STRATIFICATIONS FOR MODULI OF SHEAVES AND MODULI OF QUIVER REPRESENTATIONS

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Abstract

We study the relationship between two stratifications on parameter spaces for coherent sheaves and for quiver representations: a stratification by Harder–Narasimhan types and a stratification arising from the geometric invariant theory construction of the associated moduli spaces of semistable objects. For quiver representations, both stratifications coincide, but this is not quite true for sheaves. We explain why the stratifications on various Quot schemes do not coincide and that the correct parameter space to compare such stratifications is the stack of coherent sheaves, where we construct an asymptotic GIT stratification and prove that this coincides with the Harder–Narasimhan stratification. Then we relate these stratifications for sheaves and quiver representations using a generalisation of a construction of Álvarez-Cónsul and King.

1. Introduction

Many moduli spaces can be described as a quotient of a reductive group G acting on a scheme Y using the methods of geometric invariant theory (GIT) developed by Mumford [21]. This depends on a choice of linearisation of the G-action which determines an open subscheme Y^{ss} of Y of semistable points such that the GIT quotient $Y//G$ is a good quotient of Y^{ss}. In this article, we are interested in two moduli problems with such a GIT construction:

(1) moduli of representations of a quiver Q with relations $\mathcal{R}$;
(2) moduli of coherent sheaves on a polarised projective scheme $(X, \mathcal{O}_X(1))$.

In the first case, King [17] uses a stability parameter θ to construct moduli spaces of θ-semistable quiver representations of dimension vector d as a GIT quotient of an affine scheme by a reductive group action linearised by a character ρ_θ. In the second case, Simpson [27] constructs a moduli space of semistable sheaves with Hilbert polynomial P as a GIT quotient of a closed subscheme of a Quot scheme by a reductive group action linearised by an ample line bundle.

For both moduli problems, we compare a Hesselink stratification arising from the GIT construction and a stratification by Harder–Narasimhan (HN) types. An overview of these stratifications is given in §1.2 and §1.3 and we give an outline of our main results in §1.4 and §1.5.

1.1. Moduli and Stratifications.

One of the most famous uses of stratifications in moduli problems is the Yang-Mills stratification on an infinite dimensional affine space $\mathcal{A}$ of unitary connections on a $\mathcal{C}^\infty$-vector bundle over a smooth complex projective curve due to Atiyah and Bott [2]. The space $\mathcal{A}$ arises in a gauge theoretic construction of a moduli space of unitary connections, which is diffeomorphic to a moduli space of semistable (algebraic) vector bundles over C. This moduli space of connections is constructed as a symplectic reduction using a moment map for the action of a gauge group and the Yang-Mills stratification is defined by considering the gradient flow of the norm square of this moment map. Remarkably, the Yang-Mills stratification coincides with a stratification by Harder–Narasimhan types of vector bundles, which arises from the notion of stability for vector bundles. Furthermore, these stratifications are used to give formulae for the Betti numbers of the vector bundle moduli space [2, 5].

Bifet, Ghione and Letizia [4] provide an algebraic approach to calculating these Betti numbers: they use elementary modifications of vector bundles over a curve to characterise vector bundles by matrix divisors, and construct a HN stratification on an ind-variety of matrix divisors.

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However, this approach does not generalise to sheaves on higher dimensional bases, as one cannot use elementary transformations.

A finite dimensional version of the work of Atiyah and Bott was proved by Kirwan [18] and Ness [22]: for a linear action of a reductive group on a projective variety, the finite stratification associated to the norm square of a moment map agrees with a finite instability stratification arising from GIT (the Hesselink stratification introduced below).

1.2. **Harder–Narasimhan stratifications.** In both of the above moduli problems, there is a notion of semistability for objects that involves verifying an inequality for all sub-objects; typically, this arises from the GIT notion of semistability appearing in the construction of these moduli spaces. The idea of a Harder–Narasimhan (HN) filtration is to construct a unique maximally destabilising filtration for each object in a moduli problem [11]. The HN type of an object is a tuple of invariants for the subquotients appearing in its HN filtration.

Every coherent sheaf has a unique HN filtration: for pure sheaves, this result is well-known (see [15] Theorem 1.3.4) and the extension to coherent sheaves is due to Rudakov [25]. The HN type of a sheaf is the tuple of Hilbert polynomials of the subquotients in its HN filtration. For quiver representations, there is no canonical notion of HN filtration with respect to the stability parameter θ. Rather, the notion of HN filtration depends on a collection of positive integers α_v indexed by the vertices v of the quiver (see Definition 3.4); although often only the choice $\alpha_v = 1$, for all vertices v, is considered. The HN type of a quiver representation is the tuple of dimension vectors of the subquotients in its HN filtration.

For both quiver representations and sheaves, we can stratify the associated moduli stacks (and parameter spaces used in the GIT construction of these moduli spaces) by HN types.

1.3. **Hesselink stratifications.** Let G be a reductive group acting on a scheme Y, where

(1) Y is affine, G acts linearly and the action is linearised using a character of G, or
(2) Y is projective with an ample G-linearisation.

Then, by the Hilbert–Mumford criterion [21], it suffices to check GIT semistability on 1-parameter subgroups (1-PSs) of G. In §2, we explain that associated to this linearised action and a choice of norm on the set of conjugacy classes of 1-PSs, there is a Hesselink stratification [12]

$$Y = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

into finitely many disjoint G-invariant locally closed subschemes indexed by conjugacy classes of rational 1-PS of G, whose lowest stratum is Y^{ss}. In order to construct this stratification, the idea is to associate to each unstable point a conjugacy class of adapted 1-PSs of G, in the sense of Kempf [16], that are most responsible for the instability of this point (this class depends on the choice of norm). This stratification also depends on the choice of linearisation.

These stratifications have a combinatorial nature: the weights of the action of a maximal torus of G determine the index set $\mathcal{B}$ and the strata S_β can be constructed from simpler limit sets Z_β, which are semistable sets for smaller reductive group actions (cf. §2).

Hesselink stratifications have a diverse range of applications. For a smooth projective variety Y, the cohomology of $Y/\!/G$ can be studied via these stratifications [18]. In variation of GIT, these stratifications are used to describe the birational transformations between quotients [7, 28] and to provide orthogonal decompositions in the derived categories of GIT quotients [3, 10]. More recently, Halpern-Leistner abstracted the properties of these instability stratifications in GIT to define instability stratifications on certain classes of 'weakly reductive' stacks [9].

1.4. **Comparison results.** As both the HN stratification and the Hesselink stratification are concerned with (in)stability, it is natural to ask whether these stratifications agree. Furthermore, the work of [2] suggests we might expect these stratifications to agree.

For quiver representations, the choice of parameter α used to define HN filtrations corresponds to a choice of norm $|| - ||_\alpha$ used to define the Hesselink stratification. The corresponding HN and Hesselink stratifications on the space of quiver representations agree: for a quiver without relations, this result is [13] Theorem 5.5 and we deduce the corresponding result for a quiver with relations in Theorem 3.8.

Simpson constructs the moduli space of (Gieseker) semistable sheaves on $(X, \mathcal{O}_X(1))$ with Hilbert polynomial P as a GIT quotient of a closed subscheme R_n of the Quot scheme

$$\mathrm{Quot}_n := \mathrm{Quot}(k^{P(n)} \otimes \mathcal{O}_X(-n), P)$$

by the natural $\mathrm{SL}_{P(n)}$-action for n sufficiently large. The action is linearised by embedding the Quot scheme in a Grassmannian (which depends on choosing $m >> n$). However, the resulting moduli space does not depend on the choice of $m >> n >> 0$.

One can consider the Hesselink stratification of Quot_n and R_n associated to this action, and the stratifications by HN types of the quotient sheaves. These stratifications were first compared set-theoretically by the author in joint work with Kirwan [14, §6], where we showed, for a pure HN type τ (that is, a HN type of a pure sheaf) and for $m >> n >> 0$ (depending on τ), there is a corresponding Hesselink stratum of R_n indexed by $\beta_{n,m}(\tau)$ such that the HN stratum of R_n indexed by τ is contained in this Hesselink stratum. In general, this containment is strict. This result raises the following questions.

i) Why do we only have a containment of a HN stratum in a Hesselink stratum, rather than an agreement of these stratifications?
ii) As this is an asymptotic result for $m >> n >> 0$ depending on τ, is there an asymptotic version of the Hesselink stratifications which agrees with the HN stratification?

The primary aim of this paper is to answer these questions. In this paper, we work with coherent sheaves rather than pure sheaves and study the Hesselink stratifications on Quot_n. First, we enhance and generalise the result of [14] stated above. The proof in loc. cit. is set theoretic, whereas we now give a scheme theoretic proof, and we extend the argument to include non-pure HN types. In Theorem 4.19, we show for any HN type τ and $m >> n >> 0$, that the τ-HN stratum of Quot_n is a closed subscheme of a Hesselink stratum indexed by $\beta_{n,m}(\tau)$.

The first question is answered by Lemma 4.22: distinct HN types τ and τ' can give the same Hesselink index β for finitely many values of n and m (although, this only happens when X has dimension greater than one). Since there are infinitely many HN types, it is not possible to pick finite n and m such that the assignment $\tau \mapsto \beta_{n,m}(\tau)$ is injective. To resolve this issue, we consider a refined version of the Hesselink stratification, where the refined strata $S_{\beta,\nu}$ are further indexed by a tuples of polynomials ν and are (unions of) connected components of the unrefined strata S_β. However, this refined Hesselink stratification still does not agree with the HN stratification (as not all quotient sheaves parametrised by Quot_n are n-regular).

The second question is natural, as the Hesselink stratification is a finite stratification, whereas in general there are infinitely many HN types for sheaves on X with Hilbert polynomial P. This suggests that to compare these stratifications properly, one should construct an infinite asymptotic Hesselink stratification. Since every coherent sheaf is n-regular for n sufficiently large (depending on the sheaf), one would like to compare the Hesselink stratifications on Quot_n for different n. This is not as straight forward as in the case of curves studied in [4], as there are no natural maps between these quot schemes for different values of n. However, if we consider the open subscheme $Q^{n-\mathrm{reg}} \subset \mathrm{Quot}_n$ parametrising n-regular quotient sheaves, then there are natural equivariant morphisms $Q^{n-\mathrm{reg}} \to Q^{n'-\mathrm{reg}}$ for $n' > n$, as every n-regular sheaf is n'-regular. In fact, the quotient stack of the natural action of $\mathrm{GL}_{P(n)}$ on $Q^{n-\mathrm{reg}}$ is isomorphic to the stack $\mathcal{C}oh_P^{n-\mathrm{reg}}(X)$ of n-regular coherent sheaves on X with Hilbert polynomial P and the above equivariant morphism of schemes gives an open immersion

$$\mathcal{C}oh_P^{n-\mathrm{reg}}(X) \hookrightarrow \mathcal{C}oh_P^{n'-\mathrm{reg}}(X).$$

The direct limit of these open immersions is the stack $\mathcal{C}oh_P(X)$ of coherent sheaves on X with Hilbert polynomial P; thus $Q^{n-\mathrm{reg}}$ (and their associated quotient stacks) are only finite dimensional approximations of $\mathcal{C}oh_P(X)$. The natural space on which to construct an asymptotic Hesselink stratification is the stack $\mathcal{C}oh_P(X)$.

Let us outline our construction of an asymptotic Hesselink stratification on $\mathcal{C}oh_P(X)$. We restrict the refined Hesselink stratification of Quot_n to $Q^{n-\mathrm{reg}}$ and take the stack quotient of each stratum by $\mathrm{GL}_{P(n)}$ to give a (finite) Hesselink stratification

$$\mathcal{C}oh_P^{n-\mathrm{reg}}(X) = \bigsqcup_{(\beta,\nu)\in\mathcal{B}_n} \mathcal{S}^n_{\beta,\nu}.$$

For a tuple of Hilbert polynomials ν, we define an associated Hesselink index $\beta_n(\nu)$ for each n and write $\mathcal{S}^n_\nu := \mathcal{S}^n_{\beta_n(\nu),\nu}$, and, for $n' > n$, we construct fibre products

$$\mathcal{S}^{n,n'}_\nu := \mathcal{S}^n_\nu \times_{\mathcal{C}oh_P^{n'-\mathrm{reg}}(X)} \mathcal{S}^{n'}_\nu,$$

which we prove stabilise to an asymptotic stratum denoted $\mathcal{S}_\nu$. Let us state the main result.

Theorem 1.1 (cf. Theorem 4.27)**.** *The strata $\mathcal{S}_\nu$ determine an asymptotic Hesselink stratification of $\mathcal{C}oh_P(X)$ which coincides with the HN stratification as stacks.*

In fact, the asymptotic Hesselink strata $\mathcal{S}_\nu$ and the HN strata $\mathcal{C}oh_P^\tau(X)$ can both be realised as global quotient stacks for the same group and this agreement as stacks comes from a scheme theoretic agreement of the atlases.

The reason why the Hesselink stratification and HN stratifications on the quot schemes do not agree is that we are only seeing a finite dimensional approximation of the full picture. One may hope to be able to use this result to study the cohomology ring of the moduli space of semistable sheaves (analogously to [2, 4]), but there is a major obstruction to such a programme: the subschemes $Q^{n-\mathrm{reg}}$ of the quot schemes used for curves are all smooth, whereas for higher dimensional X, these quot schemes can have arbitrarily bad singularities, in which case the results of Kirwan [18] do not apply. Hence different techniques (such as using derived algebraic geometry) are required. Another application of this work is in the construction of moduli spaces of sheaves of a fixed HN type, as initiated in [14].

The relationship between filtrations arising from adapted 1-PSs and the Harder–Narasimhan filtration is also studied for torsion–free sheaves on a smooth complex projective variety in [8] and for quiver representations in [29]; in both cases, an algebraic approach is taken which involves studying convexity properties of a (normalised) Hilbert–Mumford weight.

In this paper and in [13, 14], we utilise the additional geometric perspective provided by the stratifications and exploit the structure of these stratifications to reduce the proofs to comparisons between simpler limit sets associated to the strata. The use of stratifications enables a direct comparison between the notions of instability coming from GIT and from the Harder–Narasimhan filtration, from which we see that these notions do not always coincide (as is the case for the Quot scheme). A key motivation for this work was to explain why this happens. Furthermore, stratifications can be used to study GIT situations where 1-PSs do not give rise to filtrations by sub-objects (for example, in the study of moduli of curves, there is no natural notion of sub-object).

1.5. **The functorial construction of Álvarez-Cónsul and King.** In §5, we relate these stratifications for sheaves and quivers using a generalisation of [1]. In loc. cit. a functor

$$\Phi_{n,m} := \mathrm{Hom}(\mathcal{O}_X(-n) \oplus \mathcal{O}_X(-m),\ -\ ) : \mathbf{Coh}(X) \to \mathbf{Rep}(K_{n,m})$$

from the category of coherent sheaves on X to the category of representations of a Kronecker quiver $K_{n,m}$ with two vertices n, m and $\dim H^0(\mathcal{O}_X(m-n))$ arrows from n to m is used (when $m >> n >> 0$) to embed the subcategory of semistable sheaves with Hilbert polynomial P into a subcategory of $\theta_{n,m}(P)$-semistable quiver representations of dimension $d_{n,m}(P)$ and to construct the moduli space of semistable sheaves on X with Hilbert polynomial P using [17].

In this set up, distinct HN strata for sheaves may be sent via $\Phi_{n,m}$ to the same HN stratum for quiver representations. We resolve this problem by generalising the above construction to allow additional vertices as follows (cf. Remark 5.16). For a tuple of strictly increasing positive integers $\underline{n} = (n_0, \dots, n_d)$, we consider a functor

$$\Phi_{\underline{n}} := \mathrm{Hom}(\bigoplus_{i=0}^{d} \mathcal{O}_X(-n_i),\ -\ ) : \mathbf{Coh}(X) \to \mathbf{Rep}(K_{\underline{n}}),$$

where $K_{\underline{n}}$ is a quiver with $d+1$ vertices and $\dim H^0(\mathcal{O}_X(n_{i+1}-n_i))$ arrows from vertex i to vertex $i+1$ and relations as described in §5.4. We provide stability parameters $(\theta_{\underline{n}}(P), \alpha_{\underline{n}}(P))$, such that a HN stratum for sheaves is mapped via $\Phi_{\underline{n}}$ to a HN stratum for quiver representations, provided the tuple of natural numbers $\underline{n}$ are sufficiently large (see Theorem 5.15).

Notation and conventions. We work over an algebraically closed field k of characteristic zero and by scheme, we mean scheme of finite type over k. By sheaf, we always mean coherent algebraic sheaf. For schemes X and S, by a family of sheaves on X parametrised by S, we mean a sheaf $\mathcal{F}$ over $X \times S$ that is flat over S and we write $\mathcal{F}_s := \mathcal{F}|_{X\times\{s\}}$.

We use the term stratification in a weaker sense than usual to mean a decomposition into disjoint locally closed subschemes with a partial order on the strata such that the closure of a given stratum is contained in the union of all higher strata (usually for a stratification, one requires the closure of a given stratum to be the union of all higher strata).

For $m, n \in \mathbb{N}$, we write 'for $m >> n$' to mean there exists $N \geq n$ such that for all $m \geq N$. Similarly, 'for $n_r >> \cdots >> n_0$' means $\exists N_0 \geq n_0\, \forall n_1 \geq N_0 \dots \exists N_{r-1} \geq n_{r-1}\, \forall n_r \geq N_{r-1}$.

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2. Background on GIT and Hesselink stratifications

Let G be a reductive group acting on a scheme Y with respect to a linearisation L over Y. Throughout this section, we assume that we are in one of the two following settings:

1. $Y \subset \mathbb{A}^n$ is affine, G acts linearly on Y and the linearisation L is given by using a character $\rho : G \to \mathbb{G}_m$ to twist the trivial linearisation on the structure sheaf of Y,
2. Y is projective and L is ample.

There are two reasons for working in such settings: i) in both these settings, there is a Hilbert–Mumford criterion, as explained below, and ii) these two settings are used to give the GIT constructions of the two moduli spaces considered in this paper.

2.1. GIT and the construction of Hesselink stratifications.

Definition 2.1. A point $y \in Y$ is L-semistable if there exists $\sigma \in H^0(Y, L)^G$ for some $n > 0$ such that $\sigma(y) \neq 0$. We let $Y^{\mathrm{ss}}(L)$ denote the subset of L-semistable points in Y.

By [21] Theorem 1.10, the GIT quotient of G acting on Y with respect to L

$$Y^{\mathrm{ss}}(L) \to Y/\!/_L G := \mathrm{Proj}(\oplus_{n\geq 0} H^0(Y, L^{\otimes n})^G)$$

is a good quotient. In Setting (2), this quotient is projective and in Setting (1), this quotient is projective over the affine GIT quotient $Y/\!/G := \mathrm{Spec}\, k[Y]^G$.

Definition 2.2. Let $y \in Y$ be a closed point and $\lambda : \mathbb{G}_m \to G$ be a 1-parameter subgroup (1-PS) of G; then the Hilbert–Mumford weight $\mu^L(y, \lambda) \in \mathbb{Z}$ is defined to be the weight of the $\mathbb{G}_m$-action induced by λ on the fibre of L over the fixed point $\lim_{t\to 0} \lambda(t) \cdot y$, if this limit exists.

The limit is unique, as in both settings Y is separated and, in setting (2), the limit always exists as Y is proper. In Setting (1), by using the pairing $(-,-) : X^*(G) \times X_*(G) \to \mathbb{Z}$ between characters and cocharacters, King [17] shows that $\mu^L(y, \lambda) = (\rho, \lambda)$.

Theorem 2.3 (The Hilbert–Mumford criterion [17, 21])**.** *A closed point $y \in Y$ is L-semistable if and only we have $\mu^L(y, \lambda) \geq 0$, for every 1-PS $\lambda : \mathbb{G}_m \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot y$ exists.*

The unstable locus $Y^{\mathrm{us}}(L) := Y - Y^{\mathrm{ss}}(L)$ can be stratified by instability types by assigning to each unstable point a conjugacy class of (rational) 1-PSs that is 'most responsible' for the instability of this point [16, 12]. Here, the term 'most responsible' is measured using a norm $||-||$ on the set $\overline{X}_*(G)$ of conjugacy classes of 1-PSs of G. More precisely, we fix a maximal torus T of G and a Weyl invariant norm $||-||_T$ on $X_*(T)_{\mathbb{R}}$; then, for $\lambda \in X_*(G)$, we let $||\lambda|| := ||g\lambda g^{-1}||_T$ for $g \in G$ such that $g\lambda g^{-1} \in X_*(T)$. Equivalently, the norm is given by a Weyl invariant inner product on the Lie algebra of T.

Example 2.4. (a) Let T be the diagonal maximal torus in $G = \mathrm{GL}_n$; then the norm associated to the dot product on $\mathbb{R}^n \cong X_*(T)_{\mathbb{R}}$ is invariant under the Weyl group $W = S_n$.

(b) For a product of general linear groups $G = \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$, we can use positive numbers $\alpha \in \mathbb{N}^r$ to weight the norms $||-||$ for each factor constructed by part (a); that is,

$$||(\lambda_1, \dots, \lambda_r)||^2_\alpha := \sum_{i=1}^r \alpha_i ||\lambda_i||^2.$$

We fix a norm $||-||$ on $\overline{X}_*(G)$ such that $||-||^2$ is $\mathbb{Z}$-valued. For a closed point $y \in Y$, we let

$$M^L(y) := \inf \left\{ \frac{\mu^L(y,\lambda)}{||\lambda||} : \text{ 1-PSs } \lambda \text{ of } G \text{ such that } \lim_{t \to 0} \lambda(t) \cdot y \text{ exists} \right\}.$$

By the Hilbert–Mumford criterion, y is L-semistable if and only if $M^L(y) \geq 0$.

Definition 2.5. (Kempf) A 1-PS λ is adapted to $y \in Y^{\mathrm{us}}(L)(k)$ if $\lim_{t\to 0} \lambda(t) \cdot y$ exists and

$$M^L(y) = \frac{\mu^L(y,\lambda)}{||\lambda||}.$$

Let $\wedge^L(y)$ denote the set of primitive 1-PSs which are adapted to y; this is a full conjugacy class of 1-PSs for a parabolic subgroup of G [16]. Kempf proves that $\wedge^L(g \cdot y) = g \wedge^L(y) g^{-1}$ for $g \in G$; thus, in order to have G-invariant strata, we need to stratify the unstable locus by conjugacy classes of primitive 1-PSs (and an additional index d which keeps track of the value of M^L). We first define the k-points of the strata; the scheme structure is given in Remark 2.9.

Definition 2.6 (Hesselink [12]; Kirwan [18] and Ness [22])**.** For a conjugacy class $[\lambda]$ of 1-PSs of G, a choice of representative $\lambda \in [\lambda]$ and $d \in \mathbb{Q}_{>0}$, we define the following subsets of $Y(k)$.

i) $S_{[\lambda],d} := \{y \in Y(k) : M^L(y) = -d \text{ and } \wedge^L(y) \cap [\lambda] \neq \emptyset\}$ 'the strata,'
ii) $S^\lambda_d = \{y \in Y(k) : M^L(y) = -d \text{ and } \lambda \in \wedge^L(y)\})$ 'the blades,'
iii) $Z^\lambda_d = \{y \in Y^\lambda(k) : M^L(y) = -d \text{ and } \lambda \in \wedge^L(y)\}$ 'the limit sets,'

where Y^λ denotes the λ-fixed locus in Y.

Remark 2.7. The additional index d is redundant in the affine setting (1), as d can be determined from the character ρ and 1-PS λ (more precisely, we have $d = -(\rho, \lambda)/||\lambda||$).

Let $Y^\lambda_+ \subset Y$ denoted the closed subscheme of points $y \in Y$ such that $\lim_{t\to 0} \lambda(t) \cdot y$ exists; then there is a natural retraction $p_\lambda : Y^\lambda_+ \to Y^\lambda$. Let $Y^\lambda_d \subset Y^\lambda$ be the union of connected components on which $\mu^L(-,\lambda) = -d||\lambda||$. Let G_λ be the commutator of λ in G and let $P(\lambda) := \{g \in G : \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists}\}$.

Theorem 2.8 ([12, 18, 22])**.** *Let G be a reductive group acting on a scheme Y with respect to a linearisation L, as in either setting (1) or (2) above, and let $||-||$ be a norm on conjugacy classes of 1-PSs of G. Then there is a stratification*

$$Y - Y^{ss}(L) = \bigsqcup_{([\lambda],d)} S_{[\lambda],d}$$

into finitely many disjoint G-invariant locally closed subschemes $S_{[\lambda],d}$ of Y. Moreover:

(1) $S_{[\lambda],d} = G \cdot S^\lambda_d$;
(2) $S^\lambda_d = p_\lambda^{-1}(Z^\lambda_d)$ *and* S^λ_d *is* $P(\lambda)$*-invariant;*

(3) Z_d^λ *is* G_λ*-invariant and is the GIT-semistable locus for* G_λ *acting on* Y_d^λ *with respect to a modified linearisation;*
(4) $\partial S_{[\lambda],d} \cap S_{[\lambda'],d'} \neq \emptyset$ *only if* $d > d'$.

Remark 2.9. This theorem was proved in [12, 18, 22] for a projective variety Y, where the strata are shown to be locally closed subvarieties of Y. However, as observed in [14] §4, this can be extended to the case where Y is a scheme by using the ample linearisation L on Y to give a projective embedding $Y \hookrightarrow \mathbb{P}^N$; then we obtain the strata in Y by taking the fibre product of the strata in $\mathbb{P}^N$ with Y (and similarly for the blades and limit sets). This gives each stratum the structure of a locally closed subscheme of Y.

This theorem was proved in [13] for an affine space Y; however, the result for an affine subscheme $Y \subset \mathbb{A}^n$ with a linear G-action can be deduced from this, by using the fact that taking invariants for a reductive group G is exact; thus $Y^{\mathrm{ss}}(L) = Y \times_{\mathbb{A}^n} (\mathbb{A}^n)^{\mathrm{ss}}(L)$ and similarly the Hesselink strata in Y are obtained as fibre products of the Hesselink strata in $\mathbb{A}^n$.

By setting $S_0 := Y^{\mathrm{ss}}(L)$, we obtain the Hesselink stratification of Y

$$Y = \bigsqcup_{\beta} S_\beta. \tag{1}$$

where the lowest stratum is S_0 and the higher strata are indexed by pairs $\beta = ([\lambda], d)$ as above or equivalently by a conjugacy class of a rational 1-PS $[\lambda_\beta]$ by the following remark.

Remark 2.10. An unstable Hesselink index $\beta = ([\lambda], d)$ determines a rational 1-PS

$$\lambda_\beta := \frac{d}{||\lambda||}\lambda$$

(where $d/||\lambda||$ is rational, as $||\lambda||^2$ and $d||\lambda||$ are both integral). In fact, we can recover $([\lambda], d)$ from λ_β, as $d = -||\lambda_\beta||$ and λ is the unique primitive 1-PS lying on the ray spanned by λ_β.

Hesselink stratifications have the following additional properties.

(1) For a closed G-invariant subscheme Y' of Y, the Hesselink strata in Y' are the fibre products of the Hesselink strata in Y with Y'.
(2) The Hesselink strata may not be connected; however, if $Z_{d,i}^\lambda$ denote the connected components of Z_d^λ, then $S_{[\lambda],d,i} := Gp_\lambda^{-1}(Z_{d,i}^\lambda)$ are the connected components of $S_{[\lambda],d}$.
(3) If $(Z_d^\lambda)'$ is a closed subscheme of Z_d^λ; then there is a naturally associated closed G-invariant subscheme $S'_{[\lambda],d}$ of $S_{[\lambda],d}$ such that $S'_{[\lambda],d} = G \cdot p_\lambda^{-1}((Z_d^\lambda)')$.

2.2. **Computing Hesselink stratifications.** The task of computing these stratifications is greatly simplified by Theorem 2.8, as we can construct the strata from the limit sets Z_λ^d. Furthermore, we will see that the index set $\mathcal{B}$ for the stratification can be determined from the weights of a maximal torus T of G.

In the projective Setting (2), this description is due to Kirwan [18]: for G acting on $\mathbb{P}^n$ linearly via a representation $\varphi : G \to \mathrm{GL}_{n+1}$, the indices can be determined from the weights of T as follows. For each subset of the T-weights, let β denote the closest point to the origin in the convex hull of these weights. Then, for $\beta \neq 0$, there is a unique primitive 1-PS λ lying on the ray through β and we let $d := ||\beta||$; then $\beta = ([\lambda], d)$ is an unstable index.

In the affine Setting (1), we obtain a similar description as follows. First, we recall that in the affine setting, the index d is redundant and, as the indices for $Y \subset V := \mathbb{A}^n$ are obtained from the indices for V, it suffices to consider the case of an affine space $V = \mathbb{A}^n$ with a linear action $G \to \mathrm{GL}_n$. For each subset W of T-weights, we consider an associated cone

$$C_W := \bigcap_{\chi \in W} H_\chi \subset X_*(T)_{\mathbb{R}}, \quad \text{where } H_\chi := \{\lambda \in X_*(T)_{\mathbb{R}} : (\lambda, \chi) \geq 0\}.$$

Let $\rho_T \in X^*(T)$ denote the restriction of ρ to T; then a subset W of the T-weights is called ρ_T-semistable if $C_W \subseteq H_{\rho_T}$ and otherwise we say W is ρ_T-unstable. If W is ρ_T-unstable, we let λ_W be the unique primitive 1-PS in $C_W \cap X_*(T)$ for which $\frac{(\lambda,\rho)}{||\lambda||}$ is minimal (for the existence and uniqueness of this 1-PS, see [13] Lemma 2.13). We will shortly prove that this gives an unstable index. For $v \in V$, we let W_v denote the set of T-weights of v.

Proposition 2.11. *Let $T = (\mathbb{G}_m)^n$ act linearly on an affine space V with respect to a character $\rho : T \to \mathbb{G}_m$ and let $||-||$ be the norm associated to the dot product on $\mathbb{Z}^n$; then the Hesselink stratification for the torus T is given by*

$$V - V^{\rho-ss} = \bigsqcup_{\{W : W \text{ is } \rho\text{-unstable}\}} S_{\lambda_W}, \quad \text{where} \quad S_{\lambda_W} = \{v \in V : W = W_v\}.$$

Therefore, the stratification is determined by the T-weights and, moreover, $v \in V$ is ρ-semistable if and only if its T-weight set W_v is ρ-semistable (that is, $C_{W_v} \subseteq H_\rho$).

Proof. By construction, C_{W_v} is the cone of (real) 1-PSs λ such that $\lim_{t\to 0} \lambda(t) \cdot v$ exists. By the Hilbert–Mumford criterion, v is ρ-semistable if and only if $(\lambda, \rho) \geq 0$ for all $\lambda \in C_{W_v}$; that is, if and only if $C_{W_v} \subseteq H_\rho$. Therefore, v is ρ-semistable if and only if W_v is ρ-semistable.

As the conjugation action is trivial, $[\lambda_W] = \lambda_W$ and every ρ-unstable point has a unique ρ-adapted primitive 1-PS. If v is ρ-unstable, then, by definition, λ_{W_v} is ρ-adapted to v. □

For a reductive group G, we fix a maximal torus T of G and positive Weyl chamber. Let $\mathcal{B}$ denote the set of 1-PSs λ_W corresponding to elements in this positive Weyl chamber where W is a ρ_T-unstable set of weights.

Corollary 2.12. *Let G be a reductive group acting linearly on an affine variety V with respect to a character $\rho : G \to \mathbb{G}_m$. Then the Hesselink stratification is given by*

$$V - V^{\rho-ss} = \bigsqcup_{\lambda_W \in \mathcal{B}} S_{[\lambda_W]}.$$

Proof. It is clear that the left hand side is contained in the right hand side. Conversely, suppose that $v \in V - V^{\rho-ss}$ and λ is an primitive 1-PS of G that is ρ-adapted to v. Then there exists $g \in G$ such that $\lambda' := g\lambda g^{-1} \in X_*(T)$ and, moreover, λ' is ρ-adapted to $v' := g \cdot v$ for the G-action on V. As every 1-PSs of T is a 1-PSs of G, it follows that $M_G^\rho(v') \leq M_T^{\rho_T}(v')$. Therefore, λ' is ρ_T-adapted to v' for the T-action on V and $\lambda' = \lambda_W \in \mathcal{B}$, where $W = W_{v'}$. In particular, we have that $v \in S_{[\lambda]} = S_{[\lambda_W]}$. □

This gives an algorithm to compute the Hesselink stratification: the index set $\mathcal{B}$ is determined from the T-weights and, for each $[\lambda] \in \mathcal{B}$, we construct $S_{[\lambda]}$ from Z_λ using Theorem 2.8.

Remark 2.13. In general, the Hesselink stratification for G and its maximal torus T are not easily comparable. The ρ-semistable locus for the G-action is contained in the ρ-semistable locus for the T-action, but the G-stratification does not refine the T-stratification. Often there are more T-strata, since 1-PSs of T may be conjugate in G but not in T.

Example 2.14. Let $G := \mathrm{GL}_r$ act on $V := \mathrm{Mat}_{r\times n} \cong \mathbb{A}^{rn}$ by left multiplication; then

$$\mathrm{Mat}_{r\times n}/\!/_{\det}\mathrm{GL}_r = \mathrm{Gr}(r, n)$$

is the Grassmannian of r-planes in $\mathbb{A}^n$ and the GIT semistable locus is given by matrices of rank r. We claim that the Hesselink stratification is the stratification given by rank.

We fix the maximal torus $T \cong \mathbb{G}_m^r$ of diagonal matrices in G and use the dot product on $\mathbb{Z}^r \cong X_*(T)$ to define our norm $||-||$. The weights of the T-action on V are $\chi_1, \dots, \chi_r$ where χ_i denotes the i^{th} standard basis vector in $\mathbb{Z}^r$ and the Weyl group S_r acts transitively on the T-weights. The restriction of the determinant to T is given by $\det_T = (1, \dots, 1) \in \mathbb{Z}^r \cong X^*(T)$. As the Weyl group is the full permutation group on the set of T-weights, it follows that, for $\det_T$-unstable weight subsets W, W', the corresponding 1-PSs λ_W and $\lambda_{W'}$ are conjugate under S_r whenever $|W| = |W'|$. Thus, it suffices to consider the weight sets

$$W_k := \{\chi_1, \dots, \chi_k\} \quad \text{for } k = 0, \dots, r.$$

The subset W_k is ρ_T-unstable if and only if $k < r$ with corresponding ρ_T-adapted 1-PS

$$\lambda_k := (\underbrace{0, \dots, 0}_{k}, \underbrace{-1, \dots, -1}_{r-k}).$$

By Corollary 2.12, the index set for the stratification is $\mathcal{B} = \{\lambda_k : 0 \leq k \leq r-1\}$. We use Theorem 2.8 to calculate the unstable strata. If we consider $\mathrm{Mat}_{k\times n}$ as the subvariety of $\mathrm{Mat}_{r\times n}$ consisting of matrices whose last $r-k$ rows are zero, then $V_+^{\lambda_k} = V^{\lambda_k} = \mathrm{Mat}_{k\times n}$ and thus $S_{\lambda_k} = Z_{\lambda_k}$. By definition, Z_{λ_k} is the GIT semistable set for $G_{\lambda_k} \cong \mathrm{GL}_k \times \mathrm{GL}_{r-k}$ acting on V^{λ_k} with respect to the character

$$\det{}_{\lambda_k} := ||\lambda_k||^2\rho - (\lambda_k,\rho)\lambda_k = (r-k)\det - (k-r)\lambda_k = (\underbrace{r-k,\dots,r-k}_{k}, \underbrace{0,\dots,0}_{r-k}).$$

As semistability is unchanged by a positive rescaling of the character, we can without loss of generality assume that ρ_{λ_k} is the product of the determinant character on GL_k and the trivial character on GL_{r-k}. Then Z_{λ_k} is the GIT semistable set for GL_k acting on $\mathrm{Mat}_{k\times n} \cong V^{\lambda_k}$ with respect to $\det : \mathrm{GL}_k \to \mathbb{G}_m$; that is, Z_{λ_k} is the set of matrices in $\mathrm{Mat}_{k\times n}$ whose final $r-k$ rows are zero and whose top k row vectors are linearly independent. Then, $S_{[\lambda_k]} = G \cdot S_{\lambda_k}$ is the locally closed subvariety of rank k matrices and the Hesselink stratification is given by rank.

3. Stratifications for moduli of quiver representations

Let $Q = (V, A, h, t)$ be a quiver with vertex set V, arrow set A and head and tail maps $h, t : A \to V$ giving the direction of the arrows. A k-representation of a quiver $Q = (V, A, h, t)$ is a tuple $W = (W_v, \phi_a)$ consisting of a k-vector space W_v for each vertex v and a linear map $\phi_a : W_{t(a)} \to W_{h(a)}$ for each arrow a. The dimension vector of W is $\dim W = (\dim W_v) \in \mathbb{N}^V$.

3.1. **Semistable quiver representations.** For quiver representations of a fixed dimension vector $d = (d_v) \in \mathbb{N}^V$, King introduced a notion of semistability depending on a stability parameter $\theta = (\theta_v) \in \mathbb{Z}^V$ such that $\sum_{v\in V} \theta_v d_v = 0$.

Definition 3.1 (King [17]). A representation W of Q of dimension d is θ-semistable if for all subrepresentations $W' \subset W$ we have $\theta(W') := \sum_v \theta_v \dim W'_v \geq 0$.

We recall King's construction of moduli of θ-semistable quiver representations [17]. Let

$$\mathrm{Rep}_d(Q) := \bigoplus_{a\in A} \mathrm{Hom}(k^{d_{t(a)}}, k^{d_{h(a)}}) \quad \text{and} \quad G_d(Q) := \prod_{v\in V} \mathrm{GL}_{d_v}.$$

Then $G_d(Q)$ acts on $\mathrm{Rep}_d(Q)$ by conjugation. Since the diagonal subgroup $\mathbb{G}_m \cong \Delta \subset G_d(Q)$, given by $t \in \mathbb{G}_m \mapsto (tI_{d_v})_{v\in V} \in G_d(Q)$, acts trivially on $\mathrm{Rep}_d(Q)$, we often consider the quotient group $\overline{G}_d(Q) := G_d(Q)/\Delta$. We note that every representation of Q of dimension d is isomorphic to an element of $\mathrm{Rep}_d(Q)$ and, moreover, two representations in $\mathrm{Rep}_d(Q)$ are isomorphic if and only if they lie in the same $\overline{G}_d(Q)$ orbit. Hence, the moduli stack $\mathcal{R}ep_d(Q)$ of representations of Q of dimension d is isomorphic to the quotient stack of $\overline{G}_d(Q)$ acting on $\mathrm{Rep}_d(Q)$. The $G_d(Q)$-action on $\mathrm{Rep}_d(Q)$ is linearised by $\rho_\theta : G_d(Q) \to \mathbb{G}_m$ where $\rho_\theta(g_v) := \Pi_v \det(g_v)^{\theta_v}$.

Theorem 3.2 (King [17]). *The moduli space of θ-semistable quiver representations is given by*

$$M_d^{\theta-ss}(Q) := \mathrm{Rep}_d(Q)//_{\rho_\theta} G_d(Q).$$

In particular, King shows that the notion of θ-semistability for quiver representations agrees with the notion of GIT semistability for $G_d(Q)$ acting on $\mathrm{Rep}_d(Q)$ with respect to ρ_θ.

Remark 3.3. We can also consider moduli spaces of θ-semistable quiver representations for quivers with relations. A non-trivial path in a quiver is a sequence of arrows in the quiver $(a_1, \dots, a_n)$ such that $h(a_i) = t(a_{i+1})$ and a relation is a linear combination of paths all of which start at some common vertex v_t and end at a common vertex v_h. Let $\mathcal{R}$ be a set of relations on Q and let $\mathrm{Rep}_d(Q, \mathcal{R})$ be the closed subscheme of $\mathrm{Rep}_d(Q)$ consisting of quiver representations that satisfy the relations $\mathcal{R}$. The moduli space of ρ-semistable representations of $(Q, \mathcal{R})$ is a closed subscheme of the moduli space of ρ-semistable representations of Q:

$$M_d^{\theta-ss}(Q, \mathcal{R}) := \mathrm{Rep}_d(Q, \mathcal{R})//_{\rho_\theta} G_d(Q) \subset M_d^{\theta-ss}(Q).$$

3.2. **HN filtrations for quivers representations.** We fix a quiver Q, a dimension vector $d \in \mathbb{N}^V$ and a stability parameter $\theta = (\theta_v) \in \mathbb{Z}^V$ such that $\sum_v \theta_v d_v = 0$. To define a Harder–Narasimhan filtration, we need a notion of semistability for quiver representations of all dimension vectors and for this we use a parameter $\alpha = (\alpha_v) \in \mathbb{N}^V$.

Definition 3.4. A representation W of Q is (θ, α)-semistable if for all $0 \neq W' \subset W$, we have

$$\frac{\theta(W')}{\alpha(W')} \geq \frac{\theta(W)}{\alpha(W)}$$

where $\alpha(W) := \sum_{v \in V} \alpha_v \dim W_v$. A Harder–Narasimhan (HN) filtration of W (with respect to θ and α) is a filtration $0 = W^{(0)} \subset W^{(1)} \subset \cdots \subset W^{(s)} = W$ by subrepresentations, such that the quotient representations $W_i := W^{(i)}/W^{(i-1)}$ are (θ, α)-semistable and

$$\frac{\theta(W_1)}{\alpha(W_1)} < \frac{\theta(W_2)}{\alpha(W_2)} < \cdots < \frac{\theta(W_s)}{\alpha(W_s)}.$$

The Harder–Narasimhan type of W (with respect to θ and α) is $\gamma(W) := (\dim W_1, \dots, \dim W_s)$.

If W is a representation of dimension d, then $\theta(W) = 0$ and so W is (θ, α)-semistable if and only if it is θ-semistable. Using the existence and uniqueness of the HN-filtration with respect to (θ, α), Reineke [24] proves there is a HN stratification (with respect to θ and α)

$$\mathrm{Rep}_d(Q) = \bigsqcup_\gamma R_\gamma$$

where R_γ is the locally closed subvariety consisting of representations with HN type γ. As isomorphic quiver representations have the same HN type, each HN stratum R_γ is invariant under the action of $\overline{G}_d(Q)$, and so we obtain a HN stratification of the moduli stack

$$\mathcal{R}ep_d(Q) = \bigsqcup_\gamma \mathcal{R}ep_d^\gamma(Q), \quad \text{where} \quad \mathcal{R}ep_d^\gamma(Q) \cong [R_\gamma / \overline{G}_d(Q)].$$

We emphasise that this stratification depends on both θ and α.

Remark 3.5. For a quiver with relations $(Q, \mathcal{R})$, there is also a HN stratification of the representation space $\mathrm{Rep}_d(Q, \mathcal{R})$. If W is a representation of Q that satisfies the relations $\mathcal{R}$, then any subrepresentation of W also satisfies these relations. Therefore, the HN strata of $\mathrm{Rep}_d(Q, \mathcal{R})$ are the intersection of the HN strata in $\mathrm{Rep}_d(Q)$ with $\mathrm{Rep}_d(Q, \mathcal{R})$.

3.3. **The HN stratification is the Hesselink stratification.** Let T be the maximal torus of $G_d(Q)$ given by the product of the maximal tori $T_v \subset \mathrm{GL}_{d_v}$ of diagonal matrices.

Definition 3.6. For a HN type $\gamma = (d_1, \dots, d_s)$ of a quiver representation of dimension d, let λ_γ be the unique primitive 1-PS on the ray through the rational 1-PS $\lambda'_\gamma = (\lambda'_{\gamma,v})$ given by

$$\lambda'_{\gamma,v}(t) = \mathrm{diag}(t^{r_1}, \dots, t^{r_1}, t^{r_2}, \dots, t^{r_2}, \dots\dots, t^{r_s}, \dots, t^{r_s})$$

where the rational weight $r_i := -\theta(d_i)/\alpha(d_i)$ appears $(d_i)_v$ times.

Remark 3.7. For $\gamma \neq \gamma'$, we note that the conjugacy classes of λ_γ and $\lambda_{\gamma'}$ are distinct.

Let $|| - ||_\alpha$ be the norm on 1-PSs of $G_d(Q)$ associated to $\alpha \in \mathbb{N}^V$ (cf. Example 2.4). For a quiver without relations, the following result is [13] Theorem 5.5. We can deduce the case with relations from this result using Theorem 2.8 and Remark 3.5.

Theorem 3.8. *Let $(Q, \mathcal{R})$ be a quiver with relations with dimension vector $d \in \mathbb{N}^V$ and let $\theta \in \mathbb{Z}^V$ and $\alpha \in \mathbb{N}^V$ be stability parameters such that $\sum_{v \in V} d_v \theta_v = 0$. Then the HN stratification with respect to with respect to (θ, α) and the Hesselink stratification with respect to ρ_θ and $|| - ||_\alpha$ coincide: if*

$$\mathrm{Rep}_d(Q, \mathcal{R}) = \bigsqcup_\gamma R_\gamma \qquad \textit{and} \qquad \mathrm{Rep}_d(Q, \mathcal{R}) = \bigsqcup_{[\lambda]} S_{[\lambda]}$$

denote these stratifications respectively, then $R_\gamma = S_{[\lambda_\gamma]}$.

In fact, in [13], we show this Hesselink stratification also coincides with a Morse type stratification associated to the norm square of a moment map.

3.4. **Stratifications on the stack of quiver representations.** The affine scheme $\mathrm{Rep}_d(Q,\mathcal{R})$ parametrises a tautological family $\mathcal{W}$ of representations of $(Q,\mathcal{R})$ of dimension vector d with the local universal property; thus, the collection of quiver representations of fixed dimension d is bounded. This is in sharp contrast with the situation for coherent sheaves on a fixed projective scheme with fixed numerical invariants: this collection of sheaves is not bounded and so cannot be parametrised by a scheme (cf. §4). Moreover, the moduli stack $\mathcal{R}ep_d(Q,\mathcal{R})$ of representations of $(Q,\mathcal{R})$ of dimension d is isomorphic to the global quotient stack

$$\mathcal{R}ep_d(Q,\mathcal{R}) \cong [\mathrm{Rep}_d(Q,\mathcal{R})/G_d(Q)].$$

As the Hesselink strata are $G_d(Q)$-invariant, we can take their stack quotients by $G_d(Q)$ to obtain a 'stacky Hesselink' stratification of $\mathcal{R}ep_d(Q,\mathcal{R})$ by locally closed substacks. Following Theorem 3.8, this stratification coincides with the stratification of the stack by HN types:

Corollary 3.9. *On $\mathcal{R}ep_d(Q,\mathcal{R})$, the stacky Hesselink strata and the HN strata agree as stacks.*

There are only finitely many HN types for representations of Q of fixed dimension vector and both stratifications are finite. In §4, we prove an analogous statement for coherent sheaves over a projective scheme; however, in this case, the stratifications are infinite and the stacky Hesselink stratification is obtained as a limit of (finite) Hesselink stratifications.

4. Stratifications for moduli of sheaves

Let $(X,\mathcal{O}_X(1))$ be a projective scheme of finite type over k with a very ample line bundle.

4.1. **Preliminaries.** For a sheaf $\mathcal{E}$ over X, we let $\mathcal{E}(n) := \mathcal{E}\otimes\mathcal{O}_X(n)$ and we let $P(\mathcal{E})$ denote the Hilbert polynomial of $\mathcal{E}$ with respect to $\mathcal{O}_X(1)$, whose value at $n\in\mathbb{N}$ is given by

$$P(\mathcal{E},n) = \chi(\mathcal{E}(n)) = \sum_{i=0}^{n}(-1)^i \dim H^i(\mathcal{E}(n)).$$

The degree of $P(\mathcal{E})$ is equal to the dimension of $\mathcal{E}$ (i.e., the dimension of the support of $\mathcal{E}$) and $\mathcal{E}$ is pure if all its non-zero subsheaves have the same dimension as $\mathcal{E}$. For a non-zero sheaf $\mathcal{E}$, the leading coefficient $r(\mathcal{E})$ of $P(\mathcal{E})$ is positive and we define the reduced Hilbert polynomial of $\mathcal{E}$ to be $P^{\mathrm{red}}(\mathcal{E}) := P(\mathcal{E})/r(\mathcal{E})$.

Definition 4.1 (Castelnuovo-Mumford [20])**.** A sheaf $\mathcal{E}$ over X is n-regular if

$$H^i(\mathcal{E}(n-i)) = 0 \quad \text{for all } \ i>0.$$

This is an open condition and, by Serre's Vanishing Theorem, any sheaf is n-regular for $n >> 0$. Furthermore, any bounded family of sheaves is n-regular for $n >> 0$.

Lemma 4.2 (cf. [20])**.** *For a n-regular sheaf $\mathcal{E}$ over X, we have the following results.*

- i) *$\mathcal{E}$ is m-regular for all $m\geq n$.*
- ii) *$\mathcal{E}(n)$ is globally generated with vanishing higher cohomology, i.e., the evaluation map $H^0(\mathcal{E}(n))\otimes\mathcal{O}_X(-n)\to\mathcal{E}$ is surjective and $H^i(\mathcal{E}(n))=0$ for $i>0$.*
- iii) *The natural multiplication maps $H^0(\mathcal{E}(n))\otimes H^0(\mathcal{O}_X(m-n))\to H^0(\mathcal{E}(m))$ are surjective for all $m\geq n$.*
- iv) *If $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ is an exact sequence of sheaves over X such that $\mathcal{F}'$ and $\mathcal{F}''$ are both n-regular, then $\mathcal{F}$ is also n-regular.*

4.2. **Construction of the moduli space of semistable sheaves.** In this section, we outline Simpson's construction [27] of the moduli space of semistable sheaves on $(X,\mathcal{O}_X(1))$ with Hilbert polynomial P. We define an ordering $\leq$ on rational polynomials in one variable by $P\leq Q$ if $P(x)\leq Q(x)$ for all sufficiently large x. For polynomials of a fixed degree with positive leading coefficient, this is equivalent to the lexicographic ordering on the coefficients.

Definition 4.3. A pure sheaf $\mathcal{F}$ over X is semistable (in the sense of Gieseker) if for all non-zero subsheaves $\mathcal{E}\subset\mathcal{F}$, we have $P^{\mathrm{red}}(\mathcal{E})\leq P^{\mathrm{red}}(\mathcal{F})$.

By the Simpson–Le Potier bounds (cf. [27] Theorem 1.1), the set of semistable sheaves over X with Hilbert polynomial P is bounded; therefore, for $n >> 0$, all semistable sheaves with Hilbert polynomial P are n-regular.

Let $V_n := k^{P(n)}$ be the trivial $P(n)$-dimensional vector space and let $\mathrm{Quot}(V_n \otimes \mathcal{O}_X(-n), P)$ denote the Quot scheme parametrising quotients sheaves of $V_n \otimes \mathcal{O}_X(-n)$ with Hilbert polynomial P. Let Q_n denote the open subscheme of this Quot scheme consisting of quotients $q : V_n \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$ such that $H^0(q(n))$ is an isomorphism. Every n-regular sheaf on X with Hilbert polynomial P can be represented as a quotient sheaf in Q_n by choosing an isomorphism $H^0(\mathcal{E}(n)) \cong V_n$ (this is possible, as $\dim H^0(\mathcal{E}(n)) = P(n)$) and using the surjective evaluation map $H^0(\mathcal{E}(n)) \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$. The group $G_n := \mathrm{GL}(V_n)$ acts on V_n and Q_n such that the G_n-orbits in Q_n are in bijection with isomorphism classes of sheaves in Q_n. As the diagonal $\mathbb{G}_m$ acts trivially, we can consider the action of $\mathrm{SL}(V_n)$.

Let $Q_n^{\mathrm{pure}} \subset Q_n$ denote the open subscheme consisting of quotient sheaves that are pure and let R_n denote the closure of Q_n^{pure} in the Quot scheme. The $\mathrm{SL}(V_n)$-action on R_n is linearised using Grothendieck's embedding of the Quot scheme into a Grassmannian by choosing $m >> n$: the corresponding line bundle is given by

$$L_{n,m} := \det(\pi_*(\mathcal{U}_n \otimes \pi_X^* \mathcal{O}_X(m))),$$

where $\mathcal{U}_n$ denotes the universal quotient sheaf on the Quot scheme and π_X and π denote the projection maps from $R_n \times X$ to X and R_n respectively.

Theorem 4.4 (Simpson [27], Theorem 1.21)**.** *Fix $(X, \mathcal{O}_X(1))$ and a Hilbert polynomial P. Then, for $m >> n >> 0$, the moduli space of semistable sheaves on X with Hilbert polynomial P is a GIT quotient:*

$$M^{ss}(X, P) = R_n//_{L_{n,m}} \mathrm{SL}(V_n).$$

In his proof, Simpson shows, for $m >> n >> 0$ (depending on X and P), that an element $q : V_n \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$ in R_n is GIT semistable if and only if q belongs to the open subscheme Q_n^{ss} of Q_n consisting of quotient sheaves $q : V_n \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$ such that $\mathcal{E}$ is a semistable sheaf. In particular, the GIT semistable locus and the GIT quotient do not depend on m and n provided these are taken sufficiently large.

4.3. **The Hesselink stratification of the Quot scheme.** For the action of $\mathrm{SL}(V_n)$ on $\mathrm{Quot}(V_n \otimes \mathcal{O}_X(-n), P)$ with respect to $L_{n,m}$ and the norm $|| - ||$ coming from the dot product for the diagonal torus $T \subset \mathrm{SL}(V_n)$ (cf. Example 2.4), there is a Hesselink stratification

$$\mathrm{Quot}_n := \mathrm{Quot}(V_n \otimes \mathcal{O}_X(-n), P) = \bigsqcup_{\beta \in \mathcal{B}(L_{n,m})} S_\beta.$$

In this section, for fixed n, we describe this stratification of Quot_n, we extend the results of [14] to non-pure sheaves and enhance the proof to a scheme theoretic statement.

By Remark 2.10, the unstable indices β can equivalently be viewed as conjugacy classes of rational 1-PSs $[\lambda_\beta]$ of $\mathrm{SL}(V_n)$. We choose a representative $\lambda_\beta \in X_*(T)$ and, by using the Weyl group action, we may assume that the weights are decreasing; that is,

$$\lambda_\beta(t) = \mathrm{diag}(t^{r_1}, \dots, t^{r_1}, t^{r_2}, \dots, t^{r_2}, \dots, t^{r_s}, \dots, t^{r_s}) \tag{2}$$

where $r_1, \dots, r_s$ are strictly decreasing rational numbers such that r_i occurs with multiplicity l_i. Thus β is equivalent to the decreasing sequence of rational weights $r(\beta) := (r_1, \dots, r_s)$ with multiplicities $l(\beta) := (l_1, \dots, l_s)$. Since λ_β is a rational 1-PSs of $\mathrm{SL}(V_n)$, we note that

$$\sum_{i=1}^{s} l_i = \dim V_n = P(n) \qquad \text{and} \qquad \sum_{i=1}^{s} r_i l_i = 0.$$

We can describe the λ_β-fixed points as follows. The 1-PS λ_β determines a filtration of $V := V_n$

$$0 = V^{(0)} \subset V^{(1)} \subset \cdots \subset V^{(s)} = V, \quad \text{where } V^{(i)} := k^{l_1 + \cdots + l_i}$$

with $V^i := V^{(i)}/V^{(i-1)}$ of dimension l_i. For a quotient $q : V \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$ in Quot, we let $\mathcal{E}^{(i)} := q(V^{(i)} \otimes \mathcal{O}_X(-n))$; then we have exact sequences of quotient sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V^{(i-1)} \otimes \mathcal{O}_X(-n) & \longrightarrow & V^{(i)} \otimes \mathcal{O}_X(-n) & \longrightarrow & V^i \otimes \mathcal{O}_X(-n) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q^{(i-1)}} & & \downarrow{\scriptstyle q^{(i)}} & & \downarrow{\scriptstyle q^i} & & \\ 0 & \longrightarrow & \mathcal{E}^{(i-1)} & \longrightarrow & \mathcal{E}^{(i)} & \longrightarrow & \mathcal{E}^i := \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)} & \longrightarrow & 0. \end{array}$$

We fix an isomorphism $V \cong \oplus_{i=1}^s V^i$; then via this isomorphism we have $\lim_{t\to 0} \lambda_\beta(t)\cdot q = \oplus_{i=1}^s q_i$ by [15] Lemma 4.4.3.

Lemma 4.5. *For $\beta = ([\lambda], d)$ with $r(\beta) = (r_1, \dots, r_s)$ and $l(\beta) = (l_1, \dots, l_s)$ defined following (2), we have*

$$\mathrm{Quot}^\lambda = \bigsqcup_{(P_1,\dots,P_s)\,:\,\sum_{i=1}^s P_i = P} \prod_{i=1}^s \mathrm{Quot}(V^i \otimes \mathcal{O}_X(-n), P_i).$$

Furthermore, let Quot_d^λ be the union of connected components of the fixed point locus on which $\mu^{L_{n,m}}(-, \lambda_\beta) = -||\lambda_\beta||^2$; then

$$\mathrm{Quot}_d^\lambda \cong \bigsqcup_{\substack{(P_1,\dots,P_s)\,:\,\sum_{i=1}^s P_i = P, \\ \sum_{i=1}^s r_i P_i(m) + r_i^2 l_i = 0}} \prod_{i=1}^s \mathrm{Quot}(V^i \otimes \mathcal{O}_X(-n), P_i).$$

Proof. The description of the λ-fixed locus follows from the discussion above. For a λ-fixed quotient sheaf $q : V \otimes \mathcal{O}_X(-n) \twoheadrightarrow \oplus_{i=1}^s \mathcal{E}^i$, we have that $\mu(q, \lambda_\beta) = \sum_{i=1}^s r_i P(\mathcal{E}^i, m)$ by [15] Lemma 4.4.4. The second statement then follows as $||\lambda_\beta||^2 = \sum_{i=1}^s r_i^2 l_i$. □

By Theorem 2.8, for $\beta = ([\lambda], d)$ as above, the associated unstable stratum S_β can be constructed from the limit set Z_d^λ which is the GIT semistable set for the subgroup

$$G_\lambda \cong \left\{ (g_1, \dots, g_s) \in \prod_{i=1}^s \mathrm{GL}(V^i) : \prod_{i=1}^s \det g_i = 1 \right\}$$

acting on Quot_d^λ with respect to the linearisation L_β obtained by twisting $L_{n,m}$ by the (rational) character $\chi_\beta : G_\lambda \to \mathbb{G}_m$ where

$$\chi_\beta(g_1, \dots, g_s) = \prod_{i=1}^s (\det g_i)^{r_i}.$$

Proposition 4.6. *Let $\beta = ([\lambda], d)$ with $r(\beta) = (r_1, \dots, r_s)$ and $l(\beta) = (l_1, \dots, l_s)$ as above; then*

$$Z_d^\lambda \cong \bigsqcup_{(P_1,\dots,P_s)} \prod_{i=1}^s \mathrm{Quot}(V^i \otimes \mathcal{O}_X(-n), P_i)^{\mathrm{SL}(V^i)-ss}(L_{n,m})$$

where this is a union over tuples $(P_1, \dots, P_s)$ such that $\sum_{i=1}^s P_i = P$ and

$$r_i = \frac{P(m)}{P(n)} - \frac{P_i(m)}{l_i} \qquad \textit{for } i = 1, \dots, s. \tag{3}$$

Proof. The centre $Z(G_\lambda) \cong \{(t_1, \dots, t_s) \in (\mathbb{G}_m)^s : \Pi_{i=1}^s t_i^{l_i} = 1\}$ acts trivially on Quot^λ and acts on the fibre of L over $q \in \mathrm{Quot}^\lambda$, lying in a component indexed by $(P_1, \dots, P_s)$, by a character

$$\chi_q : Z(G_\lambda) \to \mathbb{G}_m \quad \text{given by} \quad \chi_q(t_1, \dots, t_s) = \prod_{i=1}^s t_i^{P_i(m)}.$$

Therefore, $Z(G_\lambda)$ acts on the fibre of L_β over q by $\chi_q\chi_\beta(t_1, \dots, t_s) = \prod_{i=1}^s t_i^{P_i(m)+r_i l_i}$. By the Hilbert–Mumford criterion, q is unstable for the action of G_λ with respect to L_β unless $\chi_q\chi_\beta$ is trivial; i.e., there is a constant C such that $Cl_i = P_i(m) + r_i l_i$, for $i = 1, \ldots, s$. In this case

$$CP(n) = \sum_{i=1}^{s} Cl_i = \sum_{i=1}^{s} P_i(m) + r_i l_i = P(m)$$

and we see that the conditions given at (3) are necessary for q to belong to Z_d^λ.

We suppose that $(P_1, \ldots, P_s)$ is a tuple satisfying $\sum_{i=1}^s P_i = P$ and the conditions given at (3). Since G_λ is, modulo a finite group, $\Pi_{i=1}^s \mathrm{SL}(V^i) \times Z(G_\lambda)$ and $Z(G_\lambda)$ acts on both

$$\prod_{i=1}^{s} \mathrm{Quot}(V^i \otimes \mathcal{O}_X(-n), P_i)$$

and L_β trivially, the semistable locus for G_λ is equal to the semistable locus for $\Pi_{i=1}^s \mathrm{SL}(V^i)$. As $\Pi_{i=1}^s \mathrm{SL}(V^i)$-linearisations, we have that $L_\beta = L$. It then follows that

$$\left(\prod_{i=1}^{s} \mathrm{Quot}(V^i \otimes \mathcal{O}_X(-n), P_i)\right)^{\Pi_{i=1}^s \mathrm{SL}(V^i)-ss}(L) = \prod_{i=1}^{s} \mathrm{Quot}(V^i \otimes \mathcal{O}_X(-n), P_i)^{\mathrm{SL}(V^i)-ss}(L)$$

(for example, see [14] Lemma 6.6) which completes the proof of the proposition. □

The following corollary is an immediate consequence of Proposition 4.6 and Theorem 2.8.

Corollary 4.7. *For β as at (2), the scheme S_β parametrises quotients $q : V \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$ with a filtration $0 = W^{(0)} \subset W^{(1)} \subset \cdots \subset W^{(s)} = V$ such that, for $i = 1, \ldots, s$, we have*

1. *$\dim W^i = l_i$, where $W^i := W^{(i)}/W^{(i-1)}$,*
2. *$P(n)P(\mathcal{E}^i, m) = l_i(P(m) - r_i P(n))$, and*
3. *the quotient sheaf $q^i : W^i \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}^i$ is $\mathrm{SL}(W^i)$-semistable with respect to $L_{n,m}$,*

where $\mathcal{E}^{(i)} := q(W^{(i)} \otimes \mathcal{O}_X(-n))$ and $\mathcal{E}^i := \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$. Furthermore, $S_\beta = \mathrm{SL}(V) \cdot Y_d^\lambda$ where Y_λ^d parametrises quotients q with a filtration $W^\bullet$ as above such that $W^\bullet = V^\bullet$.

Remark 4.8. As already noted, the Hesselink strata may not be connected and so it is natural to further decompose the limit sets Z_d^λ using tuples of Hilbert polynomials $\nu = (P_1, \ldots, P_s)$; that is, we can write

$$Z_d^\lambda = \bigsqcup_{\nu \in \mathcal{C}_\beta} Z_{d,\nu}^\lambda \qquad \text{where } Z_{d,\nu}^\lambda \cong \prod_{i=1}^{s} \mathrm{Quot}(V^i \otimes \mathcal{O}_X(-n), P_i)^{\mathrm{SL}(V^i)-ss}(L_{n,m})$$

for $\nu = (P_1, \ldots, P_s)$, is a union of connected components of Z_d^λ. Here the index set $\mathcal{C}_\beta$ is the set of tuples $\nu = (P_1, \ldots, P_s)$ such that $\sum_{i=1}^s P_i = P$, the conditions (3) hold and $Z_{d,\nu}^\lambda$ is non-empty. Then S_β is a disjoint union of the refined strata $S_{\beta,\nu} := \mathrm{SL}(V) \cdot p_\lambda^{-1}(Z_{d,\nu}^\lambda)$ indexed by $\beta \in \mathcal{C}_\beta$. This gives a refinement of the Hesselink stratification

$$\mathrm{Quot}(V \otimes \mathcal{O}_X(-n), P) = \bigsqcup_{\substack{\beta \in \mathcal{B} \\ \nu \in \mathcal{C}_\beta}} S_{\beta,\nu}.$$

We note that we may have strata $S_{\beta,\nu}$ and $S_{\beta',\nu}$ with different Hesselink indices β and β' but the same tuple of Hilbert polynomials ν. However, we will later see in Definition 4.18 that for a tuple ν, there is a natural associated Hesselink index $\beta_{n,m}(\nu)$ depending on $m >> n >> 0$.

4.4. **HN filtrations for coherent sheaves.** In this section, we describe a canonical destabilising filtration for each coherent sheaf, known as its Harder–Narasimhan (HN) filtration [11].

Definition 4.9. A pure HN filtration of a sheaf $\mathcal{F}$ is a filtration by subsheaves

$$0 = \mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \cdots \subset \mathcal{F}^{(s)} = \mathcal{F}$$

such that $\mathcal{F}_i := \mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ are semistable and $P^{\mathrm{red}}(\mathcal{F}_1) > P^{\mathrm{red}}(\mathcal{F}_2) > \cdots > P^{\mathrm{red}}(\mathcal{F}_s)$.

By [15] Theorem 1.3.4, every pure sheaf has a unique pure HN filtration. To define HN filtrations for non-pure sheaves, we need an alternative definition of semistability that does not involve reduced Hilbert polynomials, as every non-pure sheaf $\mathcal{E}$ has a non-zero subsheaf whose Hilbert polynomial has degree strictly less than that of $\mathcal{E}$. For this, we use an extended notion of semistability due to Rudakov [25]. We define a partial ordering $\preccurlyeq$ on $\mathbb{Q}[x]$ by

$$P \preccurlyeq Q \iff \frac{P(n)}{P(m)} \leq \frac{Q(n)}{Q(m)} \qquad \text{for} \quad m >> n >> 0$$

and similarly define a strict partial order $\prec$ by replacing $\leq$ with $<$. This ordering allows us to compare polynomials with positive leading coefficient of different degrees in a way that polynomials of lower degree are larger with respect to this ordering.

Remark 4.10. Rudakov formulated this preordering using the coefficients of the polynomials: for rational polynomials $P(x) = p_d x^d + \cdots + p_0$ and $Q(x) = q_e x^e + \cdots + q_0$, let

$$\Lambda(P, Q) := (\lambda_{f,f-1}, \ldots, \lambda_{f,0}, \lambda_{f-1,f-2}, \ldots, \lambda_{f-1,0}, \ldots, \lambda_{1,0}) \quad \text{where} \quad \lambda_{i,j} := p_i q_j - q_i p_j$$

and $f := \max(d, e)$. We say $\Lambda(P, Q) > 0$ if the first non-zero $\lambda_{i,j}$ appearing in $\Lambda(P, Q)$ is positive. Then $P \preccurlyeq Q$ is equivalent to $\Lambda(P, Q) \geq 0$.

Definition 4.11. A sheaf $\mathcal{F}$ is semistable if $P(\mathcal{E}) \preccurlyeq P(\mathcal{F})$ for all non-zero subsheaves $\mathcal{E} \subset \mathcal{F}$.

This definition of semistability implies purity, as polynomials of smaller degree are larger with respect to $\preccurlyeq$. Moreover, for Hilbert polynomials $P(\mathcal{E})$ and $P(\mathcal{F})$ of the same degree, we have $P(\mathcal{E}) \preccurlyeq P(\mathcal{F})$ if and only if $P^{\text{red}}(\mathcal{E}) \leq P^{\text{red}}(\mathcal{F})$. Thus, a sheaf is semistable in the sense of Definition 4.11 if and only if it is semistable in the sense of Definition 4.3.

Definition 4.12. A HN filtration of a sheaf $\mathcal{E}$ is a filtration by subsheaves

$$0 = \mathcal{E}^{(0)} \subset \mathcal{E}^{(1)} \subset \cdots \subset \mathcal{E}^{(s)} = \mathcal{E}$$

such that $\mathcal{E}_i := \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$ are semistable and $P(\mathcal{E}_1) \succ P(\mathcal{E}_2) \succ \cdots \succ P(\mathcal{E}_s)$.

Using this definition of HN filtration, every coherent sheaf has a unique HN filtration (cf. [25], Corollary 28). We can construct the HN filtration of a sheaf from its torsion filtration and the pure HN filtrations of the subquotients in the torsion filtration as follows.

Proposition 4.13. *Let* $0 \subset T^{(0)}(\mathcal{E}) \subset \cdots \subset T^{(d)}(\mathcal{E}) = \mathcal{E}$ *be the torsion filtration of* $\mathcal{E}$ *and let*

$$0 = \mathcal{F}_i^{(0)} \subset \mathcal{F}_i^{(1)} \subset \cdots \subset \mathcal{F}_i^{(s_i)} = T_i(\mathcal{E}) := T^{(i)}(\mathcal{E})/T^{(i-1)}(\mathcal{E}) \tag{4}$$

be the pure HN filtrations of the subquotients in the torsion filtration. Then $\mathcal{E}$ *has HN filtration*

$$0 = \mathcal{E}_0^{(0)} \subset \mathcal{E}_0^{(1)} \subset \mathcal{E}_1^{(1)} \subset \cdots \subset \mathcal{E}_1^{(s_1)} \subset \cdots \subset \mathcal{E}_d^{(1)} \subset \cdots \subset \mathcal{E}_d^{(s_d)} = \mathcal{E}$$

where $\mathcal{E}_i^{(j)}$ *is the preimage of* $\mathcal{F}_i^{(j)}$ *under the quotient map* $T^{(i)}(\mathcal{E}) \to T_i(\mathcal{E})$.

Proof. As the HN filtration is unique, it suffices to check that the subquotients are semistable with decreasing Hilbert polynomials for $\prec$. First, we note that $\mathcal{E}_i^{(0)} = T^{(i-1)}(\mathcal{E}) = \mathcal{E}_{i-1}^{(s_{i-1})}$ and

$$\mathcal{E}_i^j := \mathcal{E}_i^{(j)}/\mathcal{E}_i^{(j-1)} \cong \mathcal{F}_i^{(j)}/\mathcal{F}_i^{(j-1)} =: \mathcal{F}_i^j.$$

Since (4) is the pure HN filtration of $T_i(\mathcal{E})$, we have inequalities $P^{\text{red}}(\mathcal{F}_i^1) > \cdots > P^{\text{red}}(\mathcal{F}_i^{s_i})$ and the subquotients $\mathcal{F}_i^j$ are semistable. It follows that

$$P(\mathcal{E}_0^1) \succ P(\mathcal{E}_1^1) \succ \cdots \succ P(\mathcal{E}_1^{s_1}) \succ \cdots \succ P(\mathcal{E}_d^1) \succ \cdots \succ P(\mathcal{E}_d^{s_d}),$$

as $\deg P(\mathcal{F}_i^j) = i$ and polynomials of lower degree are larger with respect to this ordering. □

Definition 4.14. Let $\mathcal{E}$ be a sheaf with HN filtration $0 = \mathcal{E}^{(0)} \subset \mathcal{E}^{(1)} \subset \cdots \subset \mathcal{E}^{(s)} = \mathcal{E}$; then the HN type of $\mathcal{E}$ is the tuple $\tau(\mathcal{E}) := (P(\mathcal{E}_1), \ldots, P(\mathcal{E}_s))$ where $\mathcal{E}_i := \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$. We say $\tau = (P_1, \ldots, P_s)$ is a pure HN type if all polynomials P_i have the same degree.

4.5. **The HN stratification of the stack of coherent sheaves.** Let $\mathcal{C}oh_P(X)$ denote the stack of sheaves on X with Hilbert polynomial P; this is an Artin stack such that

$$\mathcal{C}oh_P(X) \cong \bigcup_n [Q_n^o/G_n] \tag{5}$$

where $G_n = \mathrm{GL}(V_n)$ and Q_n^o is the open subscheme of $\mathrm{Quot}(V_n \otimes \mathcal{O}_X(-n), P)$ consisting of quotient sheaves $q : V_n \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$ such that $H^0(q(n))$ is an isomorphism and $H^i(\mathcal{E}(n)) = 0$ for $i > 0$ (cf. [19] Theorem 4.6.2.1).

Let $\mathrm{HNT}(X, P)$ denote the set of HN types of sheaves over X with Hilbert polynomial P. Thus, if $\tau = (P_1, \dots, P_s) \in \mathrm{HNT}(X, P)$, then $\sum_{i=1}^s P_i = P$ and $P_1 \succ P_2 \succ \cdots \succ P_s$ and, moreover, there exists a sheaf with this HN type. For such a HN type τ, we define the (n, m)th Shatz polygon $\Gamma(\tau, n, m)$ to be the union of the line segments joining $x_k := (\sum_{j=1}^k P_j(m), \sum_{j=1}^k P_j(n))$ to x_{k+1} for $k = 0, \dots, s-1$. We define a partial order $\leq$ on $\mathrm{HNT}(X, P)$ by $\tau \leq \tau'$ if $\Gamma(\tau, n, m)$ lies above $\Gamma(\tau', n, m)$ for $m >> n >> 0$.

Theorem 4.15 (Shatz [26] and Nitsure [23] Theorem 5)**.** *Let $\mathcal{F}$ be a family of sheaves on X with Hilbert polynomial P parametrised by a scheme S; then the HN type function $S \to \mathrm{HNT}(X, P)$ given by $s \mapsto \tau(\mathcal{F}_s)$ is upper semi-continuous. Furthermore,*

i) *$S_{>\tau} = \{s \in S : \tau(\mathcal{F}_s) > \tau\}$ is closed in S;*
ii) *$S_\tau = \{s \in S : \tau(\mathcal{F}_s) = \tau\}$ is locally closed in S;*
iii) *S_τ has a unique scheme structure with the following property: a morphism $T \to S$ factors via S_τ if and only if the pullback $\mathcal{F}_T$ of $\mathcal{F}$ to $X \times T$ has a filtration by coherent sheaves whose successive quotients are flat over T and for each $t \in T$ this filtration gives the HN filtration of $(\mathcal{F}_T)_t$, which is of type τ;*
iv) *there is a finite HN (or Shatz) stratification of S into disjoint subschemes S_τ such that*

$$\overline{S_\tau} \subset \bigsqcup_{\tau' \geq \tau} S_{\tau'}.$$

The universal quotient sheaf $\mathcal{U}_n$ over $X \times Q_n^o$ is a family of sheaves on X with Hilbert polynomial P parametrised by Q_n^o; therefore, we have an associated HN stratification

$$Q_n^o = \bigsqcup_\tau Q_{n,\tau}. \tag{6}$$

As the HN strata $Q_{n,\tau}$ are G_n-invariant, this stratification descends to the stack quotient

$$[Q_n^o/G_n] = \bigsqcup_\tau [Q_{n,\tau}/G_n].$$

From the description (5) of $\mathcal{C}oh_P(X)$, we have the following corollary.

Corollary 4.16. *There is a HN stratification on the stack of coherent sheaves*

$$\mathcal{C}oh_P(X) = \bigsqcup_{\tau \in \mathrm{HNT}(X,P)} \mathcal{C}oh_P^\tau(X)$$

into disjoint locally closed substacks $\mathcal{C}oh_P^\tau(X)$ such that $\overline{\mathcal{C}oh_P^\tau(X)} \subset \bigsqcup_{\tau' \geq \tau} \mathcal{C}oh_P^{\tau'}(X)$.

Remark 4.17. If $\tau = (P)$ is the trivial HN type, then $\mathcal{C}oh_{X,P}^\tau = \mathcal{C}oh_{X,P}^{ss}$ and, for $n >> 0$,

$$\mathcal{C}oh_{X,P}^{ss} \cong [Q_n^{ss}/G_n]$$

where $Q_n^{ss} = Q_{n,(P)}$ is an open subscheme of $\mathrm{Quot}(V_n \otimes \mathcal{O}_X(-n), P)$. In fact, an analogous statement holds for all HN types (cf. Proposition 4.21).

4.6. **The Hesselink and HN stratifications.** In this section, we prove, for each HN type τ, that the HN stratum $Q_{n,\tau}$ is contained in a Hesselink stratum of $\mathrm{Quot}(V_n \otimes \mathcal{O}_X(-n), P)$ for $m >> n >> 0$; this generalises an analogous result for pure HN types given in [14].

Definition 4.18. For a tuple of Hilbert polynomials $\nu = (P_1, \dots, P_s)$ which sum to P and natural numbers (n, m), we let $\beta_{n,m}(\nu)$ denote the conjugacy class of the rational 1-PS

$$\lambda_{\beta_{n,m}(\nu)}(t) = \mathrm{diag}\quad (\underbrace{t^{r_1}, \dots, t^{r_1}}_{P_1(n)}, \quad \dots \quad \underbrace{t^{r_s}, \dots, t^{r_s}}_{P_s(n)}) \qquad \text{where} \quad r_i := \tfrac{P(m)}{P(n)} - \tfrac{P_i(m)}{P_i(n)}.$$

If $\nu = \tau$ is a HN type, we have $P_1 \succ P_2 \succ \cdots\cdots \succ P_s$ and thus, for $m >> n >> 0$,

$$\frac{P_1(n)}{P_1(m)} > \frac{P_2(n)}{P_2(m)} > \cdots\cdots > \frac{P_s(n)}{P_s(m)} \tag{7}$$

i.e., the weights r_i in this rational 1-PS are decreasing.

The following result simultaneously enhances and generalises a joint result with Kirwan in [14]. More precisely, Proposition 6.8 of loc. cit. states that for a pure HN type τ (i.e., a HN type of a pure sheaf) and for $m >> n >> 0$, the HN stratum of a closed subscheme $R_n \subset \mathrm{Quot}_n$ is contained in the Hesselink stratum in R_n indexed by $\beta_{n,m}(\tau)$. The proof of loc. cit. was set theoretic, whereas the proof below is scheme theoretic and applies to all HN types.

Theorem 4.19. *For $\tau = (P_1, \dots, P_s) \in \mathrm{HNT}(X, P)$ and $m >> n >> 0$, the HN stratum $Q_{n,\tau}$ is a closed subscheme of the stratum $S_{\beta_{n,m}(\tau)}$ for the Hesselink stratification of Quot_n with respect to $L_{n,m}$. Moreover, $Q_{n,\tau}$ is a closed subscheme of the refined stratum $S_{\beta_{n,m}(\tau),\tau}$.*

Proof. We take $n >> 0$ so all semistable sheaves with Hilbert polynomial P_i are n-regular for $i = 1, \dots, s$. Then every sheaf with HN type τ is n-regular, as it admits a filtration whose successive quotients are n-regular, and so can be parametrised by the HN stratum $Q_{n,\tau}$.

Let $V_n^i := k^{P_i(n)}$ and let $Q_n^i \subset \mathrm{Quot}(V_n^i \otimes \mathcal{O}_X(-n), P_i)$ be the open subscheme consisting of quotients $q : V_n^i \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}^i$ such that $H^0(q(n))$ is an isomorphism. Let $(Q_n^i)^{\mathrm{pure}}$ be the open subscheme of Q_n^i parametrising pure sheaves and let R_n^i be the closure of this subscheme in the Quot scheme. By [27] Theorem 1.19, we can take $m >> n$ so that the GIT semistable set for $\mathrm{SL}(V_n^i)$ acting on R_n^i with respect to the linearisation $L_{n,m}$ is the lowest HN stratum in Q_n^i parametrising semistable sheaves; that is,

$$(R_n^i)^{\mathrm{SL}(V_n^i)-ss}(L_{n,m}) = (Q_n^i)^{ss} := (Q_n^i)_{(P_i)}$$

for $i = 1, \dots, s$. Furthermore, we assume $m >> n >> 0$, so that the inequalities (7) hold.

As described above, the index $\beta = \beta_{n,m}(\tau) = ([\lambda], d)$ determines a filtration $0 = V_n^{(0)} \subset \cdots \subset V_n^{(s)} = V_n$ where $V_n^i := V_n^{(i)}/V_n^{(i-1)} = k^{P_i(n)}$. By construction, the conditions (3) hold for $r(\beta)$ and $l(\beta)$; hence

$$Z_{n,m}(\tau) := \prod_{i=1}^{s} (Q_n^i)^{ss} \subset \prod_{i=1}^{s} \mathrm{Quot}(V_n^i \otimes \mathcal{O}_X(-n), P_i)^{\mathrm{SL}(V_n^i)-ss}(L_{n,m}) \cong Z_{d,\tau}^{\lambda} \subset Z_d^{\lambda}.$$

Both inclusions are closed inclusions: the first, as R_n^i is closed in $\mathrm{Quot}(V_n^i \otimes \mathcal{O}_X(-n), P_i)$ and the second as $Z_{d,\tau}^{\lambda}$ is closed in Z_d^{λ}. Therefore, associated to this closed subscheme $(Z_d^{\lambda})' := Z_{n,m}(\tau)$ of the limit set Z_d^{λ}, there is a closed subscheme $S'_\beta = \mathrm{SL}(V_n) \cdot p_\lambda^{-1}((Z_d^{\lambda})')$ of the Hesselink strata S_β. In fact, S'_β is a closed subscheme of the refined Hesselink stratum $S_{\beta,\tau}$, as $(Z_d^{\lambda})'$ is a closed subscheme of $Z_{d,\tau}^{\lambda}$.

To complete the proof, we need to show that the schemes $Q_{n,\tau}$ and S'_β coincide. We first show these agree set theoretically and then use the universal property of the scheme structure on $Q_{n,\tau}$ given by Theorem 4.15 iii) to prove the scheme structures agree. By construction, a quotient sheaf $q : V_n \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{F}$ belongs to $p_\lambda^{-1}(Z_{n,m}(\tau))$ if, for the filtration

$$0 \subset \mathcal{F}^{(1)} \subset \cdots \subset \mathcal{F}^{(s)} = \mathcal{F} \quad \text{where} \quad \mathcal{F}^{(i)} := q(V_n^{(i)} \otimes \mathcal{O}_X(-n)),$$

the successive quotients $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ are semistable with Hilbert polynomial P_i. Therefore these quotient sheaves have HN type τ. Since the $\mathrm{SL}(V_n)$-action does not change the HN type, we have a set theoretic inclusion $S'_\beta := \mathrm{SL}(V_n) \cdot p_\lambda^{-1}(Z_{n,m}(\tau)) \subset Q_{n,\tau}$. Conversely, for a point $q : V_n \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{F}$ in $Q_{n,\tau}$ with HN filtration given by

$$0 = \mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \cdots \subset \mathcal{F}^{(s)} = \mathcal{F},$$

we can choose $g \in \mathrm{SL}(V_n)$ that sends the filtration $V_n^{(i)}$ to $W_n^{(i)} := H^0(q(n))^{-1}(H^0(\mathcal{F}^{(i)}(n)))$; then $g \cdot q \in p_\lambda^{-1}(Z_{n,m}(\tau))$. Hence, as sets, $Q_{n,\tau}$ and S'_β agree.

By Theorem 4.15 iii), to show that the scheme structures coincide we need to show that a morphism $T \to Q_n^o$ factors through S'_β if and only if the pullback $\mathcal{U}_T$ of the universal quotient sheaf $\mathcal{U} := \mathcal{U}_n$ over $Q_n^o \times X$ to $T \times X$ has a relative HN filtration of type τ (that is, a filtration whose successive quotients are flat families of sheaves over T which are semistable and have Hilbert polynomials given by the P_i). First of all, we note that there is a filtration of the restriction of $\mathcal{U}_n$ to $X \times S'_\beta$ induced by the filtration of $\mathcal{U}$ over $X \times p_\lambda^{-1}(Z_{n,m}(\tau))$ given by

$$\mathcal{U}^{(i)} := q(V_n^{(i)} \otimes \pi_X^* \mathcal{O}_X(-n)),$$

where q denotes the restriction of the universal quotient homomorphism to $X \times p_\lambda^{-1}(Z_{n,m}(\tau))$. Moreover, this is a filtration of HN type τ. In particular, if a morphism $T \to Q_n^o$ factors through S'_β, then the pullback $\mathcal{U}_T$ of the universal family to $X \times T$ admits a filtration $\mathcal{U}_T^{(i)}$ of HN type τ.

Conversely, suppose we have a morphism $f : T \to Q_n^o$ such that $\mathcal{U}_T$ has a relative HN filtration

$$0 = \mathcal{U}_T^{(0)} \subset \mathcal{U}_T^{(1)} \subset \cdots \subset \mathcal{U}_T^{(s)} = \mathcal{U}_T \tag{8}$$

of type τ. Then to complete the proof we need to show that the morphism f factors via the locally closed immersion $S'_\beta \hookrightarrow Q_n^o$. In particular, we note that any such factorisation is unique. The morphism f from T to the quot scheme is equivalent to a family of quotient sheaves parametrised by T; that is a surjective homomorphism of sheaves over $X \times T$

$$q_T : V_n \otimes \pi_X^* \mathcal{O}_X(-n) \to \mathcal{U}_T,$$

where $\pi_X : X \times T \to X$ denote the projection. Since f has image in Q_n^o, we have for each $t \in T$

i) $H^j(X, \mathcal{U}_{T,t}(n)) = 0$ for $j > 0$,
ii) $H^0(q_{T,t}(n)) : V_n \to H^0(\mathcal{U}_{T,t}(n))$ is an isomorphism.

Let $\pi_T : X \times T \to T$ be the projection morphism. Then i) implies $\pi_{T*}(\mathcal{U}_T(n))$ is a locally free sheaf of rank $P(n)$ by [6, III Corollaire 7.9.9], and ii) implies the homomorphism of locally free sheaves over T

$$V_n \otimes \mathcal{O}_T \cong V_n \otimes \pi_{T*}\pi_X^* \mathcal{O}_X \to (\pi_T)_*(\mathcal{U}_T(n)) \tag{9}$$

is an isomorphism. Our choice of n implies any semistable sheaf over X with Hilbert polynomial P_i is n-regular and, as the extension of two n-regular sheaves is also n-regular (see Lemma 4.2), it follows that the successive subsheaves appearing in the HN filtration of sheaf with HN type τ are n-regular. Consequently, for $1 \leq i \leq s$, we have $H^j(\mathcal{U}_{T,t}^{(i)}(n)) = 0$ for $j > 0$ and so $\pi_{T*}(\mathcal{U}_T^{(i)}(n))$ is a locally free sheaf of rank $P^{(i)}(n) := \sum_{j \leq i} P_j(n)$. We twist the relative HN filtration (8) by n and then push-forward under π_T to obtain a filtration of $\pi_{T*}(\mathcal{U}_T(n))$ by locally free sheaves $\pi_{T*}(\mathcal{U}_T^{(i)}(n))$ of rank $P^{(i)}(n)$. Under the isomorphism (9), this induces a filtration of the trivial rank $P(n)$ locally free sheaf $V_n \otimes \mathcal{O}_T$ over T by vector subbundles of ranks $P^{(1)}(n) < P^{(2)}(n) < \cdots < P^{(s)}(n) = P(n)$, which corresponds to a unique morphism $h : T \to \mathrm{SL}(V_n)/P(V_n^\bullet)$ to the partial flag variety $\mathrm{SL}(V_n)/P(V_n^\bullet)$ where $P(V_n^\bullet) \subset \mathrm{SL}(V_n)$ is the parabolic subgroup associated to the filtration $V_n^\bullet$, given by $V_n^{(i)} = k^{P^{(i)}(n)}$. By construction of the Hesselink index $\beta = ([\lambda], d)$ associated to the HN type τ, we have that $P(\lambda) = P(V_n^\bullet)$.

The pullback of the principal $P(V_n^\bullet)$-bundle

$$\mathrm{SL}(V_n) \to \mathrm{SL}(V_n)/P(V_n^\bullet)$$

to T via h is étale locally trivial. In fact, it is Zariski locally trivial as $P(V_n^\bullet)$ is a special group: the Levi factor is a product of general linear groups, which is special, and the unipotent factor is special as it is a chain of $\mathbb{G}_a$-extensions and the additive group $\mathbb{G}_a$ is special. Therefore, we have a Zariski cover $\{U_l \hookrightarrow T\}_{l \in L}$ such that the morphisms from U_l to the flag variety can be lifted as shown in the following diagram

$$\begin{array}{ccc} U_l & \xrightarrow{\tilde{h}_l} & \mathrm{SL}(V_n) \\ \downarrow & \searrow^{h_l} & \downarrow \\ T & \xrightarrow{h} & \mathrm{SL}(V_n)/P(V_n^\bullet). \end{array}$$

The morphism $\tilde{h}_l : U_l \to \mathrm{SL}(V_n)$ determines an isomorphism of sheaves

$$V_n \otimes \mathcal{O}_{U_l} \to V_n \otimes \mathcal{O}_{U_l}$$

and we let g_l denote the induced isomorphism of the sheaf $V_n \otimes \pi_X^* \mathcal{O}_X(n)$ over $X \times U_l$. Let

$$q_l : V_n \otimes \pi_X^* \mathcal{O}_X(n) \to \mathcal{U}_l$$

denote the restriction of the surjection q_T to $X \times U_l$; then the surjection $q_l \circ q_l^{-1}$ determines a morphism $s_l : U_l \to p_\lambda^{-1}(Z_{n,m}(\tau))$, as the image of the filtration $V_n^\bullet$ under $q_l \circ q_l^{-1}$ gives the relative HN filtration of $\mathcal{U}_l$. Therefore, for each l, we have a morphism

$$F_l : U_l \to \mathrm{SL}(V_n) \times p_\lambda^{-1}(Z_{n,m}(\tau)) \to \mathrm{SL}(V_n) \times^{P(\lambda)} p_\lambda^{-1}(Z_{n,m}(\tau)) \to S'_\beta,$$

where the first morphism is given by $(\tilde{h}_l, s_l)$ and the final morphism is given by the group action, which has image in S'_β as this is the smallest G-invariant subscheme of the quot scheme which contains $p_\lambda^{-1}(Z_{n,m}(\tau))$. By checking on double intersections, we see that the morphisms F_l glue to give a morphism $F : T \to S'_\beta$ (we note that the morphisms $(\tilde{h}_l, s_l)$ will not in general glue). This gives the required factorisation of $f : T \to Q_n^o$ and completes our proof. $\square$

The proof of the above theorem also proves the following Corollary.

Corollary 4.20. *Let $\tau \in \mathrm{HNT}(X, P)$; then for $m >> n >> 0$,*

$$Q_{n,\tau} = \mathrm{SL}(V_n) \cdot p_\lambda^{-1}(Z_{n,m}(\tau))$$

where $Z_{\tau,n,m} := \prod_{i=1}^s (Q_n^i)^{ss}$ and $(Q_n^i)^{ss} \subset \mathrm{Quot}(V_n^i \otimes \mathcal{O}_X(-n), P_i)$ are the subschemes defined in the above proof.

Proposition 4.21. *Let τ be a HN type. Then, for $n >> 0$, we have isomorphisms*

$$\mathcal{C}oh_P^\tau(X) \cong [Q_{n,\tau}/G_n]$$

where $Q_{n,\tau}$ is a locally closed subscheme of $\mathrm{Quot}(V_n \otimes \mathcal{O}_X(-n), P)$ *and $G_n = \mathrm{GL}(V_n)$.*

Proof. By Corollary 4.20, there exists $n >> 0$, so all sheaves with HN type τ are n-regular and so can be parametrised by a quotient sheaf in the HN stratum $Q_{n,\tau} \subset Q_n^o$.

The restriction $\mathcal{U}_{n,\tau}$ of the universal family to $Q_{n,\tau} \times X$ has the local universal property for families of sheaves on X of HN type τ by our assumption on n. Therefore, we obtain a map

$$Q_{n,\tau} \to \mathcal{C}oh_P^\tau(X)$$

that is an atlas for $\mathcal{C}oh_P^\tau(X)$. Two morphisms $f_i : S \to Q_{n,\tau}$ define isomorphic families of sheaves of HN type τ if and only if they are related by a morphism $\varphi : S \to \mathrm{GL}(V_n)$; i.e. $f_1(s) = \varphi(s) \cdot f_2(s)$ for all $s \in S$. Hence, the above morphism descends to an isomorphism $[Q_{n,\tau}/G_n] \to \mathcal{C}oh_P^\tau(X)$. $\square$

4.7. **Stratifications on the stack.** As explained in the introduction, one might naturally expect an agreement between the Hesselink and HN stratifications, following the work of Atiyah-Bott [2] and the agreement for quiver representations. However, Theorem 4.19 only gives a containment result. In this section, we explain why these stratifications do not agree, and how to compare the stratifications on Quot_n for different n to produce an asymptotic stratification on the stack of coherent sheaves on X, which coincides with the HN stratification.

The following lemma indicates that we should work with the refined Hesselink strata $S_{\beta,\nu}$, where ν is a tuple of Hilbert polynomials (see Remark 4.8), rather than S_β; the proof of this lemma follows immediately from Definition 4.18.

Lemma 4.22. *Let $\tau = (P_1, \dots, P_s)$ and $\tau' = (P'_1, \dots, P'_t)$ be HN types; then $\beta_{n,m}(\tau) = \beta_{n,m}(\tau')$ if and only if $s = t$ and, for $i = 1, \dots, s$, we have that $P_i(n) = P'_i(n)$ and $P_i(m) = P'_i(m)$.*

Remark 4.23. We note that if $\dim X \leq 1$, then the assignment $\tau \mapsto \beta_{n,m}(\tau)$ is injective for any $m > n$, as the Hilbert polynomial P_i of any sheaf on X has at most degree 1 and so P_i is uniquely determined by the pair $(P_i(n), P_i(m))$. For $\dim X > 1$, we cannot make the same conclusion. Although, for distinct HN types $\tau \neq \tau'$, we note that

$$\beta_{n,m}(\tau) \neq \beta_{n,m}(\tau') \quad \text{for } m >> n >> 0.$$

However, as there are infinitely many HN types, we cannot pick $m >> n >> 0$ so that the assignment $\tau \mapsto \beta_{n,m}(\tau)$ is injective for all HN types.

For each n (and for each $m >> n$), we have a refined Hesselink stratification of Quot_n associated to the linearisation $L_{n,m}$. If we fix n, then there are only finitely many Hesselink stratifications for the $\mathrm{SL}_{P(n)}$-action on Quot_n by standard results on variation of GIT (for example, see [7], Theorem 1.3.9). Therefore, for m_n sufficiently large, the Hesselink stratifications on Quot_n associated to $L_{n,m}$ agree. We assume we have taken m_n sufficiently large so this is the case and will refer to the refined Hesselink stratification of Quot_n with respect to L_{n,m_n} as the refined Hesselink on Quot_n which we write as follows

$$\mathrm{Quot}_n = \bigsqcup_{\beta \in \mathcal{B}_n, \nu \in \mathcal{C}_\beta} S^n_{\beta,\nu} \tag{10}$$

where $\mathcal{B}_n$ is a finite set of conjugacy classes of rational 1-PSs of $\mathrm{SL}_{P(n)}$ and $\mathcal{C}_\beta$ is a finite set of tuples of Hilbert polynomials.

We now would like to allow n to vary and study the limit of these stratifications as n tends to infinity so that we can study all sheaves with Hilbert polynomials P on X. However, there is one immediate problem: there are no natural morphisms $\mathrm{Quot}_n \to \mathrm{Quot}_{n'}$ for $n' > n$ and so we cannot naturally produce an ind-scheme from these different quot schemes. However, as every n-regular sheaf is n'-regular for all $n' \geq n$, it makes sense to restrict to an open subscheme $Q^{n-\mathrm{reg}} \subset \mathrm{Quot}_n$ parametrising n-regular quotient sheaves $q : V_n \otimes \mathcal{O}_X(-n) \to \mathcal{E}$ such that $H^0(q(n))$ is an isomorphism. Then for $n' \geq n$, we have natural equivariant morphisms

$$Q^{n-\mathrm{reg}} \to Q^{n'-\mathrm{reg}}.$$

Furthermore, the stack quotient of the $\mathrm{GL}_{P(n)}$-action on $Q^{n-\mathrm{reg}}$ is isomorphic to the stack $\mathcal{C}oh_P^{n-\mathrm{reg}}(X)$ of n-regular coherent sheaves on X with Hilbert polynomial P. Hence the above morphisms, determine a directed system of open immersions

$$\mathcal{C}oh_P^{n-\mathrm{reg}}(X) \hookrightarrow \mathcal{C}oh_P^{n'-\mathrm{reg}}(X)$$

whose limit is the stack $\mathcal{C}oh_P(X)$ of coherent sheaves on X with Hilbert polynomial P. This suggests this stack is the correct space to compare the Hesselink stratifications for different n.

We can intersect the refined Hesselink stratification (10) with the open subscheme $Q^{n-\mathrm{reg}}$ and then take the quotient by $G_n := \mathrm{GL}_{P(n)}$ to obtain finite Hesselink stratifications

$$\mathcal{C}oh_P^{n-\mathrm{reg}}(X) = \bigsqcup_{\beta \in \mathcal{B}_n, \nu \in \mathcal{C}_\beta} \mathcal{S}^n_{\beta,\nu} \tag{11}$$

where $\mathcal{S}^n_{\beta,\nu}$ are quotient stacks. More precisely, let $S^{n-\mathrm{reg}}_{\beta,\nu}$ denote the fibre product of the Hesselink stratum $S^n_{\beta,\nu}$ in Quot_n with $Q^{n-\mathrm{reg}}$; then $\mathcal{S}^n_{\beta,\nu} := [S^{n-\mathrm{reg}}_{\beta,\nu}/\mathrm{GL}_{P(n)}]$.

Definition 4.24. For each tuple ν of Hilbert polynomials, we let $\beta_n(\nu) := \beta_{n,m_n}(\nu) \in \mathcal{B}_n$ denote the associated Hesselink index given by Definition 4.18, and we write $\mathcal{S}^n_\nu := \mathcal{S}^n_{\beta_n(\nu),\nu}$ for the corresponding refined Hesselink stratum in $\mathcal{C}oh^{n-\mathrm{reg}}_P(X)$.

Proposition 4.25. *Let $\nu = (P_1, \dots, P_s)$ be a tuple of Hilbert polynomials on X such that $\sum_i P_i = P$.*

i) *If $\nu = \tau \in \mathrm{HNT}(X,P)$, then for $n >> 0$ we have a closed immersion of stacks*

$$\mathcal{C}oh^\tau_P(X) \hookrightarrow \mathcal{S}^n_\tau.$$

ii) *If $\nu = \tau \in \mathrm{HNT}(X,P)$ and $n \in \mathbb{N}$, then for $n' >> n$, we have a locally closed immersion*

$$\mathcal{S}^n_\tau \hookrightarrow \bigsqcup_{\tau' \in \mathcal{B}^n_\tau} \mathcal{C}oh^{\tau'}_P(X),$$

where $\mathcal{B}^n_\tau$ is a finite set of HN types $\tau' \geq \tau$, and moreover, for each $\tau' \in \mathcal{B}^n_\tau$, we have a closed immersion $\mathcal{C}oh^{\tau'}_P(X) \hookrightarrow \mathcal{S}^{n'}_{\tau'}$.

iii) *Let $\mathcal{F}$ be a sheaf on X with Hilbert polynomial P. Then $\mathcal{F}$ is represented by a point in $\mathcal{S}^n_\nu$ for all $n >> 0$ if and only if $\mathcal{F}$ has HN type ν (and so, in particular, ν is a HN type).*

Proof. Part i). By Theorem 4.19, for $n >> 0$, the HN stratum $Q_{n,\tau}$ indexed by τ in the open subscheme $Q^o_n \subset \mathrm{Quot}_n$ is a closed subscheme of the refined Hesselink stratum $S^n_\tau := S_{\beta_{n,m_n}(\tau),\tau}$ in Quot_n. The assumptions on n given in Theorem 4.19 imply that all sheaves with HN type τ are n-regular, so the locally closed immersion $Q_{n,\tau} \hookrightarrow Q^o_n$ factors through the open subscheme $Q^{n-\mathrm{reg}} \subset Q^o_n$. By the universal property of the fibre product, the closed immersion $Q_{n,\tau} \hookrightarrow S^n_\tau$ factors through $S^{n-\mathrm{reg}}_\tau := S^n_\tau \times_{\mathrm{Quot}_n} Q^{n-\mathrm{reg}}$. The factorisation $Q_{n,\tau} \hookrightarrow S^{n-\mathrm{reg}}_\tau$ is a closed immersion (cf. [6, I Corollaire 5.3.13]) and, after taking the stack quotient by $\mathrm{GL}_{P(n)}$, this induces a closed immersion $\mathcal{C}oh^\tau_P(X) \hookrightarrow \mathcal{S}^n_\tau$.

Part ii). Since the refined Hesselink stratum $S^{n-\mathrm{reg}}_\tau$ is a finite type scheme, it has a finite stratification by HN types

$$S^{n-\mathrm{reg}}_\tau = \bigsqcup_{\tau' \in \mathcal{B}^n_\tau} S^{n-\mathrm{reg}}_{\tau,\tau'},$$

where $\mathcal{B}^n_\tau$ is a finite set of HN types. Furthermore, every quotient sheaf $q : V_n \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$ in $S^n_\tau \subset \mathrm{Quot}_n$ admits a filtration

$$0 \subset \mathcal{E}^{(1)} \subset \mathcal{E}^{(2)} \subset \cdots \subset \mathcal{E}^{(s)} = \mathcal{E}$$

such that $P(\mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}) = P_i$; therefore, $\mathcal{E}$ has HN type at least τ. Hence, for each $\tau' \in \mathcal{B}^n_\tau$, we have $\tau' \geq \tau$. As the set $\mathcal{B}^n_\tau$ is finite, we can apply part i) to each HN type $\tau' \in \mathcal{B}^n_\tau$ to conclude that for $n(\tau') >> 0$, we have a closed immersion $\mathcal{C}oh^{\tau'}_P(X) \hookrightarrow \mathcal{S}^{n(\tau')}_{\tau'}$. Let n' be the maximum of the $n(\tau')$ for $\tau' \in \mathcal{B}^n_\tau$. Since $S^{n-\mathrm{reg}}_{\tau,\tau'} \subset S^{n-\mathrm{reg}}_\tau$ is a locally closed subscheme, after taking the stack quotient by $\mathrm{GL}_{P(n)}$, we obtain a locally closed immersion. As a composition of immersions is an immersion, the following composition is a locally closed immersion

$$\mathcal{S}^n_{\tau,\tau'} \hookrightarrow \mathcal{S}^n_\tau \hookrightarrow \mathcal{C}oh^{n-\mathrm{reg}}_P(X) \hookrightarrow \mathcal{C}oh^{n'-\mathrm{reg}},$$

which factors via $\mathcal{C}oh^{\tau'}_P(X)$, as $\mathcal{S}^n_{\tau,\tau'}$ is the HN stratum for τ' (to prove this we use the natural atlases for these stacks and apply Theorem 4.15). The factorisation $\mathcal{S}^n_{\tau,\tau'} \hookrightarrow \mathcal{C}oh^{\tau'}_P(X)$ is then also a locally closed immersion. Hence, we have a locally closed immersion

$$\mathcal{S}^n_\tau = \bigsqcup_{\tau' \in \mathcal{B}^n_\tau} \mathcal{S}^n_{\tau,\tau'} \hookrightarrow \bigsqcup_{\tau' \in \mathcal{B}^n_\tau} \mathcal{C}oh^{\tau'}_P(X)$$

and, for each $\tau' \in \mathcal{B}^n_\tau$, closed immersions $\mathcal{C}oh^{\tau'}_P(X) \hookrightarrow \mathcal{S}^{n'}_{\tau'}$.

Part iii). If $\mathcal{F}$ has HN type ν, then $\mathcal{F}$ represents a point in $\mathcal{C}oh^\nu_P(X)$ and by part i), this is contained in $\mathcal{S}^n_\nu$ for all $n >> 0$. If $\mathcal{F}$ is represented by a point $\mathcal{S}^n_\nu$ for all $n >> 0$ and has HN type $\tau \neq \nu$, then $\mathcal{F}$ represents a point in $\mathcal{C}oh^{\tau}_P(X)$, which is contained in $\mathcal{S}^n_\tau$ for all $n >> 0$. This would contradict the fact that the refined Hesselink strata are disjoint. $\square$

For a tuple ν of Hilbert polynomials, we have a refined Hesselink stratum $\mathcal{S}^n_\nu \subset \mathcal{C}oh_P^{n-\mathrm{reg}}(X)$ for each n. Unfortunately, it is not the case that for $n' >> n$, the n'th Hesselink stratification is a refinement of the nth Hesselink stratification. For example, we may have for fixed n, two n-regular sheaves $\mathcal{E}$ and $\mathcal{E}'$ with the same HN type τ, but the subquotients appearing in their HN filtrations are not necessarily n-regular. In this case, it is possible that $\mathcal{E}$ and $\mathcal{E}'$ are represented by points in two distinct refined Hesselink strata $\mathcal{S}^n_\nu$ and $\mathcal{S}^n_{\nu'}$, but for $n' >> n$, both sheaves will represent points in $\mathcal{S}^{n'}_\tau$ by Proposition 4.25. However, for each ν, from the collection of finite Hesselink strata $\mathcal{S}^n_\nu$ for different n, our aim is to construct an asymptotic 'limit' $\mathcal{S}_\nu \subset \mathcal{C}oh_P(X)$, in the sense that a sheaf $\mathcal{F}$ is represented by a point of $\mathcal{S}_\nu$ if and only if $\mathcal{F}$ is represented by a point of $\mathcal{S}^n_\nu$ for $n >> 0$.

A natural way to compare S^n_ν and $S^{n'}_\nu$ for $n' > n$ is to take their fibre product in $\mathcal{C}oh_P^{n'-\mathrm{reg}}(X)$.

Definition 4.26. For a tuple of Hilbert polynomials and $n' > n$, we let $\mathcal{S}^{n,n'}_\nu$ be the following fibre product

$$\begin{array}{ccc} \mathcal{S}^{n,n'}_\nu & \hookrightarrow & \mathcal{S}^{n'}_\nu \\ \downarrow & & \downarrow \\ \mathcal{S}^n_\nu & \hookrightarrow & \mathcal{C}oh_P^{n'-\mathrm{reg}}(X). \end{array}$$

We note that a sheaf $\mathcal{E}$ is represented by a point in $\mathcal{S}^{n,n'}_\nu$ if and only if it is n-regular and is represented by a point in $\mathcal{S}^n_\nu$ and a point in $\mathcal{S}^{n'}_\nu$.

The following theorem shows that the fibre products $\mathcal{S}^{n,n'}_\nu$ of the Hesselink strata stabilise for $n' >> n >> 0$ and so we can define an asymptotic Hesselink stratum $\mathcal{S}_\nu \subset \mathcal{C}oh_P(X)$ to be this limit. Furthermore, we see that these asymptotic strata $\mathcal{S}_\nu$ give a stratification of $\mathcal{C}oh_P(X)$ which coincides with the Harder–Narasimhan stratification.

Theorem 4.27. *Let $\nu = (P_1, \dots, P_s)$ be a tuple of Hilbert polynomials on X such that $\sum_i P_i = P$. For $n' \geq n$, let $\mathcal{S}^{n,n'}_\nu$ be the fibre product of the Hesselink strata $\mathcal{S}^n_\nu$ and $\mathcal{S}^{n'}_\nu$ in $\mathcal{C}oh_P^{n'-\mathrm{reg}}(X)$.*

i) *If $\nu \in HNT(X,P)$, then for $n' >> n >> 0$, we have $\mathcal{S}^{n,n'}_\nu = \mathcal{C}oh^\nu_P(X)$.*
ii) *If $\nu \notin HNT(X,P)$, then for $n' >> n >> 0$, we have $\mathcal{S}^{n,n'}_\nu = \emptyset$*

In either case, the refined Hesselink strata stabilise to an asymptotic stratum $\mathcal{S}_\nu \subset \mathcal{C}oh^\nu_P(X)$. Furthermore, the strata $\mathcal{S}_\nu$ give a stratification of $\mathcal{C}oh_P(X)$ which coincides with the Harder–Narasimhan stratification.

Proof. Part i). Take $n' >> n >> 0$ as required by Proposition 4.25 so that we have immersions

$$\mathcal{C}oh^\nu_P(X) \hookrightarrow \mathcal{S}^n_\nu \hookrightarrow \bigsqcup_{\nu' \in \mathcal{B}^n_\nu} \mathcal{C}oh^{\nu'}_P(X),$$

for a finite set $\mathcal{B}^n_\nu$ of HN types $\nu' \geq \nu$, and closed immersions $\mathcal{C}oh^{\nu'}_P(X) \hookrightarrow \mathcal{S}^{n'}_{\nu'}$ for each $\nu' \in \mathcal{B}^n_\nu$. By construction of $\mathcal{S}^{n,n'}_\nu$, we have locally closed immersions $\mathcal{S}^{n,n'}_\nu \hookrightarrow \mathcal{S}^n_\nu$ and $\mathcal{S}^{n,n'}_\nu \hookrightarrow \mathcal{S}^{n'}_\nu$. Hence the composition

$$\mathcal{S}^{n,n'}_\nu \hookrightarrow \mathcal{S}^n_\nu \hookrightarrow \bigsqcup_{\nu' \in \mathcal{B}^n_\nu} \mathcal{C}oh^{\nu'}_P(X)$$

is also a locally closed immersion. We claim that $\mathcal{S}^{n,n'}_\nu \hookrightarrow \mathcal{C}oh^\nu_P(X)$. Otherwise $\mathcal{S}^{n,n'}_\nu$ has non-empty intersection with $\mathcal{C}oh^{\nu'}_P(X)$ for $\nu' \neq \nu$ and this would contradict the disjointness of the (stack quotients of the refined) Hesselink strata $\mathcal{S}^{n'}_\nu$ and $\mathcal{S}^{n'}_{\nu'}$. Since also $\mathcal{C}oh^\nu_P(X) \hookrightarrow \mathcal{S}^n_\nu$ and $\mathcal{C}oh^\nu_P(X) \hookrightarrow \mathcal{S}^{n'}_\nu$, and these maps are compatible with the maps to $\mathcal{C}oh_P(X)$, we conclude that $\mathcal{S}^{n,n'}_\nu = \mathcal{C}oh^\nu_P(X)$.

Part ii) follows directly from Proposition 4.25 iii) and the construction of $\mathcal{S}^{n,n'}_\nu$.

In particular, we see that an asymptotic Hesselink stratum $\mathcal{S}_\nu$ is non-empty if and only if $\nu \in \mathrm{HNT}(X, P)$ and these asymptotic Hesselink strata coincide with the HN strata. □

Remark 4.28. A sheaf is represented by a point of $\mathcal{S}_\nu$ if and only if it is represented by a point of $\mathcal{S}_\nu^n$ for $n >> 0$. To prove this claim, we note that a sheaf $\mathcal{F}$ has HN type ν if and only if it represents a point in $\mathcal{S}_n^\nu$ for $n >> 0$ by Proposition 4.25 iii).

5. A functor from sheaves to quiver representations

In [1], Álvarez-Cónsul and King give a functorial construction of moduli spaces of semistable sheaves on $(X, \mathcal{O}_X(1))$ using moduli spaces of representations for a Kronecker quiver. In this section, we generalise this construction by extending this quiver and compare the HN strata for sheaves and quivers.

5.1. **Overview of the construction of the functor.** Let X be a projective scheme of finite type over k with very ample line bundle $\mathcal{O}_X(1)$ and let $\mathbf{Coh}(X)$ denote the category of coherent sheaves on X. For natural numbers $m > n$, we let $K_{n,m}$ be a Kronecker quiver with vertex set $V := \{n, m\}$ and $\dim H^0(\mathcal{O}_X(m-n))$ arrows from n to m, and consider the functor

$$\Phi_{n,m} := \mathrm{Hom}(\mathcal{O}_X(-n) \oplus \mathcal{O}_X(-m), -) : \mathbf{Coh}(X) \to \mathbf{Rep}(K_{n,m})$$

that sends a sheaf $\mathcal{E}$ to the representation $W_\mathcal{E}$ of $K_{n,m}$ where

$$W_{\mathcal{E},n} := H^0(\mathcal{E}(n)) \qquad W_{\mathcal{E},m} := H^0(\mathcal{E}(m))$$

and the evaluation map $H^0(\mathcal{E}(n)) \otimes H^0(\mathcal{O}_X(m-n)) \to H^0(\mathcal{E}(m))$ gives the morphisms.

Let $\mathbf{Coh}_P^{n-\mathrm{reg}}(X)$ be the subcategory of $\mathbf{Coh}(X)$ consisting of n-regular sheaves with Hilbert polynomial P. Then the image of $\Phi_{n,m}$ restricted to $\mathbf{Coh}_P^{n-\mathrm{reg}}(X)$ is contained in the subcategory of quiver representations of dimension vector $d_{n,m}(P) = (P(n), P(m))$. By [1] Theorem 3.4, if $\mathcal{O}_X(m-n)$ is regular, then the functor

$$\Phi_{n,m} : \mathbf{Coh}_P^{n-\mathrm{reg}}(X) \to \mathbf{Rep}_{d_{n,m}(P)}(K_{n,m})$$

is fully faithful. For representations of $K_{n,m}$ of dimension $d_{n,m}(P)$, we consider the stability parameter $\theta_{n,m}(P) := (-P(m), P(n))$.

Theorem 5.1 ([1] Theorem 5.10)**.** *For $m >> n >> 0$, depending on X and P, a sheaf $\mathcal{E}$ with Hilbert polynomial P is semistable if and only if it is pure, n-regular and the quiver representation $\Phi_{n,m}(\mathcal{E})$ is $\theta_{n,m}(P)$-semistable.*

We briefly describe how to pick $m >> n >> 0$ as required for this theorem to hold; for further details, we refer to the reader to the conditions (C:1) - (C:5) stated in [1] §5.1. First, we take n so all semistable sheaves with Hilbert polynomial P are n-regular and the Le Potier–Simpson estimates hold. Then we choose m so $\mathcal{O}_X(m-n)$ is regular and, for all n-regular sheaves $\mathcal{E}$ and vector subspaces $V' \subset H^0(\mathcal{E}(n))$, the subsheaf $\mathcal{E}'$ generated by V' under the evaluation map $H^0(\mathcal{E}(n)) \otimes \mathcal{O}(-n) \to \mathcal{E}$ is m-regular. Finally, we take m sufficiently large so a finite list of polynomial inequalities can be determined by evaluation at m (see (C:5) in [1]).

We can alternatively consider the functor $\Phi_{n,m}$ as a morphism of stacks. We recall that

$$\mathcal{C}oh_P^{n-\mathrm{reg}}(X) \cong [Q^{n-\mathrm{reg}}/G_n],$$

where $G_n = \mathrm{GL}(V_n)$, and

$$\mathcal{R}\mathrm{ep}_{d_{n,m}(P)}(K_{n,m}) \cong [\mathrm{Rep}_{d_{n,m}(P)}(K_{n,m})/\overline{G}_{d_{n,m}(P)}(K_{n,m})].$$

Let $\mathcal{U}_n$ be the universal quotient sheaf over $Q^{n-\mathrm{reg}} \times X$ and $p : Q^{n-\mathrm{reg}} \times X \to Q^{n-\mathrm{reg}}$ be the projection map. By definition of $Q^{n-\mathrm{reg}}$, we have that $R^i p_*(\mathcal{U}_n(n)) = 0$ for $i > 0$; therefore, by the semi-continuity theorem, $p_*(\mathcal{U}_n(n))$ is a vector bundle over $Q^{n-\mathrm{reg}}$ of rank $P(n)$ and similarly $p_*(\mathcal{U}_n(m))$ is a rank $P(m)$ vector bundle. Hence, by using the evaluation map, we obtain a family of representations of $K_{n,m}$ of dimension $d_{n,m}(P)$ parametrised by $Q^{n-\mathrm{reg}}$ that induces a morphism

$$Q^{n-\mathrm{reg}} \to \mathcal{R}\mathrm{ep}_{d_{n,m}(P)}(K_{n,m}).$$

As this morphism is G_n-invariant, it descends to a morphism of stacks

$$\Phi_{n,m} : \mathcal{C}oh_P^{n-\mathrm{reg}}(X) \to \mathcal{R}\mathrm{ep}_{d_{n,m}(P)}(K_{n,m}).$$

We study the images of Hesselink (resp. HN) strata under this morphism in Proposition 5.5 (resp. Theorem 5.7) below.

5.2. **The images of the Hesselink strata.** In this section, we fix n and assume $m >> n$ so that

(A1) $\mathcal{O}_X(m-n)$ is regular,
(A2) for any n-regular sheaf $\mathcal{E}$ and any vector subspace $V' \subset H^0(\mathcal{E}(n))$, the subsheaf $\mathcal{E}' \subset \mathcal{E}$ generated by V' is m-regular.

The second assumption is possible as such sheaves $\mathcal{E}'$ form a bounded family (see [1] Condition (C:4) in §5.1).

Let

$$\mathrm{Quot}(V_n \otimes \mathcal{O}_X(-n), P) = \bigsqcup_{\beta \in \mathcal{B}_{n,m}} S_\beta$$

be the Hesselink stratification associated to the $\mathrm{SL}(V_n)$-action on this Quot scheme with respect to $L_{n,m}$ as we described in §4.3. As in §4.7, we consider the induced stratification on the stack of n-regular sheaves

$$\mathcal{C}oh_P^{n-\mathrm{reg}}(X) = \bigsqcup_\beta \mathcal{S}_\beta^{n,m} \tag{12}$$

where $\mathcal{S}_\beta^{n,m} = [S_\beta^{n-\mathrm{reg}}/G_n]$ and $S_\beta^{n-\mathrm{reg}}$ is the fibre product of $Q^{n-\mathrm{reg}}$ and S_β in this Quot scheme.

To define a Hesselink stratification on the space (or stack) of representations of $K_{n,m}$ of dimension vector $d_{n,m}(P)$, we need to choose a parameter $\alpha \in \mathbb{N}^2$ which defines a norm $|| - ||_\alpha$ as in Example 2.4 (b). We take $\alpha = \alpha_{n,m}(P) := (P(m), P(n))$ due to the following lemma.

Lemma 5.2. *Let $\theta = \theta_{n,m}(P)$ and $\alpha = \alpha_{n,m}(P)$. Then, for sheaves $\mathcal{E}$ and $\mathcal{F}$, we have*

$$\frac{\theta(W_\mathcal{E})}{\alpha(W_\mathcal{E})} \geq \frac{\theta(W_\mathcal{F})}{\alpha(W_\mathcal{F})} \iff \frac{h^0(\mathcal{E}(n))}{h^0(\mathcal{E}(m))} \leq \frac{h^0(\mathcal{F}(n))}{h^0(\mathcal{F}(m))}.$$

The same statement holds if we replace these inequalities with strict inequalities.

Proof. By definition of these parameters, we have

$$\frac{\theta(W_\mathcal{E})}{\alpha(W_\mathcal{E})} := \frac{-P(m)h^0(\mathcal{E}(n)) + P(n)h^0(\mathcal{E}(m))}{P(m)h^0(\mathcal{E}(n)) + P(n)h^0(\mathcal{E}(m))} = 1 - \frac{2P(m)h^0(\mathcal{E}(n))}{P(m)h^0(\mathcal{E}(n)) + P(n)h^0(\mathcal{E}(m))}.$$

From this, it is easy to check the desired equivalences of inequalities. □

By Theorem 3.8, we can equivalently view the Hesselink stratification (with respect to ρ_θ and $|| - ||_\alpha$) as a stratification by HN types (with respect to θ and α):

$$\mathcal{R}ep_{d_{n,m}(P)}(K_{n,m}) = \bigsqcup_\gamma \mathcal{R}\mathrm{ep}^\gamma_{d_{n,m}(P)}(K_{n,m}).$$

The unstable Hesselink strata in (12) are indexed by conjugacy classes of rational 1-PSs λ_β of $\mathrm{SL}(V_n)$. Equivalently, the index β is given by a collection of strictly decreasing rational weights $r(\beta) = (r_1, \dots, r_s)$ and multiplicities $l(\beta) = (l_1, \dots, l_s)$ satisfying $\sum_{i=1}^s l_i = P(n)$ and $\sum_{i=1}^s r_i l_i = 0$. More precisely, we recall that the rational 1-PS associated to $r(\beta)$ and $l(\beta)$ is

$$\lambda_\beta(t) = \mathrm{diag}(t^{r_1}, \dots, t^{r_1}, \dots, t^{r_s}, \dots, t^{r_s})$$

where r_i appears l_i times.

Definition 5.3. For an index β of the Hesselink stratification (12) on the stack of n-regular sheaves as above, we let $\gamma(\beta) := (d_1(\beta), \dots, d_s(\beta))$ where

$$d_i(\beta) = \left(l_i, l_i \frac{P(m)}{P(n)} - l_i r_i \right).$$

Lemma 5.4. *Let β be an index for the Hesselink stratification (12); then $\gamma(\beta)$ is a HN type for a representation of $K_{n,m}$ of dimension $d_{n,m}$ with respect to θ and α.*

Proof. Let $r(\beta) = (r_1, \dots, r_s)$ and $l(\beta) = (l_1, \dots, l_s)$ be as above. Then

$$\sum_{i=1}^{s} d_i(\beta) = \left(\sum_{i=1}^{n} l_i, \sum_{i=1}^{n} l_i \frac{P(m)}{P(n)} - l_i r_i \right) = (P(n), P(m)),$$

as $\sum l_i = P(n)$ and $\sum l_i r_i = 0$. Furthermore, as $r_1 > \cdots > r_s$, it follows that

$$\frac{\theta(d_1(\beta))}{\alpha(d_1(\beta))} < \frac{\theta(d_2(\beta))}{\alpha(d_2(\beta))} < \cdots\cdots < \frac{\theta(d_s(\beta))}{\alpha(d_s(\beta))}.$$

To complete the proof, we must verify that

$$d_i(\beta) = \left(l_i, l_i \frac{P(m)}{P(n)} - l_i r_i \right) \in \mathbb{N}^2.$$

The first number l_i is a multiplicity and so is a natural number, but the second number is a priori only rational. As β is an index for the Hesselink stratification, it indexes a non-empty stratum S_β and, as this stratum is constructed from its associated limit set Z_d^λ, by Theorem 2.8, it follows that this limit set must also be non-empty. Hence this limit set contains a quotient sheaf $q : V_n \otimes \mathcal{O}_X(-n) \to \oplus_{i=1}^s \mathcal{E}_i$ such that, by Proposition 4.6, for $i = 1, \dots, s$, we have

$$P(\mathcal{E}_i, m) = l_i \frac{P(m)}{P(n)} - l_i r_i.$$

Then, as $P(\mathcal{E}_i, m) \in \mathbb{N}$, this completes the proof. $\square$

Proposition 5.5. *For a Hesselink index β in (12), we have*

$$\Phi_{n,m}\left(\mathcal{S}_\beta^{n,m} \right) \subset \bigsqcup_{\gamma \geq \gamma(\beta)} \mathcal{R}ep^{\gamma}_{d_{n,m}(P)}(K_{n,m}).$$

Proof. Let $q : V_n \otimes \mathcal{O}_X(-n) \twoheadrightarrow \mathcal{E}$ be a quotient sheaf in $S_\beta^{n-\mathrm{reg}}$ and let $r(\beta) = (r_1, \dots, r_s)$ and $l(\beta) = (l_1, \dots, l_s)$ be the associated rational weights and multiplicities; thus, $r_1 > \cdots > r_s$ and

$$\lambda_\beta(t) = \mathrm{diag}(t^{r_1}, \dots, t^{r_1}, \dots, t^{r_s}, \dots, t^{r_s})$$

where r_i appears l_i times. If λ denotes the unique integral primitive 1-PS associated to λ_β, then $\beta = ([\lambda], d)$ where $d = ||\lambda_\beta||$. The 1-PS λ induces a filtration $0 = V_n^{(0)} \subset V_n^{(1)} \subset \cdots \subset V_n^{(s)} = V_n$ such that the successive quotients V_n^i have dimension l_i.

By Corollary 4.7, there exists $g \in \mathrm{SL}(V_n)$ such that we have a filtration

$$0 = \mathcal{E}^{(0)} \subset \cdots \subset \mathcal{E}^{(i)} := g \cdot q(V_n^{(i)} \otimes \mathcal{O}_X(-n)) \subset \cdots \subset \mathcal{E}^{(s)} = \mathcal{E}$$

where the Hilbert polynomials of $\mathcal{E}^i := \mathcal{E}^{(i)}/\mathcal{E}^{(i-1)}$ satisfy

$$P(n)P(\mathcal{E}_i, m) = l_i(P(m) - r_i P(n)) \qquad \text{for } \ i = 1, \dots, s.$$

Let $W^{(i)} := \Phi_{n,m}(\mathcal{E}^{(i)})$; then we have a filtration

$$0 = W^{(0)} \subset W^{(1)} \subset \cdots \subset W^{(s)} = W \tag{13}$$

with $\dim W^{(i)} := (\dim H^0(\mathcal{E}^{(i)}(n)), \dim H^0(\mathcal{E}^{(i)}(m))) = (\dim V_n^{(i)}, P(\mathcal{E}^{(i)}, m))$, due to the fact that $H^0(q(n))$ is an isomorphism (because $q \in Q^{n-\mathrm{reg}}$) and $\mathcal{E}^{(i)}$ is m-regular (by the assumption (A2) on m). Let $W_i := W^{(i)}/W^{(i-1)}$; then

$$\dim W_i = (\dim V_n^i, P(\mathcal{E}_i, m)) = \left(l_i, l_i \frac{P(m)}{P(n)} - l_i r_i \right).$$

As we have a filtration (13) of W whose successive quotients have dimension vectors specified by $\gamma(\beta)$ and $\gamma(\beta)$ is a HN type for $d_{n,m}$-dimensional representation of $K_{n,m}$ by Lemma 5.4, it follows that $W = \Phi_{n,m}(\mathcal{E})$ has HN type greater than or equal to $\gamma(\beta)$. $\square$

In the above proof, we note that the subsheaves $\mathcal{E}^{(i)}$ may not be n-regular and so W_i is only isomorphic to $\Phi_{n,m}(\mathcal{E}_i)$ if $H^1(\mathcal{E}^{(i-1)}(n)) = 0$. This is not the case in general; although W_i is always a subrepresentation of $\Phi_{n,m}(\mathcal{E}_i)$. Hence, it is not possible to use GIT semistability properties of the quotients $q_i : V_n^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}_i$ to deduce (θ, α)-semistability of W_i (i.e., to show that (13) is the HN filtration of W).

5.3. **The image of the HN strata.** In this section, we study the image of the HN strata $\mathcal{C}oh_P^\tau(X)$ under the map

$$\Phi_{n,m} : \mathcal{C}oh_P^{n-\mathrm{reg}}(X) \to \mathcal{R}\mathrm{ep}_{d_{n,m}(P)}(K_{n,m})$$

for n and m sufficiently large (depending on τ). We prove that a HN stratum for sheaves is mapped to a HN stratum for quiver representations. Let $\theta = \theta_{n,m}(P) := (-P(m), P(n))$ and $\alpha = \alpha_{n,m}(P) := (P(m), P(n))$ be as above.

Definition 5.6. For a HN type $\tau = (P_1, \dots, P_s) \in \mathrm{HNT}(X, P)$ and $(n, m) \in \mathbb{N}^2$, we let

$$\gamma_{n,m}(\tau) := (d_{n,m}(P_1), \dots, d_{n,m}(P_s))$$

where $d_{n,m}(P_i) = (P_i(n), P_i(m))$.

As τ is a HN type of sheaves, we have that $P_1 \succ P_2 \succ \cdots\cdots \succ P_s$; thus,

$$\frac{P_1(n)}{P_1(m)} > \frac{P_2(n)}{P_2(m)} > \cdots\cdots > \frac{P_s(n)}{P_s(m)} \qquad \text{for } m >> n >> 0. \tag{14}$$

Therefore, by Lemma 5.2, for $m >> n >> 0$, we have

$$\frac{\theta(d_{n,m}(P_1))}{\alpha(d_{n,m}(P_1))} < \frac{\theta(d_{n,m}(P_2))}{\alpha(d_{n,m}(P_2))} < \cdots\cdots < \frac{\theta(d_{n,m}(P_s))}{\alpha(d_{n,m}(P_s))};$$

i.e., $\gamma_{n,m}(\tau)$ is a HN type for representations of $K_{n,m}$ of dimension $d_{n,m}(P)$ for $m >> n >> 0$.

Theorem 5.7. *Let $\tau = (P_1, \dots, P_s) \in \mathrm{HNT}(X, P)$ be a HN type. Then, for $m >> n >> 0$,*

$$\Phi_{n,m}\left(\mathcal{C}oh_P^\tau(X)\right) \subset \mathcal{R}ep_{d_{n,m}(P)}^{\gamma_{n,m}(\tau)}(K_{n,m}).$$

Proof. We take $m >> n >> 0$ as needed for Theorem 5.1 for the Hilbert polynomials $P_1, \dots, P_s$. Furthermore, we assume that m and n are sufficiently large so the inequalities (14) hold.

Let $\mathcal{E}$ be a sheaf on X of HN type τ and HN filtration given by

$$0 = \mathcal{E}^{(0)} \subset \mathcal{E}^{(1)} \subset \cdots \subset \mathcal{E}^{(s)} = \mathcal{E}.$$

Let $W = \Phi_{n,m}(\mathcal{E})$ and $W^{(i)} := \Phi_{n,m}(\mathcal{E}^{(i)})$; then we claim that the induced filtration

$$0 = W^{(0)} \subset W^{(1)} \subset \cdots \subset W^{(s)} = W \tag{15}$$

is the HN filtration of W with respect to (θ, α) and, moreover, that W has HN type $\gamma_{n,m}(\tau)$. Let $\mathcal{E}_i$ and W_i denote the successive subquotients in the above filtrations. Our assumptions on n imply that each $\mathcal{E}_i$ is n-regular and so, by induction, each $\mathcal{E}^{(i)}$ is n-regular. Therefore, we have exact sequences

$$0 \to H^0(\mathcal{E}^{(i-1)}(n)) \to H^0(\mathcal{E}^{(i)}(n)) \to H^0(\mathcal{E}_i(n)) \to 0$$

that give isomorphisms $\Phi_{n,m}(\mathcal{E}_i) \cong W_i$. Since $\mathcal{E}_i$ is n-regular and semistable with Hilbert polynomial P_i, the quiver representation W_i is $\theta_i := \theta_{n,m}(P_i)$-semistable by Theorem 5.1. For a subrepresentation $W' \subset W_i$, we note that

$$\theta_i(W') \geq 0 \iff P_i(n) \dim W'_{v_m} \geq P_i(m) \dim W'_{v_n} \iff \frac{\theta(W')}{\alpha(W')} \geq \frac{\theta(W_i)}{\alpha(W_i)}.$$

Therefore, θ_i-semistability of W_i implies (θ, α)-semistability of W_i. To finish the proof of the claim, i.e. to prove that (15) is the HN filtration of W, it suffices to check that

$$\frac{\theta(W_1)}{\alpha(W_1)} < \frac{\theta(W_2)}{\alpha(W_2)} < \cdots\cdots < \frac{\theta(W_s)}{\alpha(W_s)}$$

or, equivalently, by Lemma 5.2, that

$$\frac{h^0(\mathcal{E}_1(n))}{h^0(\mathcal{E}_1(m))} > \frac{h^0(\mathcal{E}_2(n))}{h^0(\mathcal{E}_2(m))} > \cdots\cdots > \frac{h^0(\mathcal{E}_s(n))}{h^0(\mathcal{E}_s(m))}.$$

Since $\mathcal{E}_i$ is n-regular, we have that $h^0(\mathcal{E}_i(n)) = P_i(n)$ and $h^0(\mathcal{E}_i(m)) = P_i(m)$; hence, the above inequalities are equivalent to (14). Moreover, this shows that W has HN type $\gamma_{n,m}(\tau)$. □

Remark 5.8. The assignment $\tau \mapsto \gamma_{n,m}(\tau)$ is not injective for exactly the same reason as in Lemma 4.22. In fact, more generally, $P \mapsto d_{n,m}(P)$ is not injective, unless $\dim X \leq 1$.

5.4. **Adding more vertices.** In this final section, we generalise the construction of Álvarez-Cónsul and King by adding more vertices so that the map $\tau \mapsto \gamma(\tau)$ is injective.

For a tuple $\underline{n} = (n_0, \cdots, n_d)$ of increasing natural numbers, we define a functor

$$\Phi_{\underline{n}} := \mathrm{Hom}(\bigoplus_{i=0}^{d} \mathcal{O}_X(-n_i),\ -\ ) : \mathcal{C}oh(X) \to \mathcal{R}\mathrm{ep}(K_{\underline{n}})$$

where $K_{\underline{n}}$ denotes the quiver with vertex set $V = \{n_0, \ldots, n_d\}$ and $\dim H^0(\mathcal{O}_X(n_{i+1} - n_i))$ arrows from n_i to n_{i+1}:

$$K_{\underline{n}} = \left(\begin{array}{ccccccc} & \longrightarrow & & & & \longrightarrow & \\ n_0 & \vdots & n_1 & & n_{d-1} & \vdots & n_d \\ \bullet & H^0(\mathcal{O}_X(n_1 - n_0)) & \bullet & \cdots & \bullet & H^0(\mathcal{O}_X(n_d - n_{d-1})) & \bullet \\ & \vdots & & & & \vdots & \\ & \longrightarrow & & & & \longrightarrow & \end{array} \right).$$

More precisely, if $\mathcal{E}$ is a coherent sheaf on X, then $\Phi_{\underline{n}}(\mathcal{E})$ is the quiver representation denoted $W_{\mathcal{E}} = (W_{\mathcal{E},n_0}, \ldots, W_{\mathcal{E},n_d}, \mathrm{ev}^1_{\mathcal{E}}, \ldots, \mathrm{ev}^d_{\mathcal{E}})$ where we let $W_{\mathcal{E},i} := H^0(\mathcal{E}(i))$ and the arrows are given by the evaluation maps $\mathrm{ev}^i_{\mathcal{E}} : H^0(\mathcal{E}(n_{i-1})) \otimes H^0(\mathcal{O}(n_i - n_{i-1})) \to H^0(\mathcal{E}(n_i))$.

We note that $\Phi_{\underline{n}}$ maps n_0-regular sheaves with Hilbert polynomial P to representations of $K_{\underline{n}}$ of dimension vector $d_{\underline{n}}(P) := (P(n_0), \ldots, P(n_d))$.

Remark 5.9. In general, for $d > 1$, the image of $\Phi_{\underline{n}}$ is contained in a substack consisting of quiver representations satisfying certain relations $\mathcal{R}_{\underline{n}}$. To describe the relations, we consider the evaluation maps $\mathrm{ev}^i_{\mathcal{O}} : H^0(\mathcal{O}_X(n_i - n_{i-1})) \otimes H^0(\mathcal{O}_X(n_{i+1} - n_i)) \to H^0(\mathcal{O}_X(n_{i+1} - n_{i-1}))$. If $H_i := H^0(\mathcal{O}_X(n_i - n_{i-1}))$, we have commutative squares

$$\begin{array}{ccc} H^0(\mathcal{E}(n_{i-1})) \otimes H_i \otimes H_{i+1} & \xrightarrow{\mathrm{ev}^i_{\mathcal{E}} \otimes \mathrm{Id}} & H^0(\mathcal{E}(n_i)) \otimes H_{i+1} \\ \Big\downarrow {\scriptstyle \mathrm{Id}\otimes \mathrm{ev}^i_{\mathcal{O}}} & & \Big\downarrow {\scriptstyle \mathrm{ev}^{i+1}_{\mathcal{E}}} \\ H^0(\mathcal{E}(n_{i-1})) \otimes H^0(\mathcal{O}_X(n_{i+1} - n_{i-1})) & \longrightarrow & H^0(\mathcal{E}(n_{i+1})) \end{array}$$

and, therefore, elements in the kernel of $\mathrm{ev}_{\mathcal{O},i}$ give rise to compositions of paths that are zero. More precisely, the kernels of the iterated evaluation maps

$$H_i \otimes H_{i+1} \otimes \cdots \otimes H_j \to H^0(\mathcal{O}_X(n_j - n_{i-1})),$$

for $i < j$, describe the set of relations $\mathcal{R}_{\underline{n}}$ on $K_{\underline{n}}$.

Remark 5.10. In this remark, we explain how many constructions of [1] can be generalised. Throughout we assume that $\mathcal{O}_X(n_i - n_{i-1})$ are 0-regular for $i = 1, \ldots, d$, so that

$$\mathrm{ev} : H^0(\mathcal{O}_X(n_i - n_{i-1})) \otimes H^0(\mathcal{O}_X(l)) \to H^0(\mathcal{O}_X(n_i - n_{i-1} + l))$$

is surjective for $l \geq 0$. Let $B_{\underline{n}}$ be the path algebra of $K_{\underline{n}}$. Then the category of representations of $K_{\underline{n}}$ is equivalent to the category of finite dimensional right $B_{\underline{n}}$-modules. As a vector space,

$$\begin{array}{rccccccccc} B_{\underline{n}} = & \bigoplus_{i=0}^{d} k\, e_i & \oplus & \bigoplus_{i=1}^{d} H_i & \oplus & \bigoplus_{i=2}^{d} H_{i-1} \otimes H_i & \oplus \cdots \oplus & H_1 \otimes H_2 \otimes \cdots \otimes H_d \\ \text{path length:} & 0 & & 1 & & 2 & \cdots & d \end{array}$$

where e_i is the trivial path at vertex i and $H_i := H^0(\mathcal{O}_X(n_i - n_{i-1}))$ is the space of arrows $i-1 \mapsto i$. Let $L_{\underline{n}}$ (resp. $H_{\underline{n}}$) denote the space of length 0 (resp. length 1) paths; then $H_{\underline{n}}$ is an $L_{\underline{n}}$-bimodule, with structure characterised by $e_{i-1}H_{\underline{n}}e_i = H_i$, for $i = 1, \dots, d$. Moreover, $B_{\underline{n}}$ is the tensor algebra of $H_{\underline{n}}$ over $L_{\underline{n}}$. The sheaf $T_{\underline{n}} := \oplus_{i=0}^d \mathcal{O}_X(-n_i)$ is a left $B_{\underline{n}}$-module: its decomposition gives it the structure of a left $L_{\underline{n}}$-module and the evaluation map $\mathrm{ev} : H_{\underline{n}} \otimes_{L_{\underline{n}}} T_{\underline{n}} \to T_{\underline{n}}$ describes the $H_{\underline{n}}$-action. Since

$$\mathrm{End}_{\mathcal{O}_X}(T_{\underline{n}}) = \bigoplus_{i \leq j} H^0(\mathcal{O}_X(n_j - n_i)),$$

the evaluation maps determine a map $B_{\underline{n}} \to \mathrm{End}_{\mathcal{O}_X}(T_{\underline{n}})$ that is surjective if $H^0(\mathcal{O}_X) = k$. Let $A_{\underline{n}}$ be the path algebra of $(K_{\underline{n}}, \mathcal{R}_{\underline{n}})$. Then $T_{\underline{n}}$ is also a left $A_{\underline{n}}$-module and $A_{\underline{n}} = \mathrm{End}_{\mathcal{O}_X}(T_{\underline{n}})$, if $H^0(\mathcal{O}_X) = k$. Moreover, the functor

$$\Phi_{\underline{n}} := \mathrm{Hom}(T_{\underline{n}}, -) : \mathbf{Coh}(X) \to \mathbf{Rep}(K_{\underline{n}}, \mathcal{R}_{\underline{n}}) \cong \mathbf{Mod} - A_{\underline{n}}$$

has left adjoint $\Phi_{\underline{n}}^\vee = - \otimes_{A_{\underline{n}}} T_{\underline{n}}$.

We want to find a stability parameter $\theta = \theta_{\underline{n}}(P) \in \mathbb{Z}^{d+1}$ for representations of $K_{\underline{n}}$ of dimension vector $d_{\underline{n}}(P)$ such that $\Phi_{\underline{n}}$ sends semistable sheaves to θ-semistable quiver representations for $\underline{n} >> 0$ (that is, for $n_d >> n_{d-1} >> \cdots >> n_0 >> 0$). Let

$$\theta_{\underline{n}}(P) := (\theta_0, \dots, \theta_d) \quad \text{where} \quad \theta_i := \sum_{j<i} P(n_j) - \sum_{j>i} P(n_j);$$

then $\sum_{i=0}^d \theta_i P(n_i) = 0$. The following lemma demonstrates that this is a suitable choice.

Lemma 5.11. *Let $\theta = \theta_{\underline{n}}(P)$ as above and $\mathcal{E}$ be a sheaf on X. If, for all $j < i$, we have*

$$\frac{h^0(\mathcal{E}(n_j))}{h^0(\mathcal{E}(n_i))} \leq \frac{P(n_j)}{P(n_i)},$$

then $\theta(W_{\mathcal{E}}) \geq 0$.

Proof. By definition $\theta(W_{\mathcal{E}}) := \sum_{i=0}^d \theta_i h^0(\mathcal{E}(n_i))$ and so it follows that

$$\theta(W_{\mathcal{E}}) = \sum_i (\sum_{j<i} P(n_j) - \sum_{j>i} P(n_j)) h^0(\mathcal{E}(n_i)) = \sum_i \sum_{j<i} P(n_j)h^0(\mathcal{E}(n_i)) - P(n_i)h^0(\mathcal{E}(n_j) \geq 0.$$

by using the given inequalities. □

Corollary 5.12. *Let P be a Hilbert polynomial; then for $\underline{n} >> 0$, we have*

$$\Phi_{\underline{n}}(\mathcal{C}oh_P^{ss}(X)) \subset \mathcal{R}ep_{d_{\underline{n}}(P)}^{\theta_{\underline{n}}(P)}(K_{\underline{n}}).$$

Finally, we need to pick a suitable stability parameter $\alpha \in \mathbb{N}^{d+1}$. The following lemma, whose proof is analogous to Lemma 5.11, shows that we should pick

$$\alpha = \alpha_{\underline{n}}(P) := (\alpha_0, \dots, \alpha_d) \quad \text{where} \quad \alpha_i := \sum_{j<i} P(n_j) + \sum_{j>i} P(n_j).$$

Lemma 5.13. *Let $\theta = \theta_{\underline{n}}(P)$ and $\alpha = \alpha_{\underline{n}}(P)$ be as above and $\mathcal{E}$ and $\mathcal{F}$ be two sheaves on X. If, for all $j < i$, we have*

$$\frac{h^0(\mathcal{E}(n_j))}{h^0(\mathcal{E}(n_i))} \leq \frac{h^0(\mathcal{F}(n_j))}{h^0(\mathcal{F}(n_i))},$$

then

$$\frac{\theta(W_{\mathcal{E}})}{\alpha(W_{\mathcal{E}})} \geq \frac{\theta(W_{\mathcal{F}})}{\alpha(W_{\mathcal{F}})}.$$

Definition 5.14. For $\tau = (P_1, \dots, P_s) \in \mathrm{HNT}(X, P)$ and a tuple $\underline{n} = (n_0, \dots, n_d)$, we let

$$\gamma_{\underline{n}}(\tau) := (d_{\underline{n}}(P_1), \dots, d_{\underline{n}}(P_s)).$$

Then, for τ as above and for $\underline{n} >> 0$ (depending on τ), the tuple $\gamma_{\underline{n}}(\tau)$ of dimension vectors of representations of $K_{\underline{n}}$ is a HN type with respect to $(\theta_{\underline{n}}(P), \alpha_{\underline{n}}(P))$.

Theorem 5.15. *For a HN type $\tau = (P_1, \dots, P_s)$ of sheaves on X with Hilbert polynomial P, we have, for $\underline{n} >> 0$, that*

$$\Phi_{\underline{n}}(\mathcal{C}oh_P^\tau(X)) \subset \mathcal{R}ep_{d_{\underline{n}}(P)}^{\gamma_{\underline{n}}(\tau)}(K_{\underline{n}})$$

where the HN stratification of $\mathcal{R}ep(K_{\underline{n}})$ is taken with respect to $(\theta_{\underline{n}}(P), \alpha_{\underline{n}}(P))$.

Proof. To choose $\underline{n}$, we first take n_0 sufficiently large so all semistable sheaves with Hilbert polynomial P_i are n_0-regular. We choose $n_1, \dots, n_d$ successively so, for $0 \leq j < l \leq d$, we have (i) for $i = 1, \dots, s$, Theorem 5.1 holds for P_i and $(n, m) = (n_j, n_l)$ and (ii) for $i = 1, \dots, s-1$, we have

$$\frac{P_i(n_j)}{P_i(n_k)} > \frac{P_{i+1}(n_j)}{P_{i+1}(n_l)}.$$

The proof is almost identical to that of Theorem 5.7, but uses Lemma 5.13. The only part that differs, retaining the notation of Theorem 5.7, is the proof that $W_i \cong \Phi_{\underline{n}}(\mathcal{E}_i)$ is $(\theta_{\underline{n}}(P), \alpha_{\underline{n}}(P))$-semistable. For any subrepresentation $W' \subset W_i$, we must check that

$$\theta_{\underline{n}}(P)(W')\alpha_{\underline{n}}(P)(W_i) - \theta_{\underline{n}}(P)(W_i)\alpha_{\underline{n}}(P)(W') \geq 0. \tag{16}$$

To establish this inequality, we use the fact that $\mathcal{E}_i$ is n_0-regular and semistable with Hilbert polynomial P_i. Given our choice of $\underline{n}$, we know, for $j < l$, that $\Phi_{n_j,n_l}(\mathcal{E}_i)$ is $\theta_{n_j,n_l}(P_i)$-semistable by Theorem 5.1. Hence, $\theta_{n_j,n_l}(P_i)(W'_{j,l}) \geq 0$ for any subrepresentation $W'_{j,l} \subset \Phi_{n_j,n_l}(\mathcal{E}_i)$. Finally to verify (16) it suffices to consider the subrepresentations $W'_{j,l}$ induced by W'. □

Remark 5.16. To determine a polynomial $P(x)$ of degree at most d, it suffices to know the value P takes at $d+1$ different values of x. Therefore, for $d \geq \dim X$, the map $P \mapsto d_{\underline{n}}(P)$ is injective for all tuples $\underline{n}$ of strictly increasing natural numbers. It follows that the mapping of HN types $\tau \mapsto \gamma_{\underline{n}}(\tau)$ is also injective for $d \geq \dim X$. For $d < \dim X$, the mapping on HN types is not injective (cf. Remark 5.8 and Lemma 4.22) and so two HN stratum for sheaves can be sent to the same quiver HN stratum. If we take $d \geq \dim X$, then this does not happen and, moreover, the above theorem could be used to investigate whether there exists a sheaf of a given HN type.

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