Moduli of Quiver Representations via GIT

Let $Q = (Q_0, Q_1, s, t)$ be a quiver and consider the moduli problem of classifying representations of Q (over k) up to isomorphism.

Fix any discrete invariants: dimension vector $\underline{d} \in \mathbb{N}^{|Q_0|}$.

Def^n: A family of representations of a quiver Q over a scheme S/k is a collection of locally free sheaves $(\mathcal{E}_v)_{v \in Q_0}$ over S and sheaf homomorphisms $(\Phi_a : \mathcal{E}_{t(a)} \to \mathcal{E}_{h(a)})_{a \in Q_1}$

ie. a representation of Q in the category of locally free sheaves over S.

The family is of dim. $\underline{d}$ if each sheaf $\mathcal{E}_v$ has rank d_v.

Example: Over the affine space $R = Rep_{\underline{d}}(Q) = \bigoplus_{a \in Q_1} Hom(k^{d_{t(a)}}, k^{d_{s(a)}})$ there is a tautological family of $\underline{d}$-dimensional Q-reps

Let $\mathcal{E}_v = \mathcal{O}_R^{\oplus d_v}$ be the trivial sheaf of rank d_v

and $\Phi_a : \mathcal{E}_{t(a)} \to \mathcal{E}_{s(a)}$ over $r = (r_a)_{a \in Q_1} \in R$

be the homomorphism r_a.

Proposition

Every $\underline{d}$-dim^e representation (W_v, ϕ_a) of Q is isomorphic to a representation parametrised by the tautological family over $R = Rep_{\underline{d}}(Q)$

ie $\exists\, r \in R$ s.t $(W_v, \phi_a) \cong (\mathcal{E}_v, \Phi_a)_r$.

Furthermore this family has the local universal property

ie for any family $(\mathcal{F}_v, \Psi_a)$ over a scheme T and any $t \in T$, $\exists$ $t \in U \underset{\text{open}}{\subseteq} T$ and $f : U \to Rep_{\underline{d}}(Q)$ such that $(\mathcal{F}_v, \Psi_a)|_U \cong f^*(\mathcal{E}_v, \Phi_a)$.

Proof: For each vertex v, fix an isomorphism $\varphi_v : W_v \cong k^{d_v}$
then (W_v, ϕ_a) is isomorphic to the representation over
$$r = (r_a := \varphi_{h(a)} \circ \phi_a \circ \varphi_{t(a)}^{-1})_{a \in Q_1} \text{ in } R = Rep_{\underline{d}}(Q).$$
For the second statement, we can pick an open neighbourhood U of t in T on which each $\mathcal{F}_v$ is trivialisable and use $\varphi_v : \mathcal{F}_v|_U \xrightarrow{\simeq} \mathcal{O}_U^{\oplus d_v}$ and ψ_a to construct f. ∎

There is a conjugation action of $GL_{\underline{d}} := \prod_{v \in Q_0} GL(d_v, k)$ on R
$$GL_{\underline{d}} \times Rep_{\underline{d}}(Q) \longrightarrow Rep_{\underline{d}}(Q)$$
$$(g_v)_{v \in Q_0} \cdot (f_a)_{a \in Q_1} = (g_{h(a)} \circ f_a \circ g_{t(a)}^{-1})_{a \in A}.$$

Lemma: There is a bijective correspondence
$$\{ GL_{\underline{d}}\text{-orbits in } Rep_{\underline{d}}(Q) \} \leftrightarrow \{ \underline{d}\text{-dim}^e \text{ reps of } Q \} / \cong$$
In particular, as $Rep_{\underline{d}}(Q)$ parametrises a family with the local universal property, any coarse moduli space for $\underline{d}$-dim^e reps of Q is a categorical quotient of $G_{\underline{d}}(Q)$ acting on $Rep_{\underline{d}}(Q)$.

Pf: The first statement is immediate.
The second follows from P. Newsteads book "Introduction to moduli problems and orbit spaces" Prop 2.13 ∎

Goal: Use geometric invariant theory (GIT) to construct a categorical quotient of the $GL_{\underline{d}}$-action on (an open subset of) $Rep_{\underline{d}}(Q)$.

Rmk: $GL_{\underline{d}}$ is a reductive group & this action is linear
ie we have $\rho : GL_{\underline{d}} \longrightarrow GL(Rep_{\underline{d}}(Q))$,
but this representation ρ is not faithful: there is a subgroup $\Delta \hookrightarrow GL_{\underline{d}}$ which is contained in every stabiliser subgp
$$\mathbb{G}_m \ni t \longmapsto (t I_{d_v})_{v \in Q_0}$$
ie/ $\Delta \subseteq \ker \rho$.

Affine Geometric Invariant Theory (in a nutshell)

For a reductive group G acting linearly on an affine space V, there is an induced G-action on $\mathcal{O}(V)$ by

$$(g \cdot f)(x) := f(g^{-1} \cdot x).$$

<u>Nagata's Thm</u>: $\mathcal{O}(V)^G$ is a finitely generated k-alg.

<u>Mumford</u>: the inclusion $\mathcal{O}(V)^G \hookrightarrow \mathcal{O}(V)$ induces a morphism of affine varieties $V \to V/\!/G = \operatorname{Spec} \mathcal{O}(V)^G$ which is a "good quotient" (and also a categorical quotient) of $G \curvearrowright V$.

We call $V \xrightarrow{\pi} V/\!/G$ the affine GIT quotient.

Furthermore, $\exists$ open set

$$V^s = \left\{ v \in V : \begin{array}{c} G \cdot v \text{ is closed} \\ \dim G_v = 0 \end{array} \right\} \subseteq V$$

which admits a geometric quotient $V^s \to \pi(V^s) = V^s/G$
(in part. its an orbit space)

<u>Bad example</u> $\mathbb{G}_m \curvearrowright \mathbb{A}^n = V$ by scalar multiplication

Every orbit contains the origin in its closure $\to$ any $\mathbb{G}_m$-inv $f : V \to \mathbb{Z}$ is constant

In fact $\mathcal{O}(V)^{\mathbb{G}_m} = k$ and $V \to V/\!/G = *$.

$\rightsquigarrow$ If we can remove the origin by imposing a non-trivial semistability notion, we get $\mathbb{P}^{n-1}$ as a geom. quotient of $\mathbb{G}_m \curvearrowright \mathbb{A}^n - \{0\}$

The affine GIT quotient parametrises closed orbits in V.

<u>Question</u>: For $GL_{\underline{d}} \curvearrowright \operatorname{Rep}_{\underline{d}}(Q)$, which orbits are closed?

What is $\mathcal{O}(\operatorname{Rep}_{\underline{d}}(Q))^{GL_{\underline{d}}}$?

<u>Ex 1</u>: $Q = \bullet \to \bullet$ $\quad \underline{d} = (n, m)$ $\quad GL_n \times GL_m \curvearrowright \operatorname{Rep}_{\underline{d}}(Q) = \operatorname{Hom}(k^n, k^m) = M_{m \times n}$

The orbits are matrices of a fixed rank

As the rank is upper semi continuous, 0 is the only closed orbit

Hence $\mathcal{O}(\operatorname{Rep}_{\underline{d}}(Q))^{GL_{\underline{d}}} = k$.

<u>Ex 2</u>: $Q = \bullet \circlearrowleft$ $\quad GL_d \curvearrowright \operatorname{Rep}_d Q = \operatorname{Hom}(k^d, k^d) = M_{d \times d}$ by conj

We can conjugate $A \in M_{d\times d}$ to its JNF.

(1) A is diagonalisable i.e. JNF of A is $\begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_n \end{pmatrix}$ for α_i (not nec. distinct) $\in k$.

or

(2) A is not diagonalisable i.e. $\exists$ Jordan block of size > 1

In the 2^{nd} case, the orbit of A is not closed

e.g. $A = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$, then $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \in \overline{GL_2 \cdot A}$ as

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} \alpha & t^2 \\ 0 & \alpha \end{pmatrix} \to \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \text{ as } t \to 0$$

But $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ and $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ are not conjugate.

In the first case, the orbit is closed.

The coefficients of the characteristic poly

$$\mathrm{char}_A(t) = \det(tI - A) = t^d + c_1(A)t^{d-1} + \dots + c_d(A)$$

give GL_d-invariant functions $c_i(A) = (-1)^i \operatorname{tr}(A^i)$ on $M_{d\times d}$:

$$k[c_1, \dots, c_d] \subseteq \mathcal{O}(M_{d\times d})^{GL_d} \subseteq k[x_{11}, \dots, x_{dd}]^{S_d}$$

$\uparrow$ G-inv. function determined by values on closed orbits (i.e. diag matrices) & can permute entries using G-action

$k[x_{11}, \dots, x_{dd}]^{S_d} = \underbrace{k[\sigma_1, \dots, \sigma_d]}_{\text{elementary symmetric polys}}$

Therefore, these are all equalities.

$$\mathcal{O}(\mathrm{Rep}_d(\cdot \circlearrowleft))^{GL_d} = k[c_1, \dots, c_d]$$

where $c_i(A) = (-1)^i \mathrm{Tr}(A^i)$ is (up to sign) the trace of the i-fold loop given by A.

<u>Theorem</u> (Le Bruyn - Procesi)

$\mathcal{O}(\mathrm{Rep}_{\underline{d}}(Q))^{GL_{\underline{d}}}$ is generated by traces of oriented cycles in Q.

In particular, if Q has no oriented cycles, then

$$\mathcal{O}(\mathrm{Rep}_{\underline{d}}(Q))^{GL_{\underline{d}}} = k \text{ and } \mathrm{Rep}_{\underline{d}}(Q) /\!/ GL_{\underline{d}} = *.$$

Affine GIT using a linearisation by a character [King]

Let $\rho: G \to GL(V)$ be a linear action of a reductive gp G on an affine space V.

Often the origin lies in every orbit closure, so $V/\!/G = *$.

<u>Solution</u>: Throw out bad orbits by introducing a non-trivial notion of semistability on V using a "linearisation".

<u>Def^n</u>: For a character $\chi: G \to \mathbb{G}_m$, define a G-action on the total space of the trivial line bundle $\mathcal{O}_V$ on V by

$$\begin{array}{ccc} G \times \mathrm{Tot}(\mathcal{O}_V) & \longrightarrow & \mathrm{Tot}(\mathcal{O}_V) \\ \parallel & & \parallel \\ V \times \mathbb{A}^1 & & V \times \mathbb{A}^1 \end{array}$$

$$g \cdot (v, z) = (g \cdot v, \chi(g) z)$$

We write $\mathcal{O}_V^\chi$ to mean this G-action on $\mathcal{O}_V$.

This is a "<u>linearisation</u>" of the G-action on V.

(As V is affine, the only line bundle is $\mathcal{O}_V$ & to lift the G-action to $\mathcal{O}_V$ corresponds to picking a character $\chi: G \to \mathbb{G}_m$).

<u>Idea of GIT</u> (Mumford) : Use invariant sections of powers of the linearisation to construct a GIT quotient.

<u>Lemma 1</u>

$$H^0(V, \mathcal{O}_V^\chi)^G = \left\{ f \in \mathcal{O}(V) : \ f(g \cdot v) = \chi(g) f(v) \ \forall \begin{array}{l} g \in G \\ v \in V \end{array} \right\}$$

$$\cong \mathcal{O}(V)^{G,\chi} \leftarrow \text{"}\chi\text{-semi invariant functions"}$$

<u>Proof</u>

On the total space $V \times \mathbb{A}^1$ of $\mathcal{O}_V$, G acts by

$$g \cdot (v, z) = (g \cdot v, \chi(g) z).$$

We have $H^0(V, \mathcal{O}_V^\chi) \cong \mathcal{O}(V)$

$$\left(\begin{array}{c} \sigma: V \longrightarrow V \times \mathbb{A}^1 \\ v \longmapsto (v, f(v)) \end{array} \right) \mapsto f$$

$$(g \cdot \sigma)(v) = (v, \chi(g) f(g^{-1} \cdot v)) = \sigma(v) \ \text{ iff } \ \chi(g) f(g^{-1} \cdot v) = f(v).$$

$\boxtimes$

The inclusion $\bigoplus_{r\geq 0} H^0(V, \mathcal{O}_V^{\chi^r})^G \hookrightarrow \bigoplus_{r\geq 0} H^0(V, \mathcal{O}_V)$

induces a rat^{e} map

$$V \overset{\varphi}{\dashrightarrow} V/\!\!/_\chi G := \mathrm{Proj} \bigoplus_{r\geq 0} H^0(V, \mathcal{O}_V^{\chi^r})^G \xrightarrow[\text{projective}]{} V/\!\!/G = \mathrm{Spec}\, H^0(V, \mathcal{O}_V)^G$$

domain of φ is the GIT semistable set w.r.t. χ.

$U|$ open

$V^{\chi\text{-ss}} := \{ v \in V : \exists\ r \geq 1$ and $f \in \mathcal{O}(V)^{G,\chi^r} \cong H^0(V, \mathcal{O}_V^{\chi^r})^G$ such that $f(v) \neq 0 \}$

$\leftarrow$ good quotient

Rmk: In GIT, we have an open subset of stable points (points whose orbits are closed in the semistable set & have finite stabiliser) which admits a geometric quotient.

For Q a quiver and $\rho : G_{\underline{d}}(Q) \to GL(\mathrm{Rep}_{\underline{d}} Q)$, we have

$\Delta \subseteq$ Stabiliser of every point
$\Delta \cong \mathbb{G}_m$

$\rightsquigarrow$ no stable points with usual def^n

Either • work with $G_{\underline{d}}(Q)/\Delta$
• take into account Δ.

Let $\Delta := \ker \rho$.

Def^n $v \in V$ is $\underline{\chi\text{-stable}}$ if $\exists\ f \in \mathcal{O}(V)^{G,\chi^r}$ for $r \geq 1$ s.t $f(v) \neq 0$ & $G \curvearrowright V_f = \{ w \in V : f(w) \neq 0\}$ is closed & $\dim G_v/\Delta = 0$.

Rmk: If $\chi(\Delta) = 1$, then χ descends to a character of G/Δ.

Hilbert-Mumford Criterion

Lemma 2: For $v \in V$, pick a lift $\tilde{v} = (v, \overset{\neq 0}{z}) \in \mathrm{Tot}(\mathcal{O}_V^{\chi^{-1}})$.

i) v is χ-ss $\iff$ $\overline{G\cdot\tilde{v}} \subseteq V \times \mathbb{A}^1$ is disjoint from the zero section $V \times \{0\}$.

ii) v is χ-stable $\iff$ $G\cdot\tilde{v} \subseteq V \times \mathbb{A}^1$ is closed and $[\Delta : G_{\tilde{v}}]$ is finite.

In particular for $V^{\chi\text{-ss}}$ to be non-empty, we need $\chi(\Delta) = 1$.

Pf: G is reductive $\iff$ G is geometrically reductive.
For G geom reductive, any two G-inv closed subsets of an affine space can be separated by a G-invariant function. ⊠

Rmk: Let $\varphi : V^{\chi-ss} \to V/\!/_\chi G$ be the GIT quotient.
Then as this is a good quotient

- For $v_1, v_2 \in V^{\chi-ss}$, $\varphi(v_1) = \varphi(v_2) \iff \overline{G\cdot\tilde{v}_1} \cap \overline{G\cdot\tilde{v}_2} \neq \phi$
- For $p \in V/\!/_\chi G$, $\varphi^{-1}(p)$ contains a unique closed orbit.

We want to translate the above result into a statement about 1-PSs $\lambda : \mathbb{G}_m \to G$ using:

Theorem (Kempf): For $G \curvearrowright W$ = affine space, for $w \in W$ & any closed G-inv $Z \subseteq W$ such that $\overline{G\cdot w} \cap Z \neq \phi$, $\exists$ 1-PS $\lambda : \mathbb{G}_m \to G$ such that $\overline{\lambda(\mathbb{G}_m)\cdot w} \in Z$.

We apply this to $W = \mathrm{Tot}(\mathcal{O}_V^{\chi^{-1}}) = V \times \mathbb{A}^1 \supseteq Z = V \times \{0\}$

Lemma 3 Let $v \in V$ and $\tilde{v} = (v, z^{\neq 0}) \in \mathrm{Tot}(\mathcal{O}_V^{\chi^{-1}})$.

(i) $v \in V^{\chi-ss} \iff \lim_{t\to 0} \lambda(t)\cdot\tilde{v} \notin V\times\{0\}$ $\forall$ 1-PS $\lambda : \mathbb{G}_m \to G$

(ii) $v \in V^{\chi-s} \iff$ the only 1-PS $\lambda : \mathbb{G}_m \to G$ for which $\lim_{t\to 0} \lambda(t)\cdot\tilde{v}$ exists lie in Δ

Pf: (i) "$\Rightarrow$" follows by Lemma 2 & "$\Leftarrow$" by Kempf's Thm & Lemma 2.
(ii) also follows from Kempf's Thm & Lemma 2. $\square$

Finally, we have a natural pairing

$$X^*(G) \times X_*(G) \to \mathbb{Z}$$

$\langle \chi, \lambda \rangle :=$ the unique $r \in \mathbb{Z}$ such that $\chi \circ \lambda : \mathbb{G}_m \to \mathbb{G}_m$ is given by $t \mapsto t^r$.

For $\tilde{v} = (v, z^{\neq 0}) \in \mathrm{Tot}\,\mathcal{O}_V^{\chi^{-1}}$, we have

$$\lambda(t)\cdot\tilde{v} = (\lambda(t)v, \chi^{-1}(\lambda(t))z).$$

Hence $\lim_{t\to 0} \lambda(t)\cdot\tilde{v}$ exists $\iff \lim_{t\to 0} \lambda(t)\cdot v$ exists & $\lim_{t\to 0} \chi^{-1}(\lambda(t))\cdot z$ exists
(ie. $\langle \chi^{-1}, \lambda \rangle \geq 0$.)

Furthermore, if $\lim_{t\to 0} \lambda(t)\cdot v$ exists, then

$$\lim_{t\to 0} \lambda(t)\cdot\tilde{v} \in V\times\{0\} \iff 0 < \langle \chi^{-1}, \lambda \rangle = -\langle \chi, \lambda \rangle.$$

Theorem (Hilbert-Mumford criterion; King)

(i) $v \in V$ is χ-ss $\Leftrightarrow$ $\chi(\Delta) = 1$ and $\forall$ 1-PS $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t \to 0} \lambda(t) \cdot v$ exists, we have $\langle \chi, \lambda \rangle \geq 0$.

(ii) v is χ-stable $\Leftrightarrow$ the only 1-PSs $\lambda : \mathbb{G}_m \to G$ for which $\lim_{t \to 0} \lambda(t) \cdot v$ exist satisfy $\langle \chi, \lambda \rangle = 0$ & lie in Δ.

Proof: This follows from the above calculations and Lemma 3.

Proposition 4

(i) $G \cdot v \subseteq V^{\chi\text{-ss}}$ is closed $\Leftrightarrow$ $\forall$ 1-PS $\lambda : \mathbb{G}_m \to G$ s.t $\langle \chi, \lambda \rangle = 0$ if $\lim_{t \to 0} \lambda(t) \cdot v$ exists it belongs to $G \cdot v$

(ii) $v_1, v_2 \in V^{\chi\text{-ss}}$, $\varphi(v_1) = \varphi(v_2)$ [Notation: $v_1 \sim v_2$ "S-equivalence"]

$\Updownarrow$

$\exists$ 1-PS $\lambda_i : \mathbb{G}_m \to G$ for $i = 1, 2$ s.t $\langle \chi, \lambda_i \rangle = 0$ and $\lim_{t \to 0} \lambda_i(t) \cdot v_i$ belong to the same closed orbit.

Pf: (i) follows from Lemma 3 & Kempf's Thm

(ii) Uses the fact that the GIT quotient is a good quotient:
$$\varphi(v_1) = \varphi(v_2) \Leftrightarrow \overline{G \cdot v_1} \cap \overline{G \cdot v_2} \cap V^{\chi\text{-ss}} \neq \emptyset$$

Then we apply Kempf's Theorem. ☒