SEMINAR: DERIVED CATEGORIES AND VARIATION OF GEOMETRIC INVARIANT THEORY QUOTIENTS

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Abstract

1. Overview

Bondal and Orlov's study of the behaviour of the bounded derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X under certain birational transformations (known as flips and flops) [4], lead to the hope that one may be able to detect minimal models of X as special subcategories of $D^b(X)$. Their work provided a surprising parallel between the minimal model program in birational geometry and semi-orthogonal decompositions of derived categories, and cemented the current point of view that problems in birational geometry should be studied using derived categories.

Many of the examples of such birational transformations come from variation of geometric invariant theory quotients following work of Dolgachev and Hu [5] and Thaddeus [14]. More precisely, given a reductive group G acting on a smooth projective variety X, there is a wall and chamber decomposition on the cone of ample G-effective line bundles over X (up to homological equivalence) such that for two G-linearisations $\mathcal{L}_+$ and $\mathcal{L}_-$ in chambers on opposite sides of a wall $\mathcal{L}_0$ with good properties (see [3, Definition 4.1.4]), the GIT quotients $X/\!/_{\mathcal{L}_+}G$ and $X/\!/_{\mathcal{L}_-}G$ are related by an explicit birational transformation known as a VGIT flip. One can thus expect the derived categories of these GIT quotients to be related to each other by semi-orthogonal decompositions. By work of Ballard-Favero-Katzarkov [3] and Halpern-Leistner [9] this expectation holds provided one replaces the GIT quotients $X/\!/_{\mathcal{L}_\pm}G$ by the stack quotients $[X^{ss}(\mathcal{L}_\pm)/G]$ (if G acts freely on the GIT semistable loci $X^{ss}(\mathcal{L}_\pm)$, then these two quotients coincide). An application of this result is discussed in [2], where the aim is to prove homological mirror symmetry for toric stacks and their Landau-Ginzburg mirrors by using the GIT fan associated to a toric stack to produce a run of the Mori program in order to produce degenerations to the boundary of the moduli space of such Landau-Ginzburg models.

The goal of this seminar is to understand the work of [3, 9] describing the relationship between $D^b([X^{ss}(\mathcal{L})/G])$ as $\mathcal{L}$ varies in terms of semi-orthogonal decompositions. We start by first reviewing the necessary prerequisites (quotient stacks, equivariant sheaves, derived categories and semi-orthogonal decompositions, variation of geometric invariant theory quotients), before moving on the proof of this result. We will primarily follow exposition given in [3].

2. Plan of the talks

Talk 1: Overview. The goal of this talk is to summarise results relating birational geometry and derived categories, as well as giving the classical results of VGIT, in order to motivate the main result in [3, 9]. We will take an example-based approach to presenting this result.

Talk 2: Quotient stacks and equivariant sheaves. The goal of this talk is to define a quotient stack and describe the category of coherent sheaves on a quotient stack as a category of equivariant coherent sheaves on an atlas for this stack.

Start with the definition of a principal G-bundle for a linear algebraic group G over a field k, then, for a k-scheme X with a G-action, define the quotient stack $[X/G]$ as a contravariant functor from the category of k-schemes to the category of groupoids (see [6, Definition 2.14] or [15, §7.12]). If there is time, a rough introduction to stacks can be given, starting from consider the functor of points of a scheme. Explain the notion of (representable) morphism of (quotient) stacks and that there is an atlas $X \to [X/G]$ and that $[X/G]$ is an Artin stack (for example, see [6, Example 2.25]).

In the second part of the talk, give the definition of a coherent sheaves on a stack $\mathfrak{X}$ (here the speaker may assume $\mathfrak{X} = [X/G]$ if they want) following [6, Definition 2.45] or [15, §7.18]. For a G-scheme X, define a G-equivariant coherent sheaf on X and homomorphisms between such sheaves following [3, Definition 2.1.1] (see also [12, Section 1.3]); carefully explain the notion of a G-equivariant vector bundle over $X = \operatorname{Spec} k$. Define the tensor product of two equivariant sheaves and the twist of a G-equivariant sheaf by a G-representation [3, Definition 2.1.3]. Finally, prove that the category $\mathrm{Coh}([X/G])$ of coherent sheaves on the quotient stack $[X/G]$ is equivalent to the category $\mathrm{Coh}^G(X)$ of G-equivariant coherent sheaves on X following for example [15, §7.21].

Talk 3: Derived categories and semi-orthogonal decompositions. The goal of this talk is to give an introduction to derived categories of coherent sheaves and semi-orthogonal decompositions, and to describe the derived category of a quotient stack as in [3, Section 2.2].

Start by recalling the construction of the bounded derived category of an abelian category $\mathcal{A}$ via Verdier localisation; explain that this is a triangulated category (it is not necessary to prove the axioms); for example, see [8, §2.1]. Explain how to define derived functors and for a (proper) morphism $f : X \to Y$ of schemes define the derived pullback (and push forward) between the derived categories of these schemes following [8, §2.1 and §3.3].

Then, for a triangulated category $\mathcal{T}$, define the notion of a (full) exceptional collection; for $\mathcal{T} = D(\mathbb{P}^n)$, give an example of a full exceptional collection. Then define a semi-orthogonal decomposition [3, Definition 2.2.2] and state [3, Proposition 2.2.4]. For example, follow the treatment in [8, §1.4] or in [3, Section 2.2].

Finally, for a smooth variety X equipped with an action of a reductive group G, describe the objects in the bounded derived category of coherent sheaves on the quotient stack $[X/G]$ (that is, prove [3, Proposition 2.2.10]).

Talk 4: Factorisation categories. The goal of this talk is to introduce the definitions of gauged Landau-Ginzberg (LG) models and factorisation categories following [3, Section 2.3] (for further details, see [1] and the references therein). Here the notion of factorisations generalises the notion of matrix factorisations after Eisenbud.

A gauged LG-model is a quadruple $(X, G, \mathcal{L}, w)$ consisting of a linear algebraic group G, a smooth quasi-projective G-variety X, a G-equivariant line bundle $\mathcal{L} \to X$ and a G-invariant section $w \in H^0(X, L)^G$. An important example is when $\mathcal{L} = \mathcal{O}_X(\chi)$ is a linearisation on the structure sheaf given by a character $\chi : G \to \mathbb{G}_m$, in which case $w : X \to \mathbb{A}^1$ is a χ-semi-invariant function. If there is time, explain the point of view given in [1, Section 2] that tensoring by $\mathcal{L}$ gives an auto-equivalence $\Phi_{\mathcal{L}}$ of $\mathrm{Coh}^G(X)$ and that the section w defines a natural transformation from the identity functor to $\Phi_{\mathcal{L}}$. Then define the abelian category of (coherent) factorisations of $(X, G, \mathcal{L}, w)$ and the derived category of coherent factorisations of the LG-model via Verdier localisation.

Briefly state the results concerning injective and projective resolutions of factorisations [3, 2.3.7 and 2.3.8] in order to define derived push forward and pull back along a morphism of gauged LG-models [3, page 17-18]; then state [3, Lemma 2.3.10]. In preparation for talk 9, explain how to define sheafy local (hyper)cohomology of a factorisation of $(X, G, \mathcal{L}, w)$ along a closed G-invariant subvariety Z and state [3, Proposition 2.3.9]. Finally present [3, Corollary 2.3.12] which allows us to replace a derived category of equivariant coherent sheaves by a derived category of factorisations (explicitly describe the functor involved in this derived equivalence).

Talk 5: Geometric invariant theory and HKKN stratifications. The goal of this talk is to give an overview of GIT and to describe HKKN stratifications in GIT following [3, Section 2.1] (see also [7, 10, 11, 13]).

For a reductive group G acting on a smooth quasi-projective variety X over an algebraically closed field k with respect to an ample G-linearisation $\mathcal{L} \to X$, define GIT (semi)stability and state the main results of GIT (for example, the existence of a categorical quotient of $X^{ss}(\mathcal{L})$)

[12]. Define and give the properties of Mumford's numerical function μ following [3, Section 2.1] and for X proper or affine, state the Hilbert–Mumford criterion (note that Theorem 2.1.21 and Proposition 2.1.22 in [3] can be simultaneously formulated). If there is time, for $G = \mathbb{G}_m$, explain the Hilbert–Mumford criterion in terms of the weights of the $\mathbb{G}_m$-action and recall the description of the Białynicki-Birula decomposition of X.

In the second part of the talk, explain how to define a G-invariant stratification of X following [3, Definition 2.1.23]; this stratification is called a HKKN stratification after work of Hesselink [7], Kempf [10], Kirwan [11] and Ness [13]. Explain the structure of the strata [3, 2.1.25-2.1.28] and prove [3, Lemma 3.2.6]. Finally define an elementary HKKN stratification [3, Definition 2.1.27] (this definition will be repeatedly used in later talks). If there is time, describe this stratification for SL_2-acting on $X = (\mathbb{P}^1)^n$ following [11, Section 16.1].

Talk 6: Variation of geometric invariant theory. The goal of this talk is to summarise the birational transformations between GIT quotients arising under variation of the linearisation following [3, Section 4.1] and [5, 14].

Start by defining the cone $C^G(X)$ of ample G-effective line bundles on X, for X a smooth quasi-projective variety equipped with an action of a reductive group G; then define GIT equivalence on $C^G(X)$ and give the notion of chambers and walls. State without proof the VGIT result on the fan structure of $C^G(X)_{\mathbb{R}}$ for X proper or affine (see [3, Theorem 4.1.2] and [5, Theorem 0.2.3]). If there is time, give the description of this fan for $G = \mathbb{G}_m$.

In the second part of the talk, describe the morphisms between the GIT quotients in adjacent chambers to a wall; more precisely, state the relationship between the (semi)stable loci on the wall and in each chamber [5, 14]. If additionally the wall satisfies the DHT condition (see [3, Definition 4.1.4]); then explain that this wall crossing is an elementary wall crossing [3, Definition 3.5.1] and describe the birational geometry of this wall crossing following [5, Section 0.2.5] and [14, Theorem 5.9].

If there is time, explicitly describe the VGIT construction relating the weighted projective plane $\mathbb{P}(1,1,r)$ and the Hirzebruch surface $\mathcal{H}_r := \mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(r))$.

Talk 7: The main result for actions of the multiplicative group. In this talk, the aim is to state and sketch the proof of the main result concerning the description of VGIT for actions of the multiplicative group $\mathbb{G}_m$ on a smooth affine variety (or more generally a smooth quasi-projective variety) X in the derived categories of coherent sheaves on the quotient stack $[X^{ss}(\mathcal{L})/\mathbb{G}_m]$ for linearisations $\mathcal{L}$ varying across an elementary wall crossing; the main reference Sections 4.1 and 4.2 in the first arXiv version of [3].

The speaker should primarily focus on providing the details for the case when $X = \mathbb{A}^n$ (as in [3, §4.1 of arXiv v1]); then if there is time at the end, they can give the statement for X smooth and quasi-projective. For a $\mathbb{G}_m$-action on $\mathbb{A}^n$, the linearisations are given by characters of $\mathbb{G}_m$, which we identify with $\mathbb{Z}$; then there are two VGIT chambers (denoted $\pm$) and one wall (denoted 0). First describe the GIT semistable sets for each of these linearisations and draw the VGIT picture between the quotients. Then formulate the main result of this talk ([3, Theorem 4.1.1 in arXiv v1]); there are three cases to consider depending on the sign of the difference between the values of Mumford's numerical function on the normal bundles on each side of the wall. Define grade restriction windows for this $\mathbb{G}_m$-action (see [3, Definition 4.1.4 in arXiv v1]); then the goal is to prove the main result by proving [3, Corollary 4.1.9 in arXiv v1], which gives equivalences between certain grade restriction windows and $D^b([X^{ss}(\pm)/\mathbb{G}_m])$, and [3, Lemma 4.1.10 in arXiv v1], which gives fully faithful functors from $D^b(X^{\mathbb{G}_m})$ to certain grade restriction windows.

If there is time, it would be nice to see an example such as the Atiyah flop or to see how one can recover the well-known exceptional collection on $\mathbb{P}^n$ by viewing $\mathbb{P}^n$ as a GIT quotient for a $\mathbb{G}_m$-action on $\mathbb{A}^{n+1}$ and applying this result (see [3, 4.1.11 and 4.1.12 in arXiv v1]).

Talk 8: Grade restriction windows and the overview of the main result. The goal of this talk is to introduce the notion of grade restriction windows in the general setting of a reductive group action; then to state the main result and give an overview of the structure of the proof; the details of the proof will be given in the following two talks.

To define the grade restriction window associated to a subset $I \subset \mathbb{Z}$ of weights and 1-PS λ of G, we must first discuss formal completions of normal bundles along the zero section. The normal bundles we will be interested in will be normal bundles for the closed HKKN stratum appearing in an elementary HKKN stratification.

Talk 9: The proof of the main theorem part I. The goal is to sketch the proof of the fully faithfulness ([3, Corollary 3.2.2]) and essentially surjectivity ([3, Proposition 3.3.2]) arguments required for the proof of the main theorem.

More precisely, for an elementary HKKN stratification $X = X_\lambda \sqcup S_\lambda$, let $i : X_\lambda \hookrightarrow X$ denote the open immersion and i^* denote the induced pull back functor on derived categories of coherent factorisations; then for subsets $I \subset \mathbb{Z}$, we study the restriction of i^* to the grade restriction window associated to I and λ. For certain subsets I, we see this restriction is fully faithful [3, Corollary 3.2.2] and for certain subsets I we see this restriction essentially surjective [3, Proposition 3.3.2].

As there will not be time to present all of the proofs; we should prioritise ideas and results needed for later talks. For example, it would be helpful if the Thomason equivalences given in [3, Corollary 3.2.4] can be stated and the description of equivariant sheaves for the parabolic group $P(\lambda)$ given by [3, Lemma 3.3.1] can be stated (and, if possible, with a sketch of the proof). One suggestion is for the fully faithfulness to omit the proof of [3, Lemma 3.2.7] and for the essentially surjectivity, to focus on [3, Lemma 3.2.7] and only give the outline of [3, Proposition 3.3.2] explained in the first paragraph of its proof.

2.1. **Talk 10: The proof of the main theorem part II.** The goal of this talk is compare grade restriction windows associated to two nested subsets of $\mathbb{Z}$. More precisely, we describe how these categories can be related by semi-orthogonal decompositions [3, 3.4.7 and 3.4.8].

We continue to assume that we have an elementary HKKN stratification $X = X_\lambda \sqcup S_\lambda$. First describe the derived category of factorisations of the gauged LG-model for the limit set of the stratum S_λ following [3, Lemmas 3.4.2, 3.4.4, 3,4,5 and 3.4.6] (due to time restrictions, the proof of Lemma 3.4.5 can be omitted or sketched). Priority should be given to explaining the idea of the proof of Lemma 3.4.2 and Proposition 3.4.7 in *loc. cit.*

2.2. **Talk 11: The main theorem and applications.** The goal of this talk is to bring together the past three talks in order to prove [3, Theorem 3.5.2] and give the application of this theorem to birational transformations arising in VGIT (see [3, Theorem 4.2.1] and also Theorem 1 in *loc. cit.*); if there is time, the speaker can return to some examples, or explain some potential applications to studying homological mirror symmetry following [2].

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