})} \to C^{(\underline{m'})}$ induced by $(x,\dots,x)\times\mathrm{id}_{C^{nl-d}} : C^{nl-d} \to C^{n(l+1)-d}$ which includes n copies of x.

The rest of the proof follows exactly as in [40, Theorem 4.6], so we simply outline the idea. For $\underline{m}^\flat = (m_1,\dots,m_{n-1}) \in \mathbb{N}^{n-1}$, we define $c_{\underline{m}^\flat} := \sum_{i=1}^{n-1} im_i$ and $m_0^\flat(l) := nl-d-\sum_{i=1}^{n-1} m_i$ and write $\underline{m}^\flat(l) := (m_0^\flat(l), \underline{m}^\flat) \in \mathbb{Z}\times\mathbb{N}^{n-1}$; then $c_{\underline{m}^\flat(l)} = c_{\underline{m}^\flat}$. From the above description of the transitions maps one obtains that

$$M(\mathfrak{Bun}_{n,L}) \simeq \bigoplus_{\underline{m}^\flat\in\mathbb{N}^{n-1}} \operatorname*{hocolim}_l P^L_{\underline{m}^\flat,l}$$

where $P^L_{\underline{m}^\flat,l} := M(C_L^{(\underline{m}^\flat(l))})\{c_{\underline{m}^\flat}\}$ if $m_0^\flat(l) \geq 0$ and is zero otherwise. Next for $\underline{m} = (m_0,\dots,m_{n-1})$, we use a generalised Abel-Jacobi map

$$C_L^{(\underline{m})} \to C^{(m_1)} \times \cdots \times C^{(m_{n-1})},$$

which is a $\mathbb{P}^{m_0-g}$-bundle if $m_0 > 2g-2$, to deduce that

$$M(\mathfrak{Bun}_{n,L}) \simeq \bigoplus_{\underline{m}^\flat \in \mathbb{N}^{n-1}} \operatorname*{hocolim}_{l:m_0^\flat(l)\geq 0} P^L_{\underline{m}^\flat,l} \simeq \bigoplus_{\underline{m}^\flat \in \mathbb{N}^{n-1}} \operatorname*{hocolim}_{l:m_0^\flat(l)>2g-2} M(\mathbb{P}^{m_0^\flat(l)-g}) \otimes M(C^{(\underline{m}^\flat)})\{c_{\underline{m}^\flat}\}$$

$$\simeq \bigoplus_{\underline{m}^\flat \in \mathbb{N}^{n-1}} M(B\mathbb{G}_m) \otimes M(C^{(\underline{m}^\flat)})\{c_{\underline{m}^\flat}\} \simeq M(B\mathbb{G}_m) \otimes \bigotimes_{i=1}^{n-1} Z(C, \mathbb{Q}\{i\}),$$

where $C^{(\underline{m}^\flat)} := C^{(m_1)} \times \cdots \times C^{(m_{n-1})}$. This completes the proof. □

In the case when $L = \mathcal{O}_C$, we have that $\mathfrak{Bun}_{SL_n} \to \mathfrak{Bun}_{n,\mathcal{O}_C}$ is a $\mathbb{G}_m$-torsor (see [40, §4.3]) and so we deduce the following corollary.

Corollary B.2. *Assume $C(k) \neq \emptyset$. Then in $\mathrm{DM}(k,\mathbb{Q})$, we have*

$$M(\mathfrak{Bun}_{\mathrm{SL}_n}) \simeq \bigotimes_{i=1}^{n-1} Z(C, \mathbb{Q}\{i\}).$$

Proof. This follows from Theorem B.1 using the arguments of the proof of [40, Theorem 4.7]. □

B.2. **Relative formulae for families of curves.** Throughout this section, we fix a (Noetherian finite-dimensional) scheme T and consider a family $\mathcal{C}$ of smooth projective geometrically connected genus g curves over T and we assume that this family admits a section. We let $\mathfrak{Bun}_{\mathcal{C}/T,n,d}$ denote the stack (over T) of vector bundles on $\mathcal{C}/T$ of rank n and degree d. We first consider the relative case without fixing the determinant and then consider the relative case with fixed determinant.

For a T-scheme X (or stack), we write $M_T(X) \in \mathrm{DM}(T,\mathbb{Q})$ for the relative motive of X over T. We write $\mathbb{Q}_T\{r\} \in \mathrm{DM}(T,\mathbb{Q})$ for the pure Tate twists. We shall also write $(X/T)^r$ to denote the T-scheme given by the r-fold fibre product of X over T and we write $\mathrm{Sym}^r(X/T)$ for the S_r-quotient of $(X/T)^r$.

Theorem B.3. *Let $\mathcal{C}/T$ be a family of smooth projective geometrically connected genus g curves over T admitting a section $\sigma : T \to \mathcal{C}$. Then in $\mathrm{DM}(T,\mathbb{Q})$, we have*

$$M_T(\mathfrak{Bun}_{\mathcal{C}/T,n,d}) \simeq M_T(\mathrm{Jac}_{\mathcal{C}/T}) \otimes M_T(B\mathbb{G}_{m,T}) \otimes \bigotimes_{i=1}^{n-1} Z_T(\mathcal{C}/T, \mathbb{Q}_T\{i\}),$$

where

$$Z_T(\mathcal{C}/T, \mathbb{Q}_T\{i\}) := \bigoplus_{j\geq 0} M_T(\mathrm{Sym}^j(\mathcal{C}/T)) \otimes \mathbb{Q}_T\{ij\} \in \mathrm{DM}(T,\mathbb{Q}).$$

Proof. Let $\mathcal{O}_{\mathcal{C}}(\sigma)$ denote the line bundle whose restriction to $t \in T$ is the degree 1 line bundle $\mathcal{O}_{\mathcal{C}_t}(\sigma(t))$. We now consider relative versions of the (flag)-Quot schemes appearing in [40, 38]: we let

$$\mathrm{Div}_{\mathcal{C}/T,n,d}(l) := \mathrm{Quot}^{0,nl-d}_{\mathcal{C}/T}(\mathcal{O}_{\mathcal{C}}(l\sigma)^{\oplus n})$$

denote the relative Quot scheme over T of rank 0, degree $nl-d$ quotient sheaves of $\mathcal{O}_{\mathcal{C}}(l\sigma)^{\oplus n}$ and we let $\mathrm{FDiv}_{\mathcal{C}/T,n,d}(l)$ denote the relative full flag version, whose points over $S \to T$ are

$$\mathrm{FDiv}_{\mathcal{C}/T,n,d}(l)(S) = \left\{ \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_{nl-d} = \mathcal{O}_{\mathcal{C}_S}(l\sigma_S)^{\oplus n} : \begin{smallmatrix} \mathcal{F}_i \\ \downarrow \\ \mathcal{C}_S \end{smallmatrix}, \mathrm{rk}(\mathcal{F}_i) = n, \deg(\mathcal{F}_i) = d+i \right\}.$$

Both $\mathrm{Div}_{\mathcal{C}/T,n,d}(l)$ and $\mathrm{FDiv}_{\mathcal{C}/T,n,d}(l)$ are smooth projective schemes over T.

The natural forgetful morphisms $\mathrm{Div}_{\mathcal{C}/T,n,d}(l) \to \mathfrak{Bun}_{\mathcal{C}/T,n,d}$ induce a morphism

$$\operatorname*{hocolim}_{l} M_T(\mathrm{Div}_{\mathcal{C}/T,n,d}(l)) \to M_T(\mathfrak{Bun}_{\mathcal{C}/T,n,d})$$

in $\mathrm{DM}(T,\mathbb{Q})$, which we claim is an isomorphism. By [12, Proposition 3.24], it suffices to check the pullback of this map along each point $t \in T$ is an isomorphism. However, for each $t \in T$, the pullback to $\mathrm{DM}(\kappa(t),\mathbb{Q})$

$$\operatorname*{hocolim}_{l} M(\mathrm{Div}_{\mathcal{C}_t,n,d}(l\sigma(t))) \to M(\mathfrak{Bun}_{\mathcal{C}_t,n,d})$$

coincides with the isomorphism given by [40, Theorem 3.5].

Extending the absolute case proved in [38, §4] to the relative setting, there is a support map $\mathrm{supp} : \mathrm{FDiv}_{\mathcal{C}/T,n,d}(l) \to (\mathcal{C}/T)^{nl-d}$ sending $\mathcal{F}_0 \subset \cdots \subset \mathcal{F}_{nl-d}$ to $\mathrm{supp}(\mathcal{F}_i/\mathcal{F}_{i-1})$. Furthermore, as in [38, §4], the support map is an $(nl-d)$-iterated $\mathbb{P}^{n-1}$-bundle and thus

$$M_T(\mathrm{FDiv}_{\mathcal{C}/T,n,d}(l)) \simeq M_T(\mathbb{P}^{n-1}_{\mathcal{C}})^{\otimes nl-d} \in \mathrm{DM}(T,\mathbb{Q}).$$

We claim that the following composition

$$M_T(\mathrm{Sym}^{nl-d}(\mathbb{P}^n_{\mathcal{C}}/T)) \hookrightarrow M_T(\mathbb{P}^{n-1}_{\mathcal{C}})^{\otimes nl-d} \simeq M_T(\mathrm{FDiv}_{\mathcal{C}/T,n,d}(l)) \to M_T(\mathrm{Div}_{\mathcal{C}/T,n,d}(l))$$

is an isomorphism in $\mathrm{DM}(T,\mathbb{Q})$, where the last morphism is induced by the natural forgetful map. Again, by [12, Proposition 3.24], it suffices to check the pullback of this map along each point $t \in T$ is an isomorphism, which is proved in [38, Theorem 1.3]. Upgrading [38, Lemma 4.4] to the relative case, we see that the transition maps $M_T(\mathrm{Div}_{\mathcal{C}/T,n,d}(l)) \to M_T(\mathrm{Div}_{\mathcal{C}/T,n,d}(l+1))$ correspond to the inclusions

$$\bigoplus_{i=0}^{nl-d} \mathrm{Sym}^i(M_{\mathcal{C}/T,n}) \hookrightarrow \bigoplus_{i=0}^{n(l+1)-d} \mathrm{Sym}^i(M_{\mathcal{C}/T,n}),$$

where $M_{\mathcal{C}/T,n} := \overline{M}_T(\mathbb{P}^{n-1}_{\mathcal{C}}/T)$ is defined by the decomposition $M_T(\mathbb{P}^{n-1}_{\mathcal{C}}) \simeq \mathbb{Q}_T\{0\} \oplus M_{\mathcal{C}/T,n}$ given by σ. Hence, similarly to [38, Theorem 4.5], we conclude that

$$\begin{aligned} M_T(\mathfrak{Bun}_{\mathcal{C}/T,n,d}) &\simeq \operatorname*{hocolim}_l \bigoplus_{i=0}^{nl-d} \mathrm{Sym}^i(M_{\mathcal{C}/T,n}) \simeq \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(M_{C,n}) \\ &\simeq M_T(\mathrm{Jac}_{\mathcal{C}/T}) \otimes M_T(B\mathbb{G}_{m,T}) \otimes \bigotimes_{i=1}^{n-1} Z_T(\mathcal{C}/T, \mathbb{Q}_T\{i\}) \end{aligned}$$

where we use the decompositions $M_T(\mathcal{C}) \simeq \mathbb{Q}_T\{0\} \oplus M_T(\mathrm{Jac}_{\mathcal{C}/T}) \oplus \mathbb{Q}_T\{1\}$ (see [59, Corollary 3.20 (iii)]) and $M_T(\mathbb{P}^{n-1}_T) \simeq \oplus_{i=0}^{n-1} \mathbb{Q}_T\{i\}$. ☐

Now fix a line bundle $\mathcal{L} \in \mathrm{Pic}^d_{\mathcal{C}/T}(T)$ and consider the stack $\mathfrak{Bun}_{\mathcal{C}/T,n,\mathcal{L}}$ of vector bundles on $\mathcal{C}/T$ of rank n with determinant $\mathcal{L}$; this is a smooth closed substack of $\mathfrak{Bun}_{\mathcal{C}/T,n,d}$.

Theorem B.4. *Let $\mathcal{C}/T$ be a family of smooth projective geometrically connected genus g curves over T admitting a section $\sigma : T \to \mathcal{C}$ and $\mathcal{L} \in \mathrm{Pic}^d_{\mathcal{C}/T}(T)$. In $\mathrm{DM}(T,\mathbb{Q})$, we have*

$$M_T(\mathfrak{Bun}_{\mathcal{C}/T,n,\mathcal{L}}) \simeq M_T(B\mathbb{G}_{m,T}) \otimes \bigotimes_{i=1}^{n-1} Z_T(\mathcal{C}/T, \mathbb{Q}_T\{i\}).$$

Proof. One defines subschemes $\mathrm{Div}_{\mathcal{C}/T,n,\mathcal{L}}(l) \hookrightarrow \mathrm{Div}_{\mathcal{C}/T,n,d}(l)$ where the subsheaf $\mathcal{E} \subset \mathcal{O}_{\mathcal{C}}(l\sigma)^{\oplus n}$ has determinant $\mathcal{L}$; this is a closed subscheme and smooth over T. One obtains an isomorphism

$$M_T(\mathfrak{Bun}_{\mathcal{C}/T,n,\mathcal{L}}) \simeq \operatorname*{hocolim}_l M_T(\mathrm{Div}_{\mathcal{C}/T,n,\mathcal{L}}(l))$$

by adapting the proof of Theorem B.3 using Theorem B.1.

We define a closed subscheme $\mathrm{FDiv}_{\mathcal{C}/T,n,\mathcal{L}}(l) \hookrightarrow \mathrm{Div}_{\mathcal{C}/T,n,d}(l)$ with fixed determinant $\mathcal{L}$ and consider the composition

$$\bigoplus_{\underline{m} \in \mathcal{B}_l} M_T((\mathcal{C}/T)^{(\underline{m})_{\mathcal{L}}}\{c_{\underline{m}}\} \hookrightarrow M_T((\mathbb{P}^{n-1})^{nl-d} \times (\mathcal{C}/T)^{nl-d}_{\mathcal{L}}) \to M_T(\mathrm{FDiv}_{\mathcal{C}/T,n,\mathcal{L}}(l))$$

with the forgetful map $M_T(\mathrm{FDiv}_{\mathcal{C}/T,n,\mathcal{L}}(l)) \to M_T(\mathrm{Div}_{\mathcal{C}/T,n,\mathcal{L}}(l))$; here $(\mathcal{C}/T)^{nl-d}_{\mathcal{L}}$ denotes the image of $\mathrm{FDiv}_{\mathcal{C}/T,n,\mathcal{L}}(l)$ under $\mathrm{supp} : \mathrm{Div}_{\mathcal{C}/T,n,\mathcal{L}}(l) \to (\mathcal{C}/T)^{nl-d}$ and for a partition $\underline{m} \in \mathcal{B}_l$, we let $(\mathcal{C}/T)^{(\underline{m})}_{\mathcal{L}}$ denote the image of $(\mathcal{C}/T)^{nl-d}_{\mathcal{L}}$ under the quotient map $(\mathcal{C}/T)^{nl-d} \to (\mathcal{C}/T)^{(\underline{m})} := \prod_{i=0}^{n-1} \mathrm{Sym}^{m_i}(\mathcal{C}/T)$. We claim that this composition is an isomorphism. Again, by [12, Proposition 3.24], it suffices to check the pullback of this map along each point $t \in T$ is an isomorphism, so we can reduce to $T = \mathrm{Spec}(k)$. But we proved that this morphism is an isomorphism in that case in the proof Theorem B.1. Furthermore, the transition maps are as in Theorem B.1 and then the rest of the proof follows in exactly the same way. □

Appendix C. Dimension formulae

We record some dimension formulae, with references either to the literature or this paper. Note that from the second line on in this table (i.e. for all the moduli spaces related to SL_n rather than GL_n-Higgs bundles), the second column is equal to the first column specialised to $\Delta := \deg(D) = 2g-2$; we chose to keep this presentation for ease of reference.

	$\Delta := \deg(D) > 2g-2$	$D = K_C$	Reference
$\dim \mathcal{M}^D$	$n^2\Delta + 1$	$n^2(2g-2)+2$	[57, Proposition 7.1]
$\dim \mathcal{A}^D$	$\frac{n(n+1)\Delta}{2} - n(g-1)$	$n^2(g-1)+1$	[22, Eq.(77) in §6.1]
$\dim \mathcal{M}_L^D$	$(n^2-1)\Delta$	$(n^2-1)(2g-2)$	[22, Eq.(78) in §6.1]
$\dim \mathcal{A}_L^D$	$\frac{n(n+1)\Delta}{2} - (n-1)(g-1) - \Delta$	$(n^2-1)(g-1)$	[22, Eq.(78) in §6.1]
$\dim \mathcal{M}_\pi^D = \dim \mathcal{M}_\gamma^D$	$(nn_\gamma - 1)\Delta$	$(nn_\gamma-1)(2g-2)$	Lemma 5.5 i)
$\dim \mathcal{A}_\pi^D = \dim \mathcal{A}_\gamma^D$	$\frac{n(n_\gamma+1)\Delta}{2} - (n-1)(g-1) - \Delta$	$(nn_\gamma-1)(g-1)$	Lemma 5.5 ii)
$\mathrm{codim}_{\mathcal{M}_L^D} \mathcal{M}_\gamma^D$	$n(n-n_\gamma)\Delta$	$n(n-n_\gamma)(2g-2)$	Lemma 5.5 iii)
$d_\gamma := \mathrm{codim}_{\mathcal{A}^D} \mathcal{A}_\gamma^D$	$\frac{n(n-n_\gamma)\Delta}{2}$	$n(n-n_\gamma)(g-1)$	Lemma 5.5 iv)

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