STRATIFICATIONS OF PARAMETER SPACES FOR COMPLEXES BY COHOMOLOGY TYPES

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Abstract

We study a collection of stability conditions (in the sense of Schmitt) for complexes of sheaves over a smooth complex projective variety indexed by a positive rational parameter. We show that the Harder–Narasimhan filtration of a complex for small values of this parameter encodes the Harder–Narasimhan filtrations of the cohomology sheaves of this complex. Finally we relate a stratification into locally closed subschemes of a parameter space for complexes associated to these stability parameters with the stratification by Harder–Narasimhan types.

1. Introduction

Let X be a smooth complex projective variety and $\mathcal{O}_X(1)$ be an ample invertible sheaf on X. We consider the moduli of (isomorphism classes of) complexes of sheaves on X, or equivalently moduli of Q-sheaves over X where Q is the quiver

$$\bullet \to \bullet \to \cdots\cdots \to \bullet \to \bullet$$

with relations imposed to ensure the boundary maps square to zero. Moduli of quiver sheaves have been studied in [1, 2, 6, 14]. There is a construction of moduli spaces of S-equivalence classes of 'semistable' complexes due to Schmitt [14] as a geometric invariant theory quotient of a reductive group G acting on a parameter space $\mathfrak{T}$ for complexes with fixed invariants. The notion of semistability is determined by a choice of stability parameters and the motivation comes from physics; it is closely related to a notion of semistability coming from a Hitchin–Kobayashi correspondence for quiver bundles due to Álvarez-Cónsul and García-Prada [2]. The stability parameters are also used to determine a linearisation of the action. The notion of S-equivalence is weaker than isomorphism and arises from the GIT construction of these moduli spaces which results in some orbits being collapsed.

As the notion of stability depends on a choice of parameters, we can ask if certain parameters reveal information about the cohomology sheaves of a complex. We show that there is a collection of stability parameters which can be used to study the cohomology sheaves of a complex. Analogously to the case of sheaves, every unstable complex has a unique maximally destabilising filtration known as its Harder–Narasimhan filtration. We give a collection of stability parameters indexed by a rational parameter $\epsilon > 0$ and show the Harder–Narasimhan filtration of a complex with respect to these parameters encodes the Harder–Narasimhan filtrations of the cohomology sheaves in this complex for ϵ sufficiently small. We then study a G-invariant stratification of the parameter space $\mathfrak{T}$ associated to these stability parameters.

Given an action of a reductive group G on a projective scheme B with respect to an ample linearisation $\mathcal{L}$, there is an associated stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of B into G-invariant locally closed subschemes for which the open stratum is the geometric invariant theory (GIT) semistable set B^{ss} [8, 12, 13]. The unstable strata have a description due to Hesselink [8] which make use of Kempf's notion of adapted 1-parameter subgroups (1-PSs) [11] as follows. The Hilbert–Mumford criterion allows us to determine which points are GIT semistable by studying the actions of 1-PSs; that is, nontrivial homomorphisms $\lambda : \mathbb{C}^* \to G$. It states

2010 Mathematics Subject Classification. Primary: 14L24, 14D20; Secondary: 14L30, 14F05.

Key words: geometric invariant theory, stratifications, complexes of sheaves.

that $b \in B$ is semistable if and only if it is for every 1-PS λ of G we have $\mu^{\mathcal{L}}(b,\lambda) \geq 0$ where $\mu^{\mathcal{L}}(b,\lambda)$ is equal to the weight of the $\mathbb{C}^*$-action induced by λ on the fibre of $\mathcal{L}$ over $\lim_{t\to 0}\lambda(t)\cdot b$. For an unstable point b, we want to measure how unstable this point is using the function μ. As $\mu^{\mathcal{L}}(b,\lambda^n) = n\mu^{\mathcal{L}}(b,\lambda)$, we pick a norm $||-||$ for 1-PS and use this to normalise the Hilbert Mumford function. We say λ is adapted to $b \in B - B^{ss}$ if

$$\frac{\mu^{\mathcal{L}}(b,\lambda)}{||\lambda||} = M^{\mathcal{L}}(b) := \min_{\lambda'} \frac{\mu^{\mathcal{L}}(b,\lambda')}{||\lambda'||}.$$

The indices β for the unstable strata correspond to rational 1-PSs λ_β (i.e. λ_β^n is a 1-PS) and S_β is the set of unstable points b such that a conjugate of λ_β is adapted to b and $M^{\mathcal{L}}(b) = -||\lambda_\beta||$.

We study the stratifications obtained in this way for the action of G on the parameter space $\mathfrak{T}$ for complexes with linearisation determined by the above collection of stability parameters. We show that for a given Harder–Narasimhan type τ, the GIT set up of the parameter scheme can be chosen so all sheaves with Harder–Narasimhan type τ are parametrised by a locally closed subscheme R_τ of the parameter space $\mathfrak{T}$. Moreover, R_τ is a union of connected components of a stratum $S_{\beta(\tau)}$ in the associated stratification. The scheme R_τ has the nice property that it parametrises complexes whose cohomology sheaves are of a fixed Harder–Narasimhan type.

The layout of this paper is as follows. In §2, we give a summary of the construction of Schmitt of moduli spaces of complexes and study the action of 1-PSs. In §3, we give the collection of stability conditions indexed by $\epsilon > 0$ and show that the Harder–Narasimhan filtration of a complex encodes the Harder–Narasimhan filtration of the cohomology sheaves for small ϵ. In §4, we study the associated GIT stratification of the parameter space for complexes and relate this to the stratification by Harder–Narasimhan types. Finally, in §5 we consider the problem of taking a quotient of the G-action on a Harder–Narasimhan stratum R_τ.

Notation and conventions. Throughout we let X be a smooth complex projective variety and $\mathcal{O}_X(1)$ be an ample invertible sheaf on X. All Hilbert polynomials $P(\mathcal{E})$ of sheaves $\mathcal{E}$ over X will be calculated with respect to $\mathcal{O}_X(1)$. We use the term complex to mean a bounded cochain complex of torsion free sheaves. We say a complex $\mathcal{E}^\cdot$ is concentrated in $[m_1, m_2]$ if $\mathcal{E}^i = 0$ for $i < m_1$ and $i > m_2$.

2. Schmitt's construction

In this section we summarise Schmitt's construction [14] of moduli space of S-equivalence classes of semistable complexes over X and calculate the weights of $\mathbb{C}^*$-actions.

If we have an isomorphism of complexes $\mathcal{E}^\cdot \cong \mathcal{F}^\cdot$, then for all i we have $\mathcal{E}^i \cong \mathcal{F}^i$ and so $P(\mathcal{E}^i) = P(\mathcal{F}^i)$. Hence we can fix a collection of Hilbert polynomials $P = (P^i)_{i\in\mathbb{Z}}$ such that $P^i = 0$ for all but finitely many i and study complexes with these invariants. In fact we can assume P is concentrated in $[m_1, m_2]$ and write $P = (P^{m_1}, \cdots, P^{m_2})$.

2.1. Semistability.

Schmitt introduces a notion of (semi)stability for complexes which depends on a collection of stability parameters $(\underline{\sigma}, \underline{\chi})$ where $\underline{\chi} := \delta\underline{\eta}$ and

- $\underline{\sigma} = (\sigma_i \in \mathbb{Z}_{>0})_{i\in\mathbb{Z}}$,
- $\underline{\eta} = (\eta_i \in \mathbb{Q})_{i\in\mathbb{Z}}$,
- δ is a positive rational polynomial such that $\deg\delta = \max(\dim X - 1, 0)$.

Definition 2.1. The reduced Hilbert polynomial of a complex $\mathcal{F}^\cdot$ with respect to $(\underline{\sigma}, \underline{\chi})$ is

$$P^{\text{red}}_{\underline{\sigma},\underline{\chi}}(\mathcal{F}^\cdot) := \frac{\sum_{i\in\mathbb{Z}} \sigma_i P(\mathcal{F}^i) - \chi_i \operatorname{rk}\mathcal{F}^i}{\sum_{i\in\mathbb{Z}} \sigma_i \operatorname{rk}\mathcal{F}^i}$$

where $P(\mathcal{F}^i)$ and $\operatorname{rk}\mathcal{F}^i$ are the Hilbert polynomial and rank of the sheaf $\mathcal{F}^i$.

A nonzero complex $\mathcal{F}^\cdot$ is $(\underline{\sigma}, \underline{\chi})$-semistable if for any nonzero proper subcomplex $\mathcal{E}^\cdot \subset \mathcal{F}^\cdot$ we have an inequality of polynomials

$$P^{\mathrm{red}}_{\underline{\sigma},\underline{\chi}}(\mathcal{E}^\cdot) \leq P^{\mathrm{red}}_{\underline{\sigma},\underline{\chi}}(\mathcal{F}^\cdot).$$

By an inequality of polynomials $R \leq Q$ we mean $R(x) \leq Q(x)$ for all $x >> 0$. We say the complex is $(\underline{\sigma}, \underline{\chi})$-stable if this inequality is strict for all such subcomplexes.

Remark 2.2. For any rational number q, if $\underline{\eta}' := \underline{\eta} - q\underline{\sigma}$ and $\underline{\chi}' := \delta\underline{\eta}'$, then the notions of $(\underline{\sigma}, \underline{\chi})$-semistability and $(\underline{\sigma}, \underline{\chi}')$-semistability agree. For invariants $P = (P^{m_1}, \cdots, P^{m_2})$, let

$$q = \frac{\sum_{i=m_1}^{m_2} \eta_i r^i}{\sum_{i=m_1}^{m_2} \sigma_i r^i}$$

where r^i is the rank determined by the leading coefficient of P^i. The associated stability parameters $(\underline{\sigma}, \underline{\chi}')$ for P satisfy $\sum_{i=m_1}^{m_2} \eta_i' r^i = 0$. We may assume our stability parameters satisfy $\sum_{i=m_1}^{m_2} \eta_i r^i = 0$ since P is fixed in this section.

2.2. **The parameter space.** The set of sheaves occurring in a $(\underline{\sigma}, \underline{\chi})$-semistable complex $\mathcal{E}^\cdot$ with invariants P is bounded by the usual arguments (see [15], Theorem 1.1) and so we may pick $n >> 0$ so that all these sheaves are n-regular. Fix complex vector spaces V^i of dimension $P^i(n)$ and let Q^i be the open subscheme of the quot scheme $\mathrm{Quot}(V^i \otimes \mathcal{O}_X(-n), P^i)$ consisting of torsion free quotient sheaves $q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i$ such that $H^0(q^i(n))$ is an isomorphism. The parameter scheme $\mathfrak{T}$ for $(\underline{\sigma}, \underline{\chi})$-semistable complexes with invariants P is constructed as a locally closed subscheme of a projective bundle $\mathfrak{D}$ over the product $Q := Q^{m_1} \times \cdots \times Q^{m_2}$.

Given a $(\underline{\sigma}, \underline{\chi})$-semistable complex $\mathcal{E}^\cdot$ with Hilbert polynomials P we can use the evaluation maps

$$H^0(\mathcal{E}^i(n)) \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i$$

along with a choice of isomorphism $V^i \cong H^0(\mathcal{E}^i(n))$ to parametrise the sheaf $\mathcal{E}^i$ by a point $q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i$ in Q^i. We can also construct a homomorphism

$$\psi := H^0(d(n)) \circ (\oplus_i H^0(q^i(n))) : \oplus_i V^i \to \oplus_i H^0(\mathcal{E}^i(n))$$

where $d : \oplus_i \mathcal{E}^i \to \oplus_i \mathcal{E}^i$ is the morphism determined by the boundary maps $d^i : \mathcal{E}^i \to \mathcal{E}^{i+1}$. Such homomorphisms ψ correspond to points in the fibres of the sheaf

$$\mathcal{R} := (\oplus_i V^i)^\vee \otimes p_* \Big(\mathcal{U} \otimes (\pi_X^{Q\times X})^* \mathcal{O}_X(n)\Big)$$

over Q where $p : Q \times X \to Q$ is the projection and $\oplus_i V^i \otimes (\pi_X^{Q\times X})^* \mathcal{O}_X(-n) \to \mathcal{U}$ is the quotient sheaf over $Q \times X$ given by taking the direct sum of the pullbacks of the universal quotients $V^i \otimes (\pi_X^{Q^i\times X})^* \mathcal{O}_X(-n) \to \mathcal{U}^i$ on $Q^i \times X$ to $Q \times X$. Note that $\mathcal{R}$ is locally free for n sufficiently large and so we can consider the projective bundle $\mathfrak{D} := \mathbb{P}(\mathcal{R} \oplus \mathcal{O}_Q)$ over Q.

A point of $\mathfrak{D}$ over $q = (q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i)_i \in Q$ is given by a nonzero pair $(\psi : \oplus_i V^i \to \oplus_i H^0(\mathcal{E}^i(n)), \zeta \in \mathbb{C})$ defined up to scalar multiplication. The parameter scheme $\mathfrak{T}$ consists of points $(q, [\psi : \zeta])$ in $\mathfrak{D}$ such that:

i) $\psi = H^0(d(n)) \circ (\oplus_i H^0(q^i(n)))$ where the homomorphism $d : \oplus_i \mathcal{E}^i \to \oplus_i \mathcal{E}^i$ is uniquely determined by sheaf homomorphisms $d^i : \mathcal{E}^i \to \mathcal{E}^{i+1}$ which satisfy $d^i \circ d^{i-1} = 0$,

ii) $\zeta \neq 0$.

The conditions given in i) are all closed (they are cut out by the vanishing locus of homomorphisms of locally free sheaves) and condition ii) is open; therefore $\mathfrak{T}$ is a locally closed subscheme of $\mathfrak{D}$. We let $\mathfrak{D}'$ denote the closed subscheme of $\mathfrak{D}$ given by points which satisfy condition i).

Remark 2.3. The construction of the parameter scheme $\mathfrak{T}$ depends on the choice of n and the Hilbert polynomials P; we write $\mathfrak{T}_P$ or $\mathfrak{T}(n)$ if we wish to emphasise this dependence.

2.3. **The group action.** For $m_1 \leq i \leq m_2$ we have fixed vector spaces V^i of dimension $P^i(n)$. The reductive group $\Pi_i \mathrm{GL}(V^i)$ acts on both Q and $\mathfrak{D}$: if $g = (g_{m_1}, \dots, g_{m_2}) \in \Pi_i \mathrm{GL}(V^i)$ and $z = ((q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i)_i, [\psi : \zeta]) \in \mathfrak{D}$, then

$$g \cdot z = ((g_i \cdot q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i)_i, [g \cdot \psi : \zeta])$$

where

$$g_i \cdot q^i : \qquad V^i \otimes \mathcal{O}_X(-n) \xrightarrow{g_i^{-1} \cdot} V^i \otimes \mathcal{O}_X(-n) \xrightarrow{q^i} \mathcal{E}^i$$

and

$$g \cdot \psi : \qquad \oplus_i V^i \xrightarrow{g^{-1} \cdot} \oplus_i V^i \xrightarrow{\psi} \oplus_i H^0(\mathcal{E}^i(n)).$$

If instead we consider $\tilde{\psi} := \oplus_i H^0(q^i(n))^{-1} \circ \psi : \oplus_i V^i \to \oplus_i V^i$ then this action corresponds to conjugating $\tilde{\psi}$ by g; that is,

$$g \circ \tilde{\psi} \circ g^{-1} = \widetilde{g \cdot \psi}.$$

This action preserves the parameter scheme $\mathfrak{T}$ and the orbits correspond to isomorphism classes of complexes. As the subgroup $\mathbb{C}^*(I_{V_{m_1}}, \dots, I_{V_{m_2}})$ acts trivially on $\mathfrak{D}$, we are interested in the action of $(\Pi_i \mathrm{GL}(V^i))/\mathbb{C}^*$. Given integers $\underline{\sigma} = (\sigma_{m_1}, \dots, \sigma_{m_2})$ we can define a character

$$\begin{array}{cccc} \det_{\underline{\sigma}} : & \Pi_i \mathrm{GL}(V^i) & \to & \mathbb{C}^* \\ & (g_i) & \mapsto & \Pi_i \det g_i^{\sigma_i} \end{array}$$

and instead consider the action of the group $G = G_{\underline{\sigma}} := \ker \det_{\underline{\sigma}}$ which maps with finite kernel onto $(\Pi_i \mathrm{GL}(V_i))/\mathbb{C}^*$.

2.4. **The linearisation.** Schmitt uses the stability parameters $(\underline{\sigma}, \underline{\chi}) := (\underline{\sigma}, \delta \underline{\eta})$ to determine a linearisation of the G-action on the parameter space $\mathfrak{T}$ in three steps. We note that the exact details of the linearisation are only needed for the calculations in §2.6. The first step is to construct an equivariant morphism from $\mathfrak{D}$ to another projective bundle $\mathfrak{B}_{\underline{\sigma}}$ over Q. The parameters $\underline{\sigma}$ are used to associate to each point $z = (q, [\psi : \zeta]) \in \mathfrak{D}$ a nonzero decoration

$$\varphi_{\underline{\sigma}}(z) : (V_{\underline{\sigma}}^{\otimes r_{\underline{\sigma}}})^{\oplus 2} \otimes \mathcal{O}_X(-r_{\underline{\sigma}} n) \to \det \mathcal{E}_{\underline{\sigma}}$$

(up to scalar multiplication) where $r_{\underline{\sigma}} = \sum_i \sigma_i r^i$ and $V_{\underline{\sigma}} := \oplus_i (V^i)^{\oplus \sigma_i}$ and $\mathcal{E}_{\underline{\sigma}} := \oplus_i (\mathcal{E}^i)^{\oplus \sigma_i}$. The fibre of $\mathfrak{B}_{\underline{\sigma}}$ over $q \in Q$ parametrises such homomorphisms $\varphi_{\underline{\sigma}}$ up to scalar multiplication and the morphism $\mathfrak{D} \to \mathfrak{B}_{\underline{\sigma}}$ is given by sending $z = (q, [\psi : \zeta]) \in \mathfrak{D}$ to $(q, [\varphi_{\underline{\sigma}}(z)]) \in \mathfrak{B}_{\underline{\sigma}}$. The group $G \cong \mathrm{SL}(V_{\underline{\sigma}}) \cap \Pi_i \mathrm{GL}(V^i)$ acts on $\mathfrak{B}_{\underline{\sigma}}$ by acting on Q and $V_{\underline{\sigma}}$ and $\mathfrak{D} \to \mathfrak{B}_{\underline{\sigma}}$ is equivariant with respect to this action.

The second step is given by constructing a projective equivariant embedding $\mathfrak{B}_{\underline{\sigma}} \to B_{\underline{\sigma}}$. This embedding is essentially given by Gieseker's embedding [5] of Q^i into a projective bundle B_i over the components R_i of the Picard scheme of X which contain the determinant of a sheaf $\mathcal{E}^i$ parametrised by Q^i. The embedding sends a quotient sheaf $q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i$ to a homomorphism $\wedge^{r^i} V^i \to H^0(\det \mathcal{E}^i(r^i n))$ which represents a point in a projective bundle B_i over R_i. In a similar way Schmitt also constructs an equivariant morphism $\mathfrak{B}_{\underline{\sigma}} \to B'_{\underline{\sigma}}$ where $B'_{\underline{\sigma}}$ is a projective bundle over the product $\Pi_i R_i$. Let $B_{\underline{\sigma}} = B_{m_1} \times \cdots \times B_{m_2} \times B'_{\underline{\sigma}}$; then Schmitt shows the map $\mathfrak{B}_{\underline{\sigma}} \to B_{\underline{\sigma}}$ is equivariant, injective and proper morphism.

The final step is to choose a linearisation on $B_{\underline{\sigma}}$ and pull this back to $\mathfrak{T}$ via

$$\mathfrak{T} \hookrightarrow \mathfrak{D} \hookrightarrow \mathfrak{B}_{\underline{\sigma}} \hookrightarrow B_{\underline{\sigma}} = B_{m_1} \times \cdots \times B_{m_2} \times B'_{\underline{\sigma}}.$$

The schemes B_i and $B'_{\underline{\sigma}}$ have natural ample linearisations given by $\mathcal{L}_i := \mathcal{O}_{B_i}(1)$ and $\mathcal{L}' := \mathcal{O}_{B'_{\underline{\sigma}}}(1)$. The linearisation on $B_{\underline{\sigma}}$ is given by taking a weighted tensor product of these linearisations and twisting by a character ρ of $G = G_{\underline{\sigma}}$. The character $\rho : G \to \mathbb{C}^*$ is the character determined by the rational numbers

$$c_i := \sigma_i \left(\frac{P_{\underline{\sigma}}(n)}{r_{\underline{\sigma}} \delta(n)} - 1 \right) \left(\frac{r_{\underline{\sigma}}}{P_{\underline{\sigma}}(n)} - \frac{r^i}{P^i(n)} \right) - \frac{r^i \eta_i}{P^i(n)}$$

where $P_{\underline{\sigma}} := \sum_i \sigma_i P^i$; that is, if these are integral we define

$$\rho(g_{m_1}, \cdots, g_{m_2}) = \Pi_{i=m_1}^{m_2} \det g_i^{c_i}$$

and if not we can scale everything by a positive integer so that they become integral. We assume n is sufficiently large so that $a_i = \sigma_i(P_{\underline{\sigma}}(n) - r_{\underline{\sigma}}\delta(n))/r_{\underline{\sigma}}\delta(n) + \eta_i$ is positive; these positive rational numbers $\underline{a} = (a_{m_1}, \dots, a_{m_2}, 1)$ are used to define a very ample linearisation

$$\mathcal{L}_{\underline{a}} := \bigotimes_i \mathcal{L}_i^{\otimes a_i} \otimes \mathcal{L}$$

on $B_{\underline{\sigma}}$ (where again if the a_i are not integral we scale everything so that this is the case). The linearisation $\mathcal{L} = \mathcal{L}(\underline{\sigma}, \underline{\chi})$ on $\mathfrak{T}$ is equal to the pullback of the very ample linearisation $\mathcal{L}_{\underline{a}}^{\rho}$ on $B_{\underline{\sigma}}$ where $\mathcal{L}_{\underline{a}}^{\rho}$ denotes the linearisation obtained by twisting $\mathcal{L}_{\underline{a}}$ by the character ρ.

2.5. **The moduli space.** The moduli space of $(\underline{\sigma}, \underline{\chi})$-semistable complexes with invariants P is constructed as an open subscheme of the projective GIT quotient

$$\mathfrak{D}' /\!/_{\mathcal{L}} G$$

given by the locus where $\zeta \neq 0$ (by definition $\mathfrak{T}$ is the open subscheme of $\mathfrak{D}'$ given by this condition).

Definition 2.4. A Jordan–Hölder filtration of a $(\underline{\sigma}, \underline{\chi})$-semistable complex $\mathcal{E}^{\cdot}$ is a filtration by subcomplexes

$$0_{\cdot} = \mathcal{E}^{\cdot}_{[0]} \subsetneq \mathcal{E}^{\cdot}_{[1]} \subsetneq \cdots \subsetneq \mathcal{E}^{\cdot}_{[k]} = \mathcal{E}^{\cdot}$$

such that the successive quotients $\mathcal{E}^{\cdot}_{[i]}/\mathcal{E}^{\cdot}_{[i-1]}$ are $(\underline{\sigma}, \underline{\chi})$-stable with reduced Hilbert polynomial equal to $P^{\mathrm{red}}_{\underline{\sigma},\underline{\chi}}(\mathcal{E}^{\cdot})$. This filtration is in general not canonical but the associated graded object

$$\mathrm{gr}_{(\underline{\sigma},\underline{\chi})}(\mathcal{E}^{\cdot}) := \bigoplus_{j=1}^{k} \mathcal{E}^{\cdot}_{[j]}/\mathcal{E}^{\cdot}_{[j-1]}$$

is canonically associated to $\mathcal{E}^{\cdot}$ up to isomorphism. We say that two $(\underline{\sigma}, \underline{\chi})$-semistable complexes are S-equivalent if their associated graded objects are isomorphic.

Theorem 2.5. *([14], p3) Let X be a smooth complex manifold, P be a collection of Hilbert polynomials of degree* $\dim X$ *and $(\underline{\sigma}, \underline{\chi})$ be stability parameters. There is a quasi-projective coarse moduli space*

$$M^{(\underline{\sigma},\underline{\chi})-ss}(X, P)$$

for S-equivalence classes of $(\underline{\sigma}, \underline{\chi})$-semistable complexes over X with Hilbert polynomials P.

2.6. **The Hilbert-Mumford criterion.** In this section we calculate the weights of $\mathbb{C}^*$-actions given by 1-PSs of $G = G_{\underline{\sigma}}$ on the parameter space $\mathfrak{T}$ for complexes (see also [14] Section 2.1).

We first study the limit of points in $\mathfrak{T}$ under the action of a 1-PS $\lambda : \mathbb{C}^* \to G$. For this limit to exist we need to work with a projective completion $\overline{\mathfrak{T}}$ of $\mathfrak{T}$. This projective completion is constructed as a closed subscheme of a projective bundle $\overline{\mathfrak{D}}$ over the projective scheme $\overline{Q} := \Pi_i \overline{Q}^i$ where $\overline{Q}^i$ is the closure of Q^i in the relevant quot scheme. The points of $\overline{\mathfrak{D}}$ over $q = (q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i)_i \in \overline{Q}$ are nonzero pairs $[\psi : \zeta]$ defined up to scalar multiplication where $\psi : \oplus_i V^i \to \oplus_i H^0(\mathcal{E}^i(n))$ and $\zeta \in \mathbb{C}$. Then $\overline{\mathfrak{T}}$ is the subscheme of points $(q, [\psi : \zeta]) \in \overline{\mathfrak{D}}$ such that $\psi = H^0(d(n)) \circ (\oplus_i H^0(q^i(n)))$ where $d : \oplus_i \mathcal{E}^i \to \oplus_i \mathcal{E}^i$ is determined by $d^i : \mathcal{E}^i \to \mathcal{E}^{i+1}$ which satisfy $d^i \circ d^{i-1} = 0$.

For us, $G \cong \mathrm{SL}(V_{\underline{\sigma}}) \cap \Pi_i \mathrm{GL}(V^i)$ where $V_{\underline{\sigma}} = \oplus_i (V^i)^{\oplus \sigma_i}$ and so a 1-PS $\lambda : \mathbb{C}^* \to G$ is given by a collection of 1-PSs $\lambda_i : \mathbb{C}^* \to \mathrm{GL}(V^i)$ which satisfy

$$\Pi_i \det \lambda_i(t)^{\sigma_i} = 1.$$

A 1-PS λ_i of $\mathrm{GL}(V^i)$ induces a $\mathbb{C}^*$-action on V^i and so we obtain weights $k_1 > \cdots > k_s$ and a weight space decomposition $V^i = \oplus_{j=1}^s V^i_j$ where $V^i_j = \{v \in V^i : \lambda_i(t) \cdot v = t^{k_j} v\}$. This gives a filtration

$$0 \subsetneq V^i_{(1)} \subsetneq \cdots \subsetneq V^i_{(s)} = V^i \tag{1}$$

where $V^i_{(j)} := V^i_1 \oplus \cdots \oplus V^i_j$ and if we take a basis of V^i which is compatible with (1), then

$$\lambda_i(t) = \begin{pmatrix} t^{k_1} I_{V^i_1} & & \\ & \ddots & \\ & & t^{k_s} I_{V^i_s} \end{pmatrix}$$

is diagonal. We diagonalise each of these 1-PSs λ_i simultaneously so there are weights $k_1 > \cdots > k_s$, decompositions $V^i = \oplus_{j=1}^s V^i_j$ (where we may have $V^i_j = 0$ for some j) and filtrations

$$0 \subset V^i_{(1)} \subset V^i_{(2)} \subset \cdots \subset V^i_{(s)} = V^i$$

with respect to which λ_i is diagonal for each i.

Let $z = (q, [\psi : 1])$ be a point in $\mathfrak{T}$ where $q = (q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i)_i \in Q$; then we can consider its limit

$$\overline{z} := \lim_{t \to 0} \lambda(t) \cdot z = (\overline{q}, [\overline{\psi} : \overline{\zeta}])$$

under the 1-PS λ. By [10] Lemma 4.4.3,

$$\overline{q}^i := \lim_{t \to 0} \lambda_i(t) \cdot q^i = \oplus_{j=1}^s q^i_j : \oplus_{j=1}^s V^i_j \otimes \mathcal{O}_X(-n) \to \oplus_{j=1}^s \mathcal{E}^i_j$$

where $\mathcal{E}^i_j$ are the successive quotients in the filtration

$$0 \subset \mathcal{E}^i_{(1)} \subset \cdots \subset \mathcal{E}^i_{(j)} := q^i(V^i_{(j)} \otimes \mathcal{O}_X(-n)) \subset \cdots \subset \mathcal{E}^i_{(s)} = \mathcal{E}^i$$

induced by λ_i. If the boundary maps of the complex preserve these sheaf filtrations we say that λ induces a filtration of the point z (or corresponding complex $\mathcal{E}^\cdot$) by subcomplexes.

Lemma 2.6. *Let z and $\overline{z}$ be as above; then $\overline{\psi}$ is determined by $\overline{d}^i : \oplus_j \mathcal{E}^i_j \to \oplus_j \mathcal{E}^{i+1}_j$ and:*

i) *If λ induces a filtration of z by subcomplexes, then $\overline{d}^i = \oplus_{j=1}^s (d^i_j : \mathcal{E}^i_j \to \mathcal{E}^{i+1}_j)$ and $\overline{\zeta} \neq 0$. In particular $\overline{z} \in \mathfrak{T}$ and the corresponding complex is the graded complex associated to the filtration induced by λ.*

ii) *Otherwise, $\overline{\zeta} = 0$ and we have that $\overline{d}^i(\mathcal{E}^i_l) \cap \mathcal{E}^{i+1}_j = 0$ unless $k_l - k_j = N$ where $N := \min_{i,j,l}\{k_l - k_j : d^i(\mathcal{E}^i_{(l)}) \nsubseteq \mathcal{E}^{i+1}_{(j-1)}\} < 0$. In particular, $\overline{z} \notin \mathfrak{T}$.*

Proof. For the proof, we study the action of λ on

$$A^i := H^0(q^{i+1}(n)) \circ H^0(d^i(n)) \circ H^0(q^i(n)) : V^i \to V^{i+1}.$$

If we write $A^i = (A^i_{jl})$ where $A^i_{jl} : V^i_l \to V^{i+1}_j$, then the weight of λ acting on A^i_{jl} is $k_l - k_j$. If λ induces a filtration of z by subcomplexes then the matrices A^i are all block upper triangular and so λ acts with positive weights and the limit of $\lambda(t) \cdot A^i$ as $t \to 0$ is a block diagonal matrix consisting of the blocks A^i_{jj} on which λ acts with weight zero. If λ does not induce a filtration by subcomplexs then λ acts on some A^i_{jl} with a negative weight. The smallest weight is then the N defined in ii) above and as we are working projectively we can multiply everything (including ζ) by t^{-N} and take the limit as $t \to 0$ to prove ii). □

Remark 2.7. Let $z = (q, [\psi : \zeta])$ be a point in $\overline{\mathfrak{T}}$ given by $q = (q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i)_i \in \overline{Q}$ and $\psi = H^0(d(n)) \circ \oplus_i H^0(q^i(n))$ where d is defined by homomorphisms $d^i : \mathcal{E}^i \to \mathcal{E}^{i+1}$. If z is fixed by λ, then $q^i = \oplus_j q^i_j : V^i \otimes \mathcal{O}_X(-n) \to \oplus_j \mathcal{E}^i_j$ and $d^i = \oplus_{l,j} d^i_{l,j}$ where $d^i_{l,j} : \mathcal{E}^i_j \to \mathcal{E}^i_l$. The fixed point locus of a $\lambda(\mathbb{C}^*)$ acting on $\overline{\mathfrak{T}}$ decomposes into 3 pieces (each piece being a union of connected components):

- A diagonal piece consisting of points z where $d^i = \oplus_j d^i_{j,j}$ is diagonal for all i and $\zeta \in \mathbb{C}$.
- A strictly lower triangular piece consisting of points z where $d^i = \oplus_{j<l} d^i_{l,j}$ is strictly lower triangular for all i and $\zeta = 0$.
- A strictly upper triangular piece consisting of points z where $d^i = \oplus_{j>l} d^i_{l,j}$ is strictly lower triangular for all i and $\zeta = 0$.

If $z \in \mathfrak{T}$, then $\overline{z} = \lim_{t\to 0} \lambda(t) \cdot z$ lies in either the diagonal or strictly lower triangular piece by Lemma 2.6. In fact, $\overline{z} \in \mathfrak{T}$ if and only if λ induces a filtration of z by subcomplexes.

The Hilbert-Mumford function $\mu^{\mathcal{L}}(z, \lambda)$ is by definition the weight of the $\lambda(\mathbb{C}^*)$-action on the fibre of $\mathcal{L}$ over $\overline{z} := \lim_{t\to 0} \lambda(t) \cdot z$. By the construction of $\mathcal{L}$ this is

$$\mu^{\mathcal{L}}(z, \lambda) = \mu^{\mathcal{L}'}(\varphi_{\underline{\sigma}}(z), \lambda) + \sum_{i=m_1}^{m_2} a_i \mu^{\mathcal{L}_i}(q^i, \lambda_i) - \rho \cdot \lambda \tag{2}$$

where $\varphi_{\underline{\sigma}}(z)$ is the decoration associated to z and a_i and ρ are the rational numbers and character used to define $\mathcal{L}$ (c.f. §2.4). Let $P_{\underline{\sigma}} = \sum_i \sigma_i P^i$ and $r_{\underline{\sigma}} = \sum_i \sigma_i r^i$.

Lemma 2.8. *(see also* [14]*) Let λ be a 1-PS of G and let $z = (q, [\psi : 1]) \in \mathfrak{T}$ as above; then*

i) *If λ induces a filtration of z by subcomplexes*
$$\mu^{\mathcal{L}}(z, \lambda) = \sum_{i=m_1}^{m_2} \sum_{j=1}^{s} k_j \left(\sigma_i \frac{P_{\underline{\sigma}}(n)}{r_{\underline{\sigma}}\delta(n)} + \eta_i \right) \operatorname{rk} \mathcal{E}^i_j$$
where $\mathcal{E}^i_j = \mathcal{E}^i_{(j)}/\mathcal{E}^i_{(j-1)}$ and $\mathcal{E}^i_{(j)} = q^i(V^i_{(j)} \otimes \mathcal{O}_X(-n))$.

ii) *If λ does not induce a filtration of z by subcomplexes*
$$\mu^{\mathcal{L}}(z, \lambda) = \sum_{i=m_1}^{m_2} \sum_{j=1}^{s} k_j \left(\sigma_i \frac{P_{\underline{\sigma}}(n)}{r_{\underline{\sigma}}\delta(n)} + \eta_i \right) \operatorname{rk} \mathcal{E}^i_j - N$$
where N is the negative integer given in Lemma 2.6.

Proof. The weight of the action of λ_i on Q^i with respect to $\mathcal{L}_i$ is calculated in [5]:
$$\mu^{\mathcal{L}_i}(q^i, \lambda_i) = \sum_{j=1}^{s} k_j \left(\operatorname{rk} \mathcal{E}^i_j - \dim V^i_j \frac{r^i}{P^i(n)} \right).$$
As λ is a 1-PS of $\mathrm{SL}(\oplus_i (V^i)^{\oplus \sigma_i})$, the equation (2) becomes
$$\mu^{\mathcal{L}}(z, \lambda) = \mu^{\mathcal{L}'}(\varphi_{\underline{\sigma}}(z), \lambda) + \sum_{i=m_1}^{m_2} \sum_{j=1}^{s} k_j \left(\sigma_i \frac{P_{\underline{\sigma}}(n)}{r_{\underline{\sigma}}\delta(n)} - \sigma_i + \eta_i \right) \operatorname{rk} \mathcal{E}^i_j.$$
Finally, by studying the construction of the decoration $\varphi_{\underline{\sigma}}(z)$ associated to z (for details see [14]), we see that
$$\mu^{\mathcal{L}'}(\varphi_{\underline{\sigma}}(z), \lambda) = \begin{cases} \sum_{i=m_1}^{m_2} \sum_{j=1}^{s} k_j \sigma_i \operatorname{rk} \mathcal{E}^i_j & \text{if } \lambda \text{ induces a filtration by subcomplexes} \\ \sum_{i=m_1}^{m_2} \sum_{j=1}^{s} k_j \sigma_i \operatorname{rk} \mathcal{E}^i_j - N & \text{otherwise} \end{cases}$$
where N is the negative integer of Lemma 2.6. □

Remark 2.9. Schmitt observes that we can rescale the stability parameters by picking a sufficiently large integer K and replacing $(\delta, \underline{\eta})$ with $(K\delta, \underline{\eta}/K)$, so that for GIT semistability we need only worry about 1-PSs which induce filtrations by subcomplexes (cf. [14], Theorem 1.7.1). This explains why subcomplexes are the test objects for (semi)stability in Definition 2.1 rather than weighted sheaf filtrations.

3. Stability conditions relating to cohomology

Throughout this section we fix strictly increasing rational numbers η_k and a positive rational polynomial δ of degree $\max(\dim X - 1, 0)$ and consider the collection of stability conditions $(\underline{1}, \delta\underline{\eta}/\epsilon)$ indexed by a small positive rational number ϵ. For a complex $\mathcal{F}^\cdot$ with torsion free cohomology sheaves

$$H^i(\mathcal{F}^\cdot) := \ker d^i / \operatorname{Im} d^{i-1},$$

we show that the Harder–Narasimhan filtration of $\mathcal{F}^\cdot$ encodes the Harder–Narasimhan filtration of the cohomology sheaves in this complex when $\epsilon > 0$ is sufficiently small.

3.1. **Harder–Narasimhan filtrations.** The Harder–Narasimhan filtration (HN filtration) of a complex $\mathcal{F}^\cdot$ with respect to $(\underline{\sigma}, \underline{\chi})$ is a filtration by subcomplexes

$$0^\cdot = \mathcal{F}^\cdot_{(0)} \subsetneq \mathcal{F}^\cdot_{(1)} \subsetneq \cdots \subsetneq \mathcal{F}^\cdot_{(s)} = \mathcal{F}^\cdot$$

such that the successive quotients $\mathcal{F}^\cdot_j = \mathcal{F}^\cdot_{(j)}/\mathcal{F}^\cdot_{(j-1)}$ are complexes of torsion free sheaves which are $(\underline{\sigma}, \underline{\chi})$-semistable and have decreasing reduced Hilbert polynomials with respect to $(\underline{\sigma}, \underline{\chi})$. The Harder–Narasimhan type of $\mathcal{F}^\cdot$ (with respect to $(\underline{\sigma}, \underline{\chi})$) is given by $\tau = (P_1, \cdots P_s)$ where $P_j = (P^i_j)_{i \in \mathbb{Z}}$ is the tuple of Hilbert polynomials of the complex $\mathcal{F}^\cdot_j$ so that

$$P^i_j := P(\mathcal{F}^i_j) = P(\mathcal{F}^i_{(j)}/\mathcal{F}^i_{(j-1)}).$$

A subcomplex $\mathcal{F}^\cdot_1 \subset \mathcal{F}^\cdot$ is a maximal destabilising subcomplex with respect to $(\underline{\sigma}, \underline{\chi})$ if

i) The complex $\mathcal{F}^\cdot_1$ is $(\underline{\sigma}, \underline{\chi})$-semistable,
ii) For every subcomplex $\mathcal{E}^\cdot$ of $\mathcal{F}^\cdot$ such that $\mathcal{F}^\cdot_1 \subsetneq \mathcal{E}^\cdot$ we have

$$P^{\mathrm{red}}_{\underline{\sigma},\underline{\chi}}(\mathcal{F}^\cdot_1) > P^{\mathrm{red}}_{\underline{\sigma},\underline{\chi}}(\mathcal{E}^\cdot).$$

The existence and uniqueness of the maximal destabilising subcomplex follows in exactly the same way as the original proof for vector bundles of Harder and Narasimhan [7] and the HN filtration can be constructed inductively from the maximal destabilising subcomplex.

3.2. **The limit as ϵ tends to zero.** We consider the limit as ϵ tends to zero of the collection of stability conditions $(\underline{1}, \delta\underline{\eta}/\epsilon)$. The inequality

$$P^{\mathrm{red}}_{\underline{1},\delta\underline{\eta}/\epsilon}(\mathcal{E}^\cdot) \leq P^{\mathrm{red}}_{\underline{1},\delta\underline{\eta}/\epsilon}(\mathcal{F}^\cdot)$$

is equivalent to

$$\epsilon \frac{\sum_i P(\mathcal{E}^i)}{\sum_i \operatorname{rk} \mathcal{E}^i} - \delta \frac{\sum_i \eta_i \operatorname{rk} \mathcal{E}^i}{\sum_i \operatorname{rk} \mathcal{E}^i} \leq \epsilon \frac{\sum_i P(\mathcal{F}^i)}{\sum_i \operatorname{rk} \mathcal{F}^i} - \delta \frac{\sum_i \eta_i \operatorname{rk} \mathcal{F}^i}{\sum_i \operatorname{rk} \mathcal{F}^i}.$$

If we take the limit as $\epsilon \to 0$ we obtain

$$\frac{\sum_i \eta_i \operatorname{rk} \mathcal{E}^i}{\sum_i \operatorname{rk} \mathcal{E}^i} \geq \frac{\sum_i \eta_i \operatorname{rk} \mathcal{F}^i}{\sum_i \operatorname{rk} \mathcal{F}^i}. \tag{3}$$

We say $\mathcal{F}^\cdot$ is $(\underline{0}, \delta\underline{\eta})$-semistable if every nonzero proper subcomplexes $\mathcal{E}^\cdot \subset \mathcal{F}^\cdot$ satisfies (3). This is a generalisation of the parameters consider by Schmitt where we allow $\sigma_i = 0$. These generalised stability parameters no longer define an ample linearisation on the parameter space (cf. §2.4), but we can still study the corresponding notion of semistability.

Lemma 3.1. *The only $(\underline{0}, \delta\underline{\eta})$-semistable complexes are shifts of torsion free sheaves and complexes which are isomorphic to a shift of the cone of the identity morphism of torsion free sheaves.*

Proof. It is easy to verify that a shift of a torsion free sheaf or the cone of the identity morphism of a torsion free sheaf is $(\underline{0}, \delta\underline{\eta})$-semistable. Now suppose $\mathcal{F}^\cdot$ is $(\underline{0}, \delta\underline{\eta})$-semistable. If all the boundary morphisms d^i are zero, then each nonzero sheaf $\mathcal{F}^k$ is both a subcomplex and quotient complex and so by semistability

$$\eta_k = \frac{\sum_i \eta_i \operatorname{rk} \mathcal{F}^i}{\sum_i \operatorname{rk} \mathcal{F}^i}.$$

As η_i are strictly increasing, there can be at most one k such that $\mathcal{F}^k$ is nonzero. Alternatively if there is a nonzero d^k, then the image of this boundary map can be viewed as a quotient complex (in position k) and a subcomplex (in position $k+1$) so that

$$\eta_k \leq \frac{\sum \eta_i \operatorname{rk} \mathcal{F}^i}{\sum \operatorname{rk} \mathcal{F}^i} \leq \eta_{k+1}. \tag{4}$$

Hence there can be at most one k such that d^k is nonzero. From above, we see that $\mathcal{F}^i = 0$ unless $i = k$ or $k+1$. We also see that the inequalities of (4) must be strict as $\eta_k \neq \eta_{k+1}$. The kernel and cokernel of d^k may be considered as a subcomplex and quotient complex respectively and by comparing the inequalities obtained from semistability with (4), we conclude that d^k is an isomorphism. □

Lemma 3.2. *Let k be the minimal integer for which the sheaf $\mathcal{F}^k$ in a complex $\mathcal{F}^\cdot$ is nonzero. Then the maximal destabilising subcomplex $\mathcal{F}^\cdot_{(1)} \subset \mathcal{F}^\cdot$ with respect to $(\underline{0}, \delta\underline{\eta})$ is*

$$\mathcal{F}^\cdot_{(1)} = \begin{cases} \cdots \to 0 \to \ker d^k \to 0 \to 0 \to \cdots & \text{if } \ker d^k \neq 0, \\ \cdots \to 0 \to \mathcal{F}^k \to \operatorname{Im} d^k \to 0 \to \cdots & \text{if } \ker d^k = 0. \end{cases}$$

Proof. By Lemma 3.1 both these complexes are $(\underline{0}, \delta\underline{\eta})$-semistable; thus it remains to show that whenever $\mathcal{F}^\cdot_{(1)} \subsetneq \mathcal{E}^\cdot \subset \mathcal{F}^\cdot$, we have an inequality

$$\frac{\sum_i \eta_i \operatorname{rk} \mathcal{E}^i}{\sum \operatorname{rk} \mathcal{E}^i} > \frac{\sum_i \eta_i \operatorname{rk} \mathcal{F}^i_{(1)}}{\sum \operatorname{rk} \mathcal{F}^i_{(1)}} = \begin{cases} \eta_k & \text{if } \ker d^k \neq 0 \\ (\eta_k + \eta_{k+1}) \operatorname{rk} \mathcal{F}^k/2 & \text{if } \ker d^k = 0. \end{cases} \tag{5}$$

As $\mathcal{E}^\cdot \neq \mathcal{F}^\cdot_{(1)}$, the set $I := \{i \in \mathbb{Z} : \mathcal{E}^i \neq \mathcal{F}^i_{(1)}\}$ is nonempty and if $i \in I$, then $\mathcal{E}^i \neq 0$.

We only prove the case for when $\ker d^k \neq 0$ as the case when $\ker d^k = 0$ is similar. If $k \in I$, then $\ker d^k \subsetneq \mathcal{E}^k$ and so $0 \neq d(\mathcal{E}^k) \subset \mathcal{E}^{k+1}$. Thus $k+1 \in I$ and $\eta_k \operatorname{rk} \mathcal{E}^k + \eta_{k+1} \operatorname{rk} \mathcal{E}^{k+1} > \eta_k(\operatorname{rk} \mathcal{E}^k + \operatorname{rk} \mathcal{E}^{k+1})$. In particular,

$$\sum_{i \in I} \eta_i \operatorname{rk} \mathcal{E}^i > \eta_k \sum_{i \in I} \operatorname{rk} \mathcal{E}^i \quad \text{and} \quad \sum_{i \notin I} \eta_i \operatorname{rk} \mathcal{E}^i = \eta_k \sum_{i \notin I} \operatorname{rk} \mathcal{E}^i$$

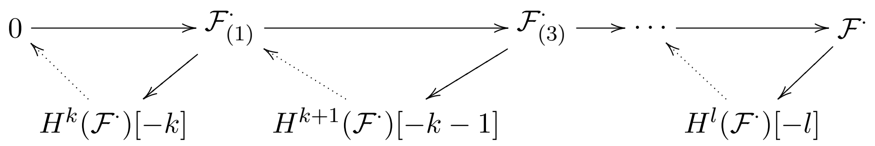
which proves (5) when $\ker d^k \neq 0$. □

Corollary 3.3. *If $\mathcal{F}^\cdot$ has torsion free cohomology sheaves, then its Harder–Narasimhan filtration with respect to $(\underline{0}, \delta\underline{\eta})$ is given by:*

$$\begin{array}{lcccccccccc}
\mathcal{F}^\cdot_{(1)}: & \cdots \to 0 & \to & \ker d^k & \to & 0 & \to & 0 & \to & 0 \cdots \\
& & & \cap & & \cap & & \cap & & \\
\mathcal{F}^\cdot_{(2)}: & \cdots \to 0 & \to & \mathcal{F}^k & \to & \operatorname{Im} d^k & \to & 0 & \to & 0 \cdots \\
& & & \cap & & \cap & & \cap & & \\
\mathcal{F}^\cdot_{(3)}: & \cdots \to 0 & \to & \mathcal{F}^k & \to & \ker d^{k+1} & \to & 0 \cdots & \to & 0 \\
& & & \cap & & \cap & & \cap & & \\
\mathcal{F}^\cdot_{(4)}: & \cdots \to 0 & \to & \mathcal{F}^k & \to & \mathcal{F}^{k+1} & \to & \operatorname{Im} d^{k+1} & \to & 0 \cdots \\
& & & \vdots & & \vdots & & \vdots & &
\end{array}$$

In particular, the successive quotients are $H^i(\mathcal{F}^\cdot)[-i]$ or isomorphic to $\operatorname{Cone}(\operatorname{Id}_{\operatorname{Im} d^i})[-(i+1)]$.

Remark 3.4. In the bounded derived category $D^b(X)$ of coherent sheaves on X, the complexes $\mathrm{Cone}(\mathrm{Id}_{\mathrm{Im}\, d^i})[-(i+1)]$ are acyclic and so this filtration corresponds to a sequence of distinguished triangles



which is the 'filtration' associated to the standard t-structure on $D^b(X)$.

3.3. **Semistability with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$.** A torsion free sheaf $\mathcal{F}$ can be viewed as a complex (by placing it in any position k) and it is easy to see that $(\underline{\sigma}, \underline{\chi})$-semistability of the associated complex is equivalent to (Gieseker) semistability of $\mathcal{F}$; that is, for all proper nonzero subsheaves we have an inequality of reduced Hilbert polynomials.

Lemma 3.5. *Suppose $\mathcal{F}^\cdot$ is a complex and there is $\epsilon_0 > 0$ such that $\mathcal{F}^\cdot$ is $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable for all positive rational $\epsilon < \epsilon_0$. Then $\mathcal{F}^\cdot$ is either a shift of a semistable torsion free sheaf or isomorphic to a shift of the cone on the identity morphism of a semistable torsion free sheaf.*

Proof. By letting ϵ tend to zero, we see that $\mathcal{F}^\cdot$ is $(\underline{0}, \delta\underline{\eta})$-semistable and so $\mathcal{F}^\cdot$ is either a shift of a torsion free sheaf or isomorphic to a shift of the cone on the identity morphism of a torsion free sheaf by Lemma 3.1. If $\mathcal{F}^\cdot$ is the shift of a sheaf, then it is semistable by $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistability for any $0 < \epsilon < \epsilon_0$. If $\mathcal{F}^\cdot$ is the cone on the identity morphism of a torsion free sheaf $\mathcal{F}$ and $\mathcal{F}' \subset \mathcal{F}$ is a subsheaf, then we can consider $\mathrm{Cone}(\mathrm{id}_{\mathcal{F}'})$ as a subcomplex. Then $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistability of $\mathcal{F}^\cdot$ implies semistability of $\mathcal{F}$. □

Remark 3.6. Conversely, a shift of a semistable torsion free sheaf or a shift of a cone on the identity morphism of a semistable torsion free sheaf is $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable for any $\epsilon > 0$.

Corollary 3.7. *The HN filtration of $\mathcal{F}[-k]$ with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$ is given by the HN filtration of the sheaf $\mathcal{F}$. Similarly, the HN filtration of* $\mathrm{Cone}(\mathrm{id}_{\mathcal{F}})$ *with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$ is given by taking cones on the identity morphism of each term in the HN filtration of $\mathcal{F}$.*

By Corollary 3.3, the successive quotients in the HN filtration of $\mathcal{F}^\cdot$ with respect to $(\underline{0}, \delta\underline{\eta})$ are either $H^i(\mathcal{F}^\cdot)[-i]$ or isomorphic to $\mathrm{Cone}(\mathrm{Id}_{\mathrm{Im}\, d^i})[-(i+1)]$.

Theorem 3.8. *Let $\mathcal{F}^\cdot$ be a complex concentrated in $[m_1, m_2]$ with torsion free cohomology sheaves. There exists $\epsilon_0 > 0$ such that for all rational $0 < \epsilon < \epsilon_0$ the Harder–Narasimhan filtration of $\mathcal{F}^\cdot$ with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$ is given by refining the Harder–Narasimhan filtration of $\mathcal{F}^\cdot$ with respect to $(\underline{0}, \delta\underline{\eta})$ by the Harder–Narasimhan filtrations of the cohomology sheaves $H^i(\mathcal{F}^\cdot)$ and image sheaves* $\mathrm{Im}\, d^i$.

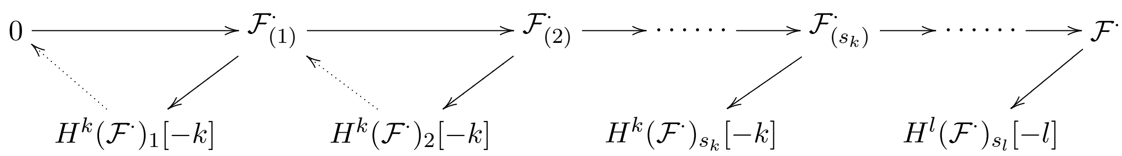
Proof. If $d := \dim X = 0$, then every sheaf is semistable and so any choice of ϵ_0 will work. Hence we assume $d = \dim X > 0$. Let $H^i(\mathcal{F}^\cdot)_j$ for $1 \leq j \leq s_i$ (resp. $\mathrm{Im}\, d^i_j$ for $1 \leq j \leq t_i$) denote the successive quotient sheaves in the Harder–Narasimhan filtration of $H^i(\mathcal{F}^\cdot)$ (resp. $\mathrm{Im}\, d^i$). The successive quotients in this filtration are either shifts of $H^i(\mathcal{F}^\cdot)_j$ or isomorphic to shifts of the cone on the identity morphism of $\mathrm{Im}\, d^i_j$ and so by Remark 3.6 are $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable for any rational $\epsilon > 0$. Thus it suffices to show there is an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ we have inequalities of reduced Hilbert polynomials of the successive quotients. Since we know that the reduced Hilbert polynomials of the successive quotients in the Harder–Narasimhan filtrations of the cohomology and image sheaves are decreasing, it suffices to show for $m_1 \leq i < m_2 - 1$ that:

$$1)\ \epsilon\frac{P(H^i(\mathcal{F}^{\cdot})_{s_i})}{\operatorname{rk} H^i(\mathcal{F}^{\cdot})_{s_i}} - \delta\eta_i > \epsilon\frac{P(\operatorname{Im} d_1^i)}{\operatorname{rk}\operatorname{Im} d_1^i} - \delta\frac{\eta_i + \eta_{i+1}}{2} \text{ and}$$

$$2)\ \epsilon\frac{P(\operatorname{Im} d_{t_i}^i)}{\operatorname{rk}\operatorname{Im} d_{t_i}^i} - \delta\frac{\eta_i + \eta_{i+1}}{2} > \epsilon\frac{P(H^{i+1}(\mathcal{F}^{\cdot})_1)}{\operatorname{rk} H^{i+1}(\mathcal{F}^{\cdot})_1} - \delta\eta_{i+1}.$$

These polynomials all have the same top coefficient and we claim we can pick ϵ_0 so that we have strict inequalities in the second to top coefficients if $0 < \epsilon < \epsilon_0$. Let $\mu(\mathcal{A})$ denote the coefficient of x^{d-1} in the reduced Hilbert polynomial of $\mathcal{A}$ and let $\delta^{\text{top}} > 0$ be the coefficient of x^{d-1} in δ. Let $M_i := \max\{\mu(\operatorname{Im} d_1^i) - \mu(H^i(\mathcal{F}^{\cdot})_{s_i}), \mu(H^{i+1}(\mathcal{F}^{\cdot})_1) - \mu(\operatorname{Im} d_{t_i}^i)\}$ for $m_1 \leq i < m_2 - 1$; then pick $\epsilon_0 > 0$ so that when $M_i > 0$ we have $\epsilon_0 < \delta^{\text{top}}(\eta_{i+1} - \eta_i)/2M_i$. □

Remark 3.9. In the bounded derived category $D^b(X)$, this filtration corresponds to a sequence of distinguished triangles



which is given by combining the 'filtration' coming from the standard t-structure with heart equal to $\operatorname{Coh}(X)$ with the (shifted) Harder–Narasimhan filtrations of the subquotients in $\operatorname{Coh}(X)$.

4. The stratification of the parameter space

In this section we study the GIT stratifications of the parameter space $\mathfrak{T}$ for complexes associated to the collection of stability conditions $(\underline{1}, \delta\underline{\eta}/\epsilon)$ given in §3 and compare these stratifications to the natural stratification by Harder–Narasimhan types.

4.1. The Hesselink-Kirwan-Ness stratification.

Given a projective G-scheme B with an ample linearisation $\mathcal{L}$, there is a stratification of B by G-invariant subschemes $\{S_\beta : \beta \in \mathcal{B}\}$ [8, 12, 13]. If we choose a compact maximal torus T of G and positive Weyl chamber $\mathfrak{t}_+$ in $\mathfrak{t} = \operatorname{Lie}T$, then the index set $\mathcal{B}$ can be identified with a finite set of rational weights in $\mathfrak{t}_+$ as follows. By fixing an invariant inner product on the Lie algebra $\mathfrak{K}$ of the maximal compact subgroup $K \subset G$, we can identify characters and cocharacters as well as weights and coweights. There are a finite number of weights for the action of T on B and the index set $\mathcal{B}$ can be identified with the set of rational weights in $\mathfrak{t}_+$ which are the closest points to 0 of the convex hull of a subset of these weights.

Associated to β there is a parabolic subgroup $P_\beta \subset G$, a rational 1-PS $\lambda_\beta : \mathbb{C}^* \to T_\mathbb{C}$ (i.e. λ_β^n is a 1-PS) and a rational character $\chi_\beta : T_\mathbb{C} \to \mathbb{C}^*$ which extends to P_β. Let Z_β be the components of the fixed point locus of λ_β acting on B on which λ_β acts with weight $-||\lambda_\beta||^2$ and Z_β^{ss} be the GIT semistable subscheme for the action of the reductive part $\operatorname{Stab}\beta$ of P_β on Z_β with respect to the linearisation $\mathcal{L}^{\chi_{-\beta}}$, which is the original linearisation $\mathcal{L}$ twisted by the character $\chi_{-\beta} : \operatorname{Stab}\beta \to \mathbb{C}^*$. Then Y_β (resp. Y_β^{ss}) is defined to be the subscheme of B consisting of points whose limit under the action of $\lambda_\beta(t)$ as $t \to 0$ lies in Z_β (resp. Z_β^{ss}) and there is a retraction $p_\beta : Y_\beta \to Z_\beta$ given by taking a point to its limit under λ_β. Then

$$S_\beta := GY_\beta^{ss} \cong G \times^{P_\beta} Y_\beta^{ss}$$

can be defined for any rational weight β, although S_β is nonempty if and only if β is an index.

4.2. **GIT set up.** We consider the collection of stability parameters $(\underline{1}, \delta\underline{\eta}/\epsilon)$ from §3. The parameter space $\mathfrak{T} = \mathfrak{T}_P(n)$ for $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable complexes with invariants P is a locally closed subscheme of a projective bundle $\mathfrak{D}$ over a product $Q = Q^{m_1} \times \cdots \times Q^{m_2}$ of open subschemes Q^i of quot schemes. The reductive group $G = \mathrm{SL}(\oplus_i V^i) \cap \Pi_i \mathrm{GL}(V^i)$ acts on $\mathfrak{T}$ and a linearisation of this action is determined by the stability parameters (see §2.4 and [14]). We work with the natural projective completion $\overline{\mathfrak{T}}$ of $\mathfrak{T}$ given in §2.6. For any $n >> 0$ and $\epsilon > 0$, associated to this action we have a stratification of $\overline{\mathfrak{T}}$ into G-invariant locally closed subschemes such that the open stratum is the GIT semistable subscheme. As we are primarily interested in complexes with torsion free cohomology sheaves, which form an open subscheme $\mathfrak{T}^{tf}$ of the parameter space $\mathfrak{T}$, we look at restriction this stratification to the closure $\overline{\mathfrak{T}}^{tf}$ of $\mathfrak{T}^{tf}$ in $\overline{\mathfrak{T}}$:

$$\overline{\mathfrak{T}}^{tf} = \bigsqcup_{\beta \in \mathcal{B}} S_\beta. \tag{6}$$

We also have a stratification by Harder-Narasimhan types with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$:

$$\mathfrak{T}^{tf} = \bigsqcup_{\tau} R_\tau \tag{7}$$

where the union is over all Harder–Narasimhan types τ.

Notation: Let us fix a complex $\mathcal{F}^\cdot$ with torsion free cohomology sheaves and invariants $\underline{P}$ and pick ϵ sufficiently small as in Theorem 3.8. We assume that $\mathcal{F}^\cdot$ has nontrivial Harder–Narasimhan type τ with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$. Let $H_{i,j}$ (resp. $I_{i,j}$) denote the Hilbert polynomial of the jth successive quotient $H^i(\mathcal{F}^\cdot)_j$ (resp. $\mathrm{Im}\, d^i_j$) in the Harder–Narasimhan filtration of the sheaf $H^i(\mathcal{F}^\cdot)$ (resp. $\mathrm{Im}\, d^i$) for $1 \leq j \leq s_i$ (resp. $1 \leq j \leq t_i$). Let $H_{i,j} = (H^k_{i,j})_{k \in \mathbb{Z}}$ and $I_{i,j} = (I^k_{i,j})_{k \in \mathbb{Z}}$ be given by

$$H^k_{i,j} = \begin{cases} H_{i,j} & \text{if } k = i, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad I^k_{i,j} = \begin{cases} I_{i,j} & \text{if } k = i, i+1, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tau = (H_{m_1,1}, \dots, H_{m_1,s_{m_1}}, I_{m_1,1}, \dots, I_{m_1,t_{m_1}}, H_{m_1+1,1}, \dots, H_{m_2,s_{m_2}})$ is the HN type of $\mathcal{F}^\cdot$ with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$, which we abbreviate to $\tau = (H, I)$. We note that a complex with Hilbert polynomials specified by the tuple $H_{i,j} = (H^k_{i,j})_{k \in \mathbb{Z}}$ is a complex concentrated in a single degree i (i.e., a shift of a sheaf), whereas a complex with Hilbert polynomials specified by $I_{i,j}$ is a two term complex concentrated in degrees i and $i+1$.

Following Lemma 3.5 and finiteness results regarding variation of GIT (for example, see [3] and [16]), we may make the following assumption about the two term complexes with Hilbert polynomials specified by the tuple $I_{i,j}$:

Assumption 4.1. We assume ϵ is sufficiently small so that the only $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable complexes with Hilbert polynomials $I_{i,j}$ are isomorphic to cones on the identity morphism of a torsion free semistable sheaf.

4.3. **Overview of the proof.** In the remainder of this section we compare the GIT stratification (6) with the Harder–Narasimhan stratification (7). We are working towards Theorem 4.15, which states for $n >> 0$ that $R_\tau \subset \mathfrak{T}^{tf}$ parametrises all complexes with Harder–Narasimhan type τ and that R_τ is a union of connected components of $S_\beta \cap \mathfrak{T}^{tf}$ where β is a weight associated to τ. The first step is to show we can pick $n >> 0$ so that $R_\tau \subset \mathfrak{T}^{tf}$ contains all complexes with Harder–Narasimhan type τ which we do in §4.4. Then in §4.5, we find the required index $\beta = \beta(\tau)$. In §4.6 we describe the components of S_β that we are interested in (i.e. which contain R_τ) and in §4.7 we complete the proof.

4.4. **Boundedness.** A result of Simpson (cf. [15] Theorem 1.1) states that a collection of torsion free sheaves on X of fixed Hilbert polynomial is bounded if the slopes of their subsheaves are bounded above by a fixed constant. It follows from this that:

Lemma 4.2. *The set of sheaves occurring in a complex of torsion free sheaves with Harder–Narasimhan type* $(P_1, \dots, P_s)$ *with respect to* $(\underline{\sigma}, \underline{\chi})$ *is bounded.*

Corollary 4.3. *Let* $(P_1, \dots, P_s)$ *be a Harder–Narasimhan type with respect to* $(\underline{\sigma}, \underline{\chi})$*. Then we can choose* n *sufficiently large so that for* $1 \leq i_1 < \cdots < i_k \leq s$ *all the sheaves occurring in a complex of torsion free sheaves with Harder–Narasimhan type* $(P_{i_1}, \dots, P_{i_k})$ *are* n*-regular.*

Assumption 4.4. We assume n is sufficiently large so that the statement of Corollary 4.3 holds for the Harder–Narasimhan type $\tau = (H, I)$ of our fixed complex $\mathcal{F}^\cdot$. In particular this means every complex $\mathcal{E}^\cdot$ with Harder–Narasimhan type τ is parametrised by $R_\tau \subset \mathfrak{T}^{tf}$.

4.5. **The associated index.** Let $z = (q, [\psi : 1])$ be a point in $\mathfrak{T}^{tf}$ which parametrises the fixed complex $\mathcal{F}^\cdot$ with HN type τ. The stratification can be described by adapted 1-PSs, so rather than searching for a rational weight β such that $z \in S_\beta$, we can look for a 1-PS λ_β which is adapted to z i.e. is most responsible for the instability of z. It is natural to expect that λ should induce the filtration of $\mathcal{F}^\cdot$ which is most responsible for the instability of this complex; that is, its HN filtration. To distinguish between the cohomology and image parts we write the HN filtration of $\mathcal{F}^\cdot$ as

$$0 \subsetneq \mathcal{H}^\cdot_{m_1,(1)} \subsetneq \cdots \subsetneq \mathcal{H}^\cdot_{m_1,(s_{m_1})} \subsetneq \mathcal{I}^\cdot_{m_1,(1)} \cdots \mathcal{I}^\cdot_{m_1,(t_{m_1})} \subsetneq \mathcal{H}^\cdot_{m_1+1,(1)} \subsetneq \cdots \subsetneq \mathcal{H}^\cdot_{m_2,(s_{m_2})} = \mathcal{F}^\cdot$$

where the quotient $\mathcal{H}^\cdot_{k,j}$ (resp. $\mathcal{I}^\cdot_{k,j}$) of $\mathcal{H}^\cdot_{k,(j)}$ (resp. $\mathcal{I}^\cdot_{k,(j)}$) by its predecessor is isomorphic to $H^k(\mathcal{F}^\cdot)_j[-k]$ (resp. $\mathrm{Cone}(\mathrm{id}_{\mathrm{Im}\, d^k_j})[-(k+1)]$). This induces vector space filtrations

$$0 \subset V^i_{m_1,(1)} \subset \cdots \subset V^i_{m_1,(s_{m_1})} \subset W^i_{m_1,(1)} \subset \cdots \subset W^i_{m_1,(t_{m_1})} \subset \cdots \subset V^i_{m_2,(s_{m_2})} = V^i \tag{8}$$

for $m_1 \leq i \leq m_2$ where

$$V^i_{k,(j)} := H^0(q^i(n))^{-1} H^0(\mathcal{H}^i_{k,(j)}(n)) \quad \text{and} \quad W^i_{k,(j)} := H^0(q^i(n))^{-1} H^0(\mathcal{I}^i_{k,(j)}(n)).$$

Let $V^i_{k,j}$ (resp. $W^i_{k,j}$) denote the quotient of $V^i_{k,(j)}$ (resp. $W^i_{k,(j)}$) by its predecessor in this filtration. By the construction of the HN filtration (see Theorem 3.8) we have that $V^i_{k,j} = 0$ unless $k = i$ and $W^i_{k,j} = 0$ unless $k = i, i-1$ and $W^i_{i,j} \cong W^{i+1}_{i,j}$.

Given integers $a_{m,1} > \cdots > a_{m_1,s_{m_1}} > b_{m_1,1} > \cdots > b_{m_1,t_{m_1}} > a_{m_1+1,1} > \cdots > a_{m_2,s_{m_2}}$ such that

$$\sum_{i=m_1}^{m_2} \sum_{j=1}^{s_i} a_{i,j} \dim V^i_{i,j} + 2 \sum_{i=m_1}^{m_2-1} \sum_{j=1}^{t_i} b_{i,j} \dim W^i_{i,j} = 0, \tag{9}$$

we can define a 1-PS $\lambda(\underline{a}, \underline{b}) : \mathbb{C}^* \to G$ which induces the above filtrations as follows. Let $V^i_k = \oplus_{j=1}^{s_k} V^i_{k,j}$ and $W^i_k = \oplus_{j=1}^{t_k} W^i_{k,j}$; then define 1-PSs $\lambda^{H,i}_k : \mathbb{C}^* \to \mathrm{GL}(V^i_k)$ and $\lambda^{I,i}_k : \mathbb{C}^* \to \mathrm{GL}(W^i_k)$ by

$$\lambda^{H,i}_k(t) = \begin{pmatrix} t^{a_{k,1}} I_{V^i_{k,1}} & & \\ & \ddots & \\ & & t^{a_{k,s_k}} I_{V^i_{k,s_k}} \end{pmatrix} \quad \lambda^{I,i}_k(t) = \begin{pmatrix} t^{b_{k,1}} I_{W^i_{k,1}} & & \\ & \ddots & \\ & & t^{b_{k,t_k}} I_{W^i_{k,t_k}} \end{pmatrix}$$

Then $\lambda(\underline{a}, \underline{b}) := (\lambda_{m_1}, \dots, \lambda_{m_2})$ is given by

$$\lambda_i(t) := \begin{pmatrix} \lambda^{I,i}_{i-1}(t) & & \\ & \lambda^{H,i}_i(t) & \\ & & \lambda^{I,i}_i(t) \end{pmatrix} \in \mathrm{GL}(V^i) = \mathrm{GL}(W^i_{i-1} \oplus V^i_i \oplus W^i_i). \tag{10}$$

For all pairs $(\underline{a},\underline{b})$ the associated 1-PS $\lambda(\underline{a},\underline{b})$ of G induces the HN filtration of $\mathcal{F}^\cdot$. The weight $\mu^{\mathcal{L}}(z,\lambda(\underline{a},\underline{b}))$ is given by Proposition 2.8 i). Let $(\underline{1},\delta\underline{\eta}'/\epsilon)$ be the stability parameters associated to $(\underline{1},\delta\underline{\eta}/\epsilon)$ which satisfy $\sum_i \eta_i' r^i = 0$ (cf. Remark 2.2) and let $C := (\sum_i P^i(n))/(\delta(n)\sum_i r^i)$; then define

$$a_{i,j} := \frac{1}{\delta(n)} - \left(C + \frac{\eta_i'}{\epsilon}\right)\frac{\mathrm{rk}(H_{i,j})}{H_{i,j}(n)} \quad \text{and} \quad b_{i,j} := \frac{1}{\delta(n)} - \left(C + \frac{\eta_i' + \eta_{i+1}'}{2\epsilon}\right)\frac{\mathrm{rk}(I_{i,j})}{I_{i,j}(n)}$$

where $\mathrm{rk}(H_{i,j})$ and $\mathrm{rk}(I_{i,j})$ are the ranks determined by the leading coefficients of the polynomials $H_{i,j}$ and $I_{i,j}$. These numbers minimise the normalised Hilbert–Mumford function $\mu^{\mathcal{L}}(z,\lambda(\underline{a},\underline{b}))/||\lambda(\underline{a},\underline{b})||$ subject to condition (9).

Take a basis of V^i which is compatible with the filtration of V^i defined at (8) and define T_i to be the maximal torus of the compact group $\mathrm{U}(V^i)$ given by taking diagonal matrices with respect to this basis and let

$$\mathfrak{t}_{i+} := \{i\mathrm{diag}(a_1,\dots,a_{\dim V^i}) : a_1 \geq \cdots \geq a_{\dim V^i}\} \subset \mathfrak{t}_i = \mathrm{Lie}T_i.$$

Let T be the compact maximal torus of G determined by the tori T_i and let $\mathfrak{t}_+$ be the positive Weyl chamber associated to the $\mathfrak{t}_{i+}$.

Definition 4.5. Let $\beta = \beta(\tau,n) \in \mathfrak{t}_+$ be the rational weight defined by the rational weights

$$\beta_i = i\mathrm{diag}(b_{i-1,1},\dots,b_{i-1,t_{i-1}},a_{i,1}\dots,a_{i,s_i},b_{i,1}\dots b_{i,t_i}) \in \mathfrak{t}_{i+}$$

where $a_{i,j}$ appears $H_{i,j}(n)$ times and $b_{k,j}$ appears $I_{k,j}(n)$ times. This rational weight defines a rational 1-PS λ_β of G by $\lambda_\beta = \lambda(\underline{a},\underline{b})$.

4.6. **Describing components of S_β.** Consider the closed subscheme F_τ of $\overline{\mathfrak{T}}^{tf}$ consisting of $z = (q,[\psi:\zeta])$ where ψ is determined by boundary maps d^i and we have decompositions

$$q^i = \bigoplus_{j=1}^{t_{i-1}} p^i_{i-1,j} \oplus \bigoplus_{j=1}^{s_i} q^i_{i,j} \oplus \bigoplus_{j=1}^{t_i} p^i_{i,j} \quad \text{and} \quad d^i = \oplus_{j=1}^{t_i} d^i_j \tag{11}$$

where $q^i_{i,j} : V^i_{i,j}\otimes\mathcal{O}_X(-n) \to \mathcal{E}^i_{i,j}$ is a point in $\mathrm{Quot}(V^i_{i,j}\otimes\mathcal{O}_X(-n),H^i_{i,j})$ and $p^i_{k,j} : W^i_{k,j}\otimes\mathcal{O}_X(-n) \to \mathcal{G}^i_{k,j}$ is a point in $\mathrm{Quot}(W^i_{k,j}\otimes\mathcal{O}_X(-n),I^i_{k,j})$ and $d^i_j : \mathcal{G}^i_{i,j} \to \mathcal{G}^{i+1}_{i,j}$.

By Remark 2.7, the $\lambda_\beta(\mathbb{C}^*)$-fixed point locus of $\overline{\mathfrak{T}}^{tf}$ decomposes into three pieces: a diagonal piece, a strictly upper triangular piece and a strictly lower triangular piece. Each of these pieces decomposes further in terms of the Hilbert polynomials of the direct summands of each sheaf in this complex. In particular:

Lemma 4.6. *F_τ is a union of connected components of the fixed point locus $(\overline{\mathfrak{T}}^{tf})^{\lambda_\beta(\mathbb{C}^*)}$ which is contained in the diagonal part of this fixed point locus.*

Remark 4.7. Every point in $\mathfrak{T}_{(\tau)} := F_\tau \cap \mathfrak{T}^{tf}$ is a direct sum of complexes with Hilbert polynomials specified by τ and it follows that there is an isomorphism

$$\mathfrak{T}_{H_{m_1,1}} \times \cdots \times \mathfrak{T}_{H_{m_1,s_{m_1}}} \times \mathfrak{T}^{tf}_{I_{m_1,1}} \times \cdots \times \mathfrak{T}^{tf}_{I_{m_1,t_{m_1}}} \times \mathfrak{T}_{H_{m_1+1,1}} \times \cdots \times \mathfrak{T}_{H_{m_2,s_{m_2}}} \cong \mathfrak{T}_{(\tau)}.$$

Lemma 4.8. *Let $z \in \mathfrak{T}_{(\tau)} := F_\tau \cap \mathfrak{T}^{tf}$; then $z \in Z_\beta$.*

Proof. The weight of the action of λ_β on $z = (q,[\psi:1])$ is equal to $\mu^{\mathcal{L}}(z,\lambda_\beta)$ and using the formula given in Proposition 2.8 i) we can check that $\mu^{\mathcal{L}}(z,\lambda_\beta) = -||\lambda_\beta||^2$ as required. □

Let F be the union of connected components of Z_β meeting $\mathfrak{T}_{(\tau)}$; then F is contained in the diagonal part of Z_β. Consider the groups

$$\mathrm{Stab}\,\beta = \left(\prod_{i=m_1}^{m_2}\prod_{j=1}^{s_i}\mathrm{GL}(V^i_{i,j}) \times \prod_{i=m_1}^{m_2-1}\prod_{j=1}^{t_i}\mathrm{GL}(W^i_{i,j})\times\mathrm{GL}(W^{i+1}_{i,j})\right) \cap \mathrm{SL}(\oplus_i V^i)$$

and $\hat{G} = \{(v_{i,j}, w_{i,j}) \in (\mathbb{C}^*)^{\sum_{i=m_1}^{m_2} s_i + \sum_{i=m_1}^{m_2-1} t_i} : \Pi_{i,j}(v_{i,j})^{H_{i,j}(n)}(w_{i,j})^{2I_{i,j}(n)} = 1\}$.

Lemma 4.9. *$\hat{G}$ is a central subgroup of* $\mathrm{Stab}\,\beta$ *which fixes every point of F. This central subgroup acts on the fibre of $\mathcal{L}$ over any point of F by a character $\chi_F : \hat{G} \to \mathbb{C}^*$ given by*

$$\chi_F(v_{i,j}, w_{i,j}) = \prod_{i,j}(v_{i,j})^{\mathrm{rk}\, H_{i,j}(C+\frac{\eta'_i}{\epsilon})}(w_{i,j})^{\mathrm{rk}\, I_{i,j}(2C+\frac{\eta'_i+\eta'_{i+1}}{\epsilon})}.$$

Proof. The inclusion $\hat{G} \hookrightarrow \mathrm{Stab}\,\beta$ is given by

$$(v_{i,j}, w_{i,j}) \mapsto (v_{i,j} I_{V^i_{i,j}}\, ,\ w_{i,j} I_{W^i_{i,j}}\, ,\ w_{i,j} I_{W^{i+1}_{i,j}}).$$

Let $z = (q, [\psi : \zeta])$ be a point of F; then we have a decomposition of q^i and d^i as at (11). A copy of $\mathbb{C}^*$ acts trivially on each quot scheme and so the central subgroup $\hat{G}$ fixes this quotient sheaf. As $(v_{i,j}\, ,\, w_{i,j}) \in \hat{G}$ acts on both $\mathcal{G}^i_{i,j}$ and $\mathcal{G}^{i+1}_{i,j}$ by multiplication by $w_{i,j}$, the boundary maps are also fixed by the action of $\hat{G}$. The calculation of the character $\chi_F : \hat{G} \to \mathbb{C}^*$ is done by modifying the calculations for $\mathbb{C}^*$-actions in §2.6 to general torus actions. □

Let $\mathcal{L}^{\chi_{-\beta}}$ denote the linearisation of the $\mathrm{Stab}\,\beta$ action on Z_β given by twisting the original linearisation $\mathcal{L}$ by the character $\chi_{-\beta}$ associated to $-\beta$ and let

$$\begin{aligned} G' &:= \left\{(g^i_j, h^i_{i,j}, h^{i+1}_{i,j}) \in \mathrm{Stab}\,\beta : \det g^i_j = 1 \text{ and } \det h^i_{i,j} \det h^{i+1}_{i,j} = 1\right\} \\ &= \prod_{i=m_1}^{m_2}\prod_{j=1}^{s_i} \mathrm{SL}(V^i_{i,j}) \times \prod_{i=m_1}^{m_2-1}\prod_{j=1}^{t_i}(\mathrm{GL}(W^i_{i,j}) \times \mathrm{GL}(W^{i+1}_{i,j})) \cap \mathrm{SL}(W^i_{i,j} \oplus W^{i+1}_{i,j}). \end{aligned}$$

Proposition 4.10. *If F is the components of Z_β which meet $\mathfrak{T}_{(\tau)}$; then*

$$F^{\mathrm{Stab}\,\beta - ss}(\mathcal{L}^{\chi_{-\beta}}) = F^{G'-ss}(\mathcal{L}).$$

Proof. There is a surjective homomorphism $\Phi : \mathrm{Stab}\,\beta \to \hat{G}$ such that the composition of $\hat{G} \hookrightarrow \mathrm{Stab}\,\beta$ with Φ is

$$(u_{i,j}, w_{i,j}) \mapsto (u^M_{i,j}, w^M_{i,j})$$

for some positive integer M. Hence, $\ker\Phi \times \hat{G}$ surjects onto $\mathrm{Stab}\,\beta$ with finite kernel and $F^{\mathrm{Stab}\,\beta-ss}(\mathcal{L}^{\chi_{-\beta}}) = F^{\ker\Phi\times\hat{G}-ss}(\mathcal{L}^{\chi_{-\beta}})$. The restriction of $\chi_{-\beta}$ to the central subgroup $\hat{G}$ is

$$\chi_{-\beta}(v_{i,j}, w_{i,j}) = \prod_{i,j} v_{i,j}^{-a_{i,j}H_{i,j}(n)} w_{i,j}^{-b_{i,j}2I_{i,j}(n)},$$

which is equal to the character $\chi_F : \hat{G} \to \mathbb{C}^*$ given in Lemma 4.9. As we are considering the action of $\ker\Phi \times \hat{G}$ on F linearised by $\mathcal{L}^{\chi_{-\beta}}$, the effects of the action of $\hat{G}$ and the modification by the character corresponding to $-\beta$ cancel so that $F^{\ker\Phi\times\hat{G}-ss}(\mathcal{L}^{\chi_{-\beta}}) = F^{\ker\Phi-ss}(\mathcal{L})$. Finally as G' injects into $\ker\Phi$ with finite cokernel, we conclude $F^{\ker\Phi-ss}(\mathcal{L}) = F^{G'-ss}(\mathcal{L})$. □

Let $z_{i,j} = (q^i_{i,j}, [0,1])$ be points in the parameter spaces $\mathfrak{T}_{H_{i,j}}$ for complexes with invariants $H_{i,j}$ which correspond to complexes $\mathcal{H}^\cdot_{i,j}$ concentrated in degree i and let $y_{i,j} = (p^i_{i,j}, p^{i+1}_{i,j}[\varphi^i_j, 1])$ be points in $\mathfrak{T}^{tf}_{I_{i,j}}$ corresponding to a complex $\mathcal{I}^\cdot_{i,j}$ concentrated in degrees i and $i+1$. Then let $z \in \mathfrak{T}_{(\tau)}$ denote the associated point under the isomorphism

$$\mathfrak{T}_{(\tau)} \cong \mathfrak{T}_{H_{m_1,1}} \times \cdots \times \mathfrak{T}_{H_{m_1,s_{m_1}}} \times \mathfrak{T}^{tf}_{I_{m_1,1}} \times \cdots \times \mathfrak{T}^{tf}_{I_{m_1,t_{m_1}}} \times \mathfrak{T}_{H_{m_1+1,1}} \times \cdots \times \mathfrak{T}_{H_{m_2,s_{m_2}}}$$

of Remark 4.7. By Proposition 4.10, we have

$$\mathfrak{T}^{ss}_{(\tau)} := \mathfrak{T}^{\mathrm{Stab}\,\beta-ss}_{(\tau)}(\mathcal{L}^{\chi_{-\beta}}) = \mathfrak{T}^{G'-ss}_{(\tau)}(\mathcal{L}|_{\mathfrak{T}_{(\tau)}});$$

therefore, z is in $\mathfrak{T}^{ss}_{(\tau)}$ if and only if $\mu^{\mathcal{L}}(z,\lambda) \geq 0$ for every 1-PS λ of G'. A 1-PS λ of G' is given by

- 1-PSs $\lambda^H_{i,j}$ of $\mathrm{SL}(V^i_{i,j})$ and
- 1-PSs $\lambda^I_{i,j} = (\lambda^{I,i}_{i,j}, \lambda^{I,i+1}_{i,j})$ of $(\mathrm{GL}(W^i_{i,j}) \times \mathrm{GL}(W^{i+1}_{i,j})) \cap \mathrm{SL}(W^i_{i,j} \oplus W^{i+1}_{i,j})$.

Lemma 4.11. *For $n >> 0$, if $z \in \mathfrak{T}_{(\tau)}$ is as above and a direct summand $\mathcal{H}^{\cdot}_{i,j}$ or $\mathcal{I}^{\cdot}_{i,j}$ is $(\underline{1}, \delta\underline{\eta}/\epsilon)$-unstable, then there is a 1-PS λ of G' such that $\mu^{\mathcal{L}}(z,\lambda) < 0$.*

Proof. We suppose n is sufficiently large so that semistability of a torsion free sheaf with Hilbert polynomial $H_{i,j}$ (respectively $I_{i,j}$) is equivalent to GIT-semistability of a point in the relevant quot scheme representing this sheaf. We also assume n is sufficiently large so for $1 \leq i \leq m-1$ and $1 \leq j \leq t_i$, we have that $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistability of a complex with Hilbert polynomials $I_{i,j}$ is equivalent to GIT semistability of a point in $\mathfrak{T}_{I_{i,j}}$.

If $\mathcal{H}^i_{i,j}$ is unstable, then the point $q^i_{i,j}$ in the quot scheme is unstable for the action of $\mathrm{SL}(V^i_{i,j})$ and so there is $\lambda^H_{i,j}$ for which $\mu(q^i_{i,j}, \lambda^H_{i,j}) < 0$. If we let all other parts of $\lambda = (\lambda^H_{i,j}, \lambda^I_{i,j})$ to be trivial, then it follows immediately from (2) that $\mu^{\mathcal{L}}(z,\lambda) < 0$.

If $\mathcal{I}^{\cdot}_{i,j}$ is unstable with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$, then it is not isomorphic to the cone on the identity map of a semistable sheaf by our assumption on ϵ. Let $d : \mathcal{I}^i_{i,j} \to \mathcal{I}^{i+1}_{i,j}$ denote the boundary morphism of this complex. If $d = 0$, we can choose a 1-PS λ to pick out the subcomplex $\mathcal{I}^{i,j}_i \to 0$. If $d \neq 0$ but has nonzero kernel, then consider the reduced Hilbert polynomial of this kernel. If the kernel has reduced Hilbert polynomial strictly larger than $\mathcal{I}^i_{i,j}$, then choose λ to pick out the subcomplex $\ker d \to 0$. If the kernel has reduced Hilbert polynomial strictly smaller than $\mathcal{I}^i_{i,j}$, then choose λ to pick out the subcomplex $0 \to \operatorname{Im} d$. If the kernel has reduced Hilbert polynomial equal to $I_{i,j}/\operatorname{rk} I_{i,j}$, then choose λ to pick out the subcomplex $\mathcal{I}^i_{i,j} \to \operatorname{Im} d$. Finally, if d is an isomorphism but $\mathcal{I}^i_{i,j}$ is unstable, then let $\mathcal{C}$ be its maximal destabilising subsheaf and choose a 1-PS λ which picks out the subcomplex $\mathcal{C} \to d^i_j(\mathcal{C})$. In all these cases we can check that $\mu^{\mathcal{L}}(z,\lambda) < 0$ using Proposition 2.8. ☐

Lemma 4.12. *Suppose n is sufficiently large and we are allow a rescaling of $(\delta, \underline{\eta})$. If $z \in \mathfrak{T}_{(\tau)}$ and all the direct summands $\mathcal{H}^{\cdot}_{i,j}$ and $\mathcal{I}^{\cdot}_{i,j}$ are $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable, then for every 1-PS λ of G' we have $\mu^{\mathcal{L}}(z,\lambda) \geq 0$.*

Proof. Let n be chosen as in Lemma 4.11. We can diagonalise the 1-PSs $\lambda^H_{i,j}$ and $\lambda^I_{i,j}$ simultaneously to get decreasing integer weights $\gamma_1 > \cdots > \gamma_u$ and weight space decompositions $V^i_{i,j} = V^{i,1}_{i,j} \oplus \cdots \oplus V^{i,u}_{i,j}$ and $W^i_{i,j} = W^{i,1}_{i,j} \oplus \cdots \oplus W^{i,u}_{i,j}$ and $W^{i+1}_{i,j} = W^{i+1,1}_{i,j} \oplus \cdots \oplus W^{i+1,u}_{i,j}$. The corresponding vector space filtrations give rise to sheaf filtrations of $\mathcal{H}^i_{i,j}$, $\mathcal{I}^i_{i,j}$ and $\mathcal{I}^{i+1}_{i,j}$ and we let $\mathcal{H}^{i,k}_{i,j}$, $\mathcal{I}^{i,k}_{i,j}$ and $\mathcal{I}^{i+1,k}_{i,j}$ denote the successive quotients. If λ induces a filtration by subcomplexes, then by Proposition 2.8 we have

$$\mu^{\mathcal{L}}(z,\lambda) = \sum_{k=1}^{u} \gamma_k \left[\sum_{\substack{m_1 \leq i \leq m_2 - 1 \\ 1 \leq j \leq t_i}} (C + \frac{\eta'_i}{\epsilon}) \operatorname{rk} \mathcal{I}^{i,k}_{i,j} + (C + \frac{\eta'_{i+1}}{\epsilon}) \operatorname{rk} \mathcal{I}^{i+1,k}_{i,j} + \sum_{\substack{m_1 \leq i \leq m_2 \\ 1 \leq j \leq s_i}} (C + \frac{\eta'_i}{\epsilon}) \operatorname{rk} \mathcal{H}^{i,k}_{i,j} \right]$$

where $C = (\sum_i P^i(n))/(\delta(n) \sum_i r^i)$. By construction of the linearisation, $C + \eta'_i/\epsilon$ is positive. As $\mathcal{H}^i_{i,j}$, $\mathcal{I}^i_{i,j}$ and $\mathcal{I}^{i+1}_{i,j}$ are semistable, the Hilbert–Mumford criterion gives

$$\sum_{k=1}^{u} \gamma_k \operatorname{rk} \mathcal{H}^{i,k}_{i,j} \geq 0 \quad \text{and} \quad \sum_{k=1}^{u} \gamma_k \left(\operatorname{rk} \mathcal{I}^{l,k}_{i,j} - \frac{\operatorname{rk} I_{i,j}}{I_{i,j}(n)} \dim W^{l,k}_{i,j} \right) \geq 0 \quad \text{for } l = i, i+1.$$

Using these inequalities and the fact that $\lambda_{i,j}^I$ is a 1-PS of $\mathrm{SL}(W_{i,j}^i \oplus W_{i,j}^{i+1})$, we obtain

$$\mu^{\mathcal{L}}(z,\lambda) \geq \frac{\mathrm{rk}\, I_{i,j}}{I_{i,j}(n)} \sum_{i=m_1}^{m_2-1} \sum_{j=1}^{t_i} \sum_{k=1}^{u} \gamma_k \left[\left(C + \frac{\eta_i'}{\epsilon} \right) \dim W_{i,j}^{i,k} + \left(C + \frac{\eta_{i+1}'}{\epsilon} \right) \dim W_{i,j}^{i+1,k} \right]$$

$$= \frac{\mathrm{rk}\, I_{i,j}}{I_{i,j}(n)} \sum_{i=m_1}^{m_2-1} \sum_{j=1}^{t_i} \frac{(\eta_{i+1}' - \eta_i')}{\epsilon} \sum_{k=1}^{u} \gamma_k \dim W_{i,j}^{i+1,k}.$$

As λ induces a filtration by subcomplexes,

$$\dim(W_{i,j}^{i,1} \oplus \cdots \oplus W_{i,j}^{i,k}) \leq \dim(W_{i,j}^{i+1,1} \oplus \cdots \oplus W_{i,j}^{i+1,k})$$

and it follows that $-\sum_{k=1}^u \gamma_k \dim W_{i,j}^{i,k} = \sum_{k=1}^u \gamma_k \dim W_{i,j}^{i+1,k} \geq 0$ and so $\mu^{\mathcal{L}}(z,\lambda) \geq 0$.

As in [14] Theorem 1.7.1 (see also Remark 2.9), we can rescale $(\delta, \underline{\eta})$ to $(K\delta, \underline{\eta}/K)$ for K a large integer so that we can verify GIT-semistability by only checking for 1-PSs which induce filtrations by subcomplexes. □

Let $\mathfrak{T}_{H_{i,j}}^{ss}$ (resp. $\mathfrak{T}_{I_{i,j}}^{ss}$) be the subscheme of $\mathfrak{T}_{H_{i,j}}$ (resp. $\mathfrak{T}_{I_{i,j}}^{tf}$) which parametrises $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable complexes with Hilbert polynomials $H_{i,j}$ (resp. $I_{i,j}$). We assume the pair $(\delta, \underline{\eta})$ have been scaled as required by Lemma 4.12. Then it follows from Proposition 4.10, Lemma 4.11 and Lemma 4.12 that:

Proposition 4.13. *For n sufficiently large, we have an isomorphism*

$$\mathfrak{T}_{(\tau)}^{ss} \cong \mathfrak{T}_{H_{m_1,1}}^{ss} \times \cdots \times \mathfrak{T}_{H_{m_1,s_{m_1}}}^{ss} \times \mathfrak{T}_{I_{m_1,1}}^{ss} \times \cdots \times \mathfrak{T}_{I_{m_1,t_{m_1}}}^{ss} \times \mathfrak{T}_{H_{m_1+1,1}}^{ss} \times \cdots \times \mathfrak{T}_{H_{m_2,s_{m_2}}}^{ss}.$$

Lemma 4.14. *Let F^{ss} denote the connected components of Z_β^{ss} meeting $\mathfrak{T}_{(\tau)}^{ss}$; then for n sufficiently large*

$$p_\beta^{-1}(F^{ss}) \cap \mathfrak{T}^{tf} = p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss})$$

where $p_\beta : Y_\beta \to Z_\beta$ is the retraction given by $p_\beta(y) = \lim_{t \to 0} \lambda_\beta(t) \cdot y$.

Proof. Let n be chosen as in Proposition 4.13. Let $y \in p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss})$ so that $p_\beta(y) \in \mathfrak{T}_{(\tau)}^{ss} \subset F^{ss}$. If $y \notin \mathfrak{T}^{tf}$, then for all $t \neq 0$ we have $\lambda_\beta(t) \cdot y \notin \mathfrak{T}^{tf}$ which would contradict the openness of $\mathfrak{T}^{tf} \cap F^{ss}$ in F^{ss}.

Let $y = (q, [\varphi : 1]) \in \mathfrak{T}^{tf}$ be given by $q^i : V^i \otimes \mathcal{O}_X(-n) \to \mathcal{E}^i$ and $d^i : \mathcal{E}^i \to \mathcal{E}^{i+1}$. If $z = p_\beta(y) \in F^{ss}$, then as F^{ss} is contained in the diagonal components of Z_β^{ss}, the 1-PS λ_β induces a filtration of y by subcomplexes and the associated graded point z represents a direct sum of complexes (cf. Lemma 2.6). In particular, $z = p_\beta(y) \in \mathfrak{T}_{(\tau)}$ and is GIT semistable for the action of G' on Z_β with respect to $\mathcal{L}$ by Proposition 4.10. We can use similar arguments to those used in the proof of Lemma 4.11 to show that the direct summands $z_{i,j} \in \mathfrak{T}_{H_{i,j}}$ and $y_{i,j} \in \mathfrak{T}_{I_{i,j}}$ of z are $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable. □

4.7. **A comparison of the stratifications.** We are now ready to prove the main result relating the GIT stratification (6) with the Harder–Narasimhan stratification (7) with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$ of the parameter space $\mathfrak{T}^{tf}$. Let $\beta(\tau, n)$ denote the rational weight given in Definition 4.5.

Theorem 4.15. *For n sufficiently large we have:*

i) *$\beta = \beta(\tau, n)$ belongs to the index set $\mathcal{B}$ for the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $\overline{\mathfrak{T}}^{tf}$.*

ii) *$R_\tau = G p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss})$.*

iii) *The subscheme $R_\tau = G p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss})$ of the parameter scheme $\mathfrak{T}^{tf}$ parametrising complexes with Harder–Narasimhan type τ is a union of connected components of $S_\beta \cap \mathfrak{T}^{tf}$.*

Proof. Suppose n is sufficiently large as in Proposition 4.13. We defined β by fixing a point $z = (q^{m_1}, \dots, q^{m_2}, [\varphi : 1]) \in R_\tau$ corresponding to the complex $\mathcal{F}^\cdot$ with HN type τ. For i), it suffices to show $S_\beta \neq \phi$ and we claim $\overline{z} := p_\beta(z) \in Z_\beta^{ss}$. As λ_β induces the HN filtration of $\mathcal{F}^\cdot$, $\overline{z} = \lim_{t \to 0} \lambda_\beta(t) \cdot z$ is the graded object associated to this filtration. Then by Proposition 4.13 it suffices to show that each summand in the associated graded object is $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable, which follows by definition of the HN filtration.

The above argument shows that $p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss}) \subset R_\tau$ and since R_τ is G-invariant we have $Gp_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss}) \subset R_\tau$. For ii), suppose $y = (q^{m_1}, \dots, q^{m_2}, [\varphi : 1]) \in R_\tau$ corresponds to a complex $\mathcal{E}^\cdot$ with Harder–Narasimhan filtration

$$0 \subsetneq \mathcal{H}^\cdot_{m_1,(1)} \subsetneq \cdots \subsetneq \mathcal{H}^\cdot_{m_1,(s_{m_1})} \subsetneq \mathcal{I}^\cdot_{m_1,(1)} \cdots \mathcal{I}^\cdot_{m_1,(t_{m_1})} \subsetneq \cdots \subsetneq \mathcal{H}^\cdot_{m_2,(s_{m_2})} = \mathcal{E}.$$

of type τ. Then this filtration induces a filtration of each vector space V^i and we can choose a change of basis matrix g which switches this filtration with the filtration of V^i given at (8) used to define β. Then $g \cdot y \in p_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss})$ which completes the proof of ii).

Since F^{ss} is a union of connected components of Z_β^{ss}, the scheme $Gp_\beta^{-1}(F^{ss})$ is a union of connected components of S_β. Therefore, $Gp_\beta^{-1}(F^{ss}) \cap \mathfrak{T}^{tf}$ is a union of connected components of $S_\beta \cap \mathfrak{T}^{tf}$. Finally, by Lemma 4.14 we have $R_\tau = Gp_\beta^{-1}(\mathfrak{T}_{(\tau)}^{ss}) = Gp_\beta^{-1}(F^{ss}) \cap \mathfrak{T}^{tf}$ which proves iii). □

5. Quotients of the Harder–Narasimhan strata

In the previous section we saw for small ϵ, there is a parameter space R_τ for complexes of Harder–Narasimhan type τ with respect to $(\underline{1}, \delta\underline{\eta}/\epsilon)$ and R_τ is a union of connected components of a stratum $S_\beta(\tau) \cap \mathfrak{T}^{tf}(n)$ when n is sufficiently large. In this section we consider the problem of constructing a quotient of the G-action on this Harder–Narasimhan stratum R_τ. If a suitable quotient did exist, then it would provide a moduli space for complexes of this Harder–Narasimhan type. In particular, it would have the desirable property that for two complexes to represent the same point it is necessary that their cohomology sheaves have the same Harder–Narasimhan type.

By [9] Proposition 3.6, any stratum in a stratification associated to a linearised G-action on a projective scheme B has a categorical quotient. We can apply this to our situation and produce a categorical quotient of the G-action on R_τ.

Proposition 5.1. *The categorical quotient of the G-action on R_τ is isomorphic to*

$$\prod_{i=m_1}^{m_2} \prod_{j=1}^{s_i} M^{(\underline{1}, \delta\underline{\eta}/\epsilon)-ss}(X, H_{i,j}) \times \prod_{i=m_1}^{m_2-1} \prod_{j=1}^{t_i} M^{(\underline{1}, \delta\underline{\eta}/\epsilon)-ss}(X, I_{i,j})$$

where $M^{(\underline{1}, \delta\underline{\eta}/\epsilon)-ss}(X, P)$ denotes the moduli space of $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable complexes with invariants P. Moreover:

1. *A complex with invariants $H_{i,j}$ is just a shift of a sheaf and it is $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable if and only if the corresponding sheaf is Gieseker semistable.*
2. *A complex with invariants $I_{i,j}$ is concentrated in degrees $[i, i+1]$ and it is $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable if and only if it is isomorphic to a shift of the cone on the identity morphism of a Gieseker semistable sheaf.*

Proof. By [9] Proposition 3.6, the categorical quotient is equal to the GIT quotient of Stab β acting on $\mathfrak{T}_{(\tau)}$ with respect to the twisted linearisation $\mathcal{L}^{\chi_{-\beta}}$. It follows from Proposition 4.10 this is the same as the GIT quotient of the group G' acting on $\mathfrak{T}_{(\tau)}$ with respect to $\mathcal{L}$ and by Theorem 2.5, this is the product of moduli spaces of $(\underline{1}, \delta\underline{\eta}/\epsilon)$-semistable complexes with invariants given by τ. The final statement follows from Lemma 3.5, Remark 3.6 and the assumption on ϵ (cf. Assumption 4.1). □

In general this categorical quotient has lower dimension than expected and so is not a suitable quotient of the G-action on R_τ. Instead, we should perturb the linearisation used to construct the categorical quotient and take a quotient with respect to this perturbed linearisation. However, it is not always possible to find a way to perturb this linearisation and get an ample linearisation with nonempty semistable locus.

Since $R_\tau = GY^{ss}_{(\tau)} \cong G \times^{P_\beta} Y^{ss}_{(\tau)}$ where $Y^{ss}_{(\tau)} := p_\beta^{-1}(\mathfrak{T}^{ss}_{(\tau)})$, a categorical quotient of G acting on R_τ is equivalent to a categorical quotient of P_β acting on $Y^{ss}_{(\tau)}$. If we instead consider P_β acting on $Y^{ss}_{(\tau)}$, then there are perturbed linearisations which are ample although P_β is not reductive. In fact moduli spaces of objects with filtrations is one of the motivations for the work on non-reductive GIT (for example, see [4]).

Following [9], it is possible to take a quotient of the reductive part $\operatorname{Stab}\beta$ of P_β acting on $Y^{ss}_{(\tau)}$ with respect to an ample perturbed linearisation and get a moduli space for complexes of Harder–Narasimhan type with τ some additional data. The additional data is an 'n-rigidification' of the complex, which generalises the notion for sheaves given in [9] §7, and compensates for the fact that we are not quotienting by the full group P_β. In fact it is easy to see that an n-rigidification of a complex of HN type τ is equivalent to n-rigidifications of its cohomology sheaves $H^k(\mathcal{F}^\cdot)$ and images $\operatorname{Im} d^k$. The perturbation of the linearisation is given by a tuple $\underline{\theta}$ of rational numbers and this also determines a notion of semistability for complexes of HN type τ via the Hilbert–Mumford criterion.

Acknowledgements. This work forms part of my thesis which was funded by the EPSRC. I am very grateful to my supervisor Frances Kirwan for her support and guidance over the last few years. The final touches to this paper were made whilst at the Freie Universität Berlin and so I would also like to thank Alexander Schmitt for his hospitality and the SFB 647 "Space-Time-Matter" for their financial support.

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