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Quotients of unstable subvarieties and moduli spaces of sheaves of fixed Harder–Narasimhan type

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Abstract

When a reductive group G acts linearly on a complex projective scheme X, there is a stratification of X into G-invariant locally closed subschemes, with an open stratum X^{ss} formed by the semistable points in the sense of Mumford's geometric invariant theory which has a categorical quotient $X^{\mathrm{ss}} \to X//G$. In this article, we describe a method for constructing quotients of the unstable strata. As an application, we construct moduli spaces of sheaves of fixed Harder–Narasimhan type with some extra data (an 'n-rigidification') on a projective base.

1. *Introduction*

Let X be a complex projective scheme and G be a complex reductive group acting linearly on X with respect to an ample line bundle. Mumford's geometric invariant theory (GIT) [**19**] provides us with a projective scheme $X//G$ which is a categorical quotient of an open subscheme X^{ss} of X, whose geometric points are the semistable points of X, by the action of G. This GIT quotient $X//G$ contains an open subscheme X^s/G which is a geometric quotient of the scheme X^s of stable points for the linear action.

Associated to the linear action of G on X there is a stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of X into disjoint G-invariant locally closed subschemes, one of which is X^{ss} [**10**, **11**]. In this paper, we consider the problem of finding quotients for each unstable stratum S_β separately. For each $\beta \in \mathcal{B}$, we find a categorical quotient of the G-action on the stratum S_β. However, this categorical quotient is far from an orbit space in general. We attempt to rectify this by making small perturbations to a canonical linearization on a projective completion $\hat{S}_\beta$ of S_β and an associated affine bundle over S_β and considering GIT quotients with respect to these perturbed linearizations.

We then apply this to construct moduli spaces of unstable sheaves on a complex projective scheme W which have some additional data (depending on a choice of any sufficiently positive integer n) called an n-rigidification. There is a well-known construction due to Simpson [**21**] of the moduli space of semistable pure sheaves on W of fixed Hilbert polynomial as the GIT quotient of a linear action of a special linear group G on a scheme Q (closely related to a quot-scheme) which is G-equivariantly embedded in a projective space. This construction can be chosen so that elements of Q which parametrize sheaves of a fixed Harder–Narasimhan type form a stratum in the stratification of Q associated to the linear action of G (modulo taking connected components of strata). As above, we consider perturbations of the canonical linearization on a projective completion of this stratum using a parameter θ which defines for us a notion of semistability for sheaves of this fixed Harder–Narasimhan type τ. Finally, for each τ, we construct a moduli space of S-equivalence classes of θ-semistable n-rigidified sheaves of fixed Harder–Narasimhan type τ.

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The layout of this paper is as follows. Section 2 summarizes the properties of the stratifications introduced in [**10**, **11**] when X is a non-singular complex projective variety with a linear G-action. In § 3, we construct linearizations on a projective completion of a given stratum in this stratification and provide a categorical quotient of each unstable stratum. In § 4, we observe that this construction can be extended without any difficulty from varieties to schemes. Section 5 summarizes Simpson's construction of moduli spaces of semistable sheaves and calculates the associated Hilbert–Mumford functions for one-parameter subgroups, while § 6 relates the stratification of the parameter scheme Q to Harder–Narasimhan type. In § 7, we define what we mean by an n-rigidified sheaf. Finally, in § 8, we construct moduli spaces for n-rigidified sheaves of fixed Harder–Narasimhan type which are semistable with respect to a given parameter θ.

2. *Stratifications of X*

In this section, we state the results needed from [**11**] for linear reductive group actions on non-singular projective varieties. Let G be a complex reductive group acting linearly on a smooth complex projective variety X with respect to an ample line bundle $\mathcal{L}$. Abusing notation, we will use $\mathcal{L}$ to denote both the linearization (the lift of the G-action to the line bundle) and the line bundle itself. For the purposes of GIT, we can assume, without loss of generality, that $\mathcal{L}$ is very ample, so that X is embedded in a projective space $\mathbb{P}^n = \mathbb{P}(H^0(X,\mathcal{L})^*)$ and the action of G is given by a homomorphism $\rho : G \to \mathrm{GL}(n+1)$. The associated GIT quotient $X/\!/G = X/\!/_{\mathcal{L}}G$ is topologically the semistable set $X^{\mathrm{ss}} = X^{\mathrm{ss}}(\mathcal{L})$ modulo S-equivalence, where x and y in X^{ss} are S-equivalent if and only if the closures of their G-orbits meet in X^{ss}. The fact that G is a complex reductive group means that it is the complexification of a maximal compact subgroup K, and we assume, without loss of generality, that K acts unitarily on $\mathbb{P}^n$ via $\rho : K \to \mathrm{U}(n+1)$.

Since X is non-singular, the Fubini–Study metric on $\mathbb{P}^n$ gives X a Kähler structure and the Kähler form ω is a K-invariant symplectic form on X. Let $\mathfrak{K}$ denote the Lie algebra of K; the action of K on the symplectic manifold (X,ω) is Hamiltonian with moment map $\mu : X \to \mathfrak{K}^*$ defined by

$$\mu(x) := \rho^*\left(\frac{x^\star \bar{x}^{\star t}}{2\pi i\|x^\star\|^2}\right),$$

where $x^\star \in \mathbb{C}^{n+1}$ lies over $x \in \mathbb{P}^n$ and $\rho^* : \mathfrak{u}(n+1)^* \to \mathfrak{K}^*$ is dual to $\mathrm{Lie}\,\rho$. Then $x \in X$ is semistable if and only if the closure of its G-orbit meets $\mu^{-1}(0)$, and the inclusion of $\mu^{-1}(0)$ in X^{ss} induces a homeomorphism from the symplectic quotient $\mu^{-1}(0)/K$ to the GIT quotient $X/\!/G$.

We fix an inner product on the Lie algebra $\mathfrak{K}$ which is invariant under the adjoint action of K, and use it to identify $\mathfrak{K}^*$ with $\mathfrak{K}$. The norm square of the moment map $\|\mu\|^2 : X \to \mathbb{R}$ with respect to this inner product induces a Morse-type stratification of X into G-invariant locally closed non-singular subvarieties

$$X = \bigsqcup_{\beta\in\mathcal{B}} S_\beta,$$

where the indexing set $\mathcal{B}$ is a finite set of adjoint orbits in the Lie algebra $\mathfrak{K}$ (or equivalently a finite set of points in a fixed positive Weyl chamber $\mathfrak{t}_+$ in $\mathfrak{K}$). In particular, $0 \in \mathcal{B}$ indexes the open stratum S_0, which is equal to the semistable subset X^{ss}.

Remark 1. It is important to note that this stratification depends on the choice of linearization and the choice of invariant inner product on $\mathfrak{K}$. However, the stratification is unchanged if the ample line bundle $\mathcal{L}$ is replaced with $\mathcal{L}^{\otimes m}$ for any integer $m > 0$, which means that we can work with rational linearizations $\mathcal{L}^{\otimes q}$ for $q \in \mathbb{Q} \cap (0, \infty)$.

REMARK 2. The gradient flow of $\|\mu\|^2$ from any $x \in X$ is contained in the G-orbit of x, and so the stratification of X is given by intersecting X with the stratification of the ambient projective space $\mathbb{P}^n$.

REMARK 3. If X is singular (and/or quasi-projective rather than projective), we still get a stratification of X into G-invariant locally closed subvarieties, which may be singular, by intersecting X with the stratification of the ambient projective space $\mathbb{P}^n$. Indeed, as we will see in § 4, we can allow X to be any G-invariant projective subscheme of $\mathbb{P}^n$ and obtain a stratification of X by intersecting X with the stratification of $\mathbb{P}^n$.

The strata indexed by non-zero $\beta \in \mathcal{B}$ have an inductive description in terms of semistable sets for actions of reductive subgroups of G on subvarieties of X (see [**11**]). We fix a maximal torus T of K and let $H := T_{\mathbb{C}}$ be the complexification of T, which is a maximal torus of $G = K_{\mathbb{C}}$. We also fix a positive Weyl chamber $\mathfrak{t}_+$ in the Lie algebra $\mathfrak{t}$ of T. The restriction $\rho|_T : T \to \mathrm{U}(n+1)$ is diagonalizable with weights

$$\alpha_0, \ldots, \alpha_n : T \longrightarrow S^1.$$

If we identify the tangent space of S^1 at the identity with the line $2\pi i\mathbb{R}$ in the complex plane, and identify $2\pi i\mathbb{R}$ with $\mathbb{R}$ in the natural way, then by taking the derivative of α_j at the identity we obtain an element of the dual of the Lie algebra $\mathfrak{t}$ which we also call α_j. The index set $\mathcal{B}$ is defined in [**11**] to be the set of $\beta \in \mathfrak{t}_+$ such that β is the closest point to zero of the convex hull in $\mathfrak{t}$ of some non-empty subset of the set of weights $\{\alpha_0, \ldots \alpha_n\}$.

REMARK 4. Since the set of weights $\{\alpha_0, \ldots \alpha_n\}$ is invariant under the Weyl group, $\mathcal{B}$ can also be identified with the set of K-orbits in $\mathfrak{K}$ of closest points to 0 of convex hulls of subsets of $\{\alpha_0, \ldots, \alpha_n\}$.

If $\beta \in \mathcal{B}$, we define Z_β to be

$$Z_\beta := X \cap \{[x_0 : \cdots : x_n] \in \mathbb{P}^n : x_i = 0 \text{ if } \alpha_i \cdot \beta \neq \|\beta\|^2\}. \tag{2.1}$$

Z_β also has a symplectic description as the set of critical points for the function $\mu_\beta(x) := \mu(x) \cdot \beta$ on which μ_β takes the value $\|\beta\|^2$. By [**11**, Lemma 3.15] the critical point set of $\|\mu\|^2$ is the disjoint union over $\beta \in \mathcal{B}$ of the closed subsets

$$C_\beta := K(Z_\beta \cap \mu^{-1}(\beta)). \tag{2.2}$$

The stratum S_β corresponding to the critical point set C_β is the set of points in X whose path of steepest descent under $\|\mu\|^2$ has a limit point in C_β.

REMARK 5. The stratum S_β depends only on the adjoint orbit of β, but in order to define Z_β we need to fix an element in that adjoint orbit.

The strata have an alternative algebraic description. Let $\operatorname{Stab}\beta$ be the stabilizer of β under the adjoint action of G on its Lie algebra $\mathfrak{g}$; then Z_β is $\operatorname{Stab}\beta$-invariant [**11**, § 4.8]. We consider the action of $\operatorname{Stab}\beta$ on Z_β with respect to the original linearization twisted by the character

$-\beta$ of $\operatorname{Stab}\beta$, so that the semistable set Z_β^{ss} with respect to this modified linearization is equal to the open stratum for the Morse stratification of the function $\|\mu-\beta\|^2$ on Z_β. Let

$$Y_\beta := X \cap \left\{ [x_0 : \cdots : x_n] \in \mathbb{P}^n : \begin{array}{l} x_i = 0 \text{ if } \alpha_i \cdot \beta < \|\beta\|^2 \text{ and } x_i \neq 0 \\ \text{for some } i \text{ such that } \alpha_i \cdot \beta = \|\beta\|^2 \end{array} \right\} \tag{2.3}$$

be the set of points in X whose corresponding weights are all on the opposite side to the origin of the hyperplane to β and such that at least one of the weights lies on the hyperplane to β. In the symplectic description, Y_β is the set of points in X whose path of steepest descent under μ_β has limit in Z_β. There is an obvious surjection $p_\beta : Y_\beta \to Z_\beta$ which is a retraction onto Z_β. We define $Y_\beta^{\mathrm{ss}} = p_\beta^{-1}(Z_\beta^{\mathrm{ss}})$; then by [**11**, Theorem 6.18]

$$S_\beta = GY_\beta^{\mathrm{ss}}.$$

The positive Weyl chamber $\mathfrak{t}_+$ corresponds to a choice of positive roots

$$\Phi_+ := \{\alpha \in \Phi : \alpha \cdot \eta \geqslant 0 \text{ for all } \eta \in \mathfrak{t}_+\},$$

where $\Phi \subset \mathfrak{t}^*$ is the set of roots coming from the adjoint action of T on $\mathfrak{g}$. This in turn corresponds to a Borel subgroup $B = B_+$ of G such that the Lie algebra $\mathfrak{b}_+$ of B_+ is given by

$$\mathfrak{b}_+ := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha.$$

For $\beta \in \mathfrak{t}_+$, we construct a parabolic subgroup $P_\beta := B_+ \operatorname{Stab}\beta$ which may also be defined as

$$P_\beta := \left\{ g \in G : \lim_{t \to -\infty} \exp(it\beta) g \exp(it\beta)^{-1} \text{ exists in } G \right\}.$$

The subsets Y_β^{ss} and Y_β are P_β-invariant (see [**11**, Lemma 6.10]) and by [**11**, Theorem 6.18] there is an isomorphism

$$S_\beta \cong G \times_{P_\beta} Y_\beta^{\mathrm{ss}}.$$

Remark 6. This stratification can also be described in terms of the work of Kempf and Ness [**10**] and Hesselink [**7**]. The Hilbert–Mumford criterion gives a test for (semi-)stability in terms of limits of one-parameter subgroups (1-PSs) acting on a given point $x \in X$. Given a 1-PS λ, we define $\mu(x, \lambda)$ to be the integer equal to the weight of the $\mathbb{C}^*$-action induced by this 1-PS on the fibre $\mathcal{L}_{x_0}$ where $x_0 = \lim_{t \to 0} \lambda(t) \cdot x$. We call $\mu(x, \lambda)$ the Hilbert–Mumford function and the Hilbert–Mumford criterion states that x is semistable if and only if $\mu(x, \lambda) \geqslant 0$ for all 1-PSs. A point x is unstable if and only if it fails the Hilbert–Mumford criterion for at least one 1-PS, and there is a notion of an *adapted* 1-PS for this point: that is, a non-divisible 1-PS λ for which the quantity $\mu(\lambda, x)/\|\lambda\|$ is minimized. The set $\wedge^{\mathcal{L}}(x)$ of 1-PSs which are adapted to x is studied by Kempf [**9**], who shows that $\wedge^{\mathcal{L}}(x)$ is a full conjugacy class of 1-PSs in a parabolic subgroup P_x of G. In fact, for each $\lambda \in \wedge^{\mathcal{L}}(x)$,

$$P_x = P(\lambda) := \left\{ g \in G : \lim_{t \to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists in } G \right\}.$$

These sets of 1-PSs give us a stratification of the unstable locus $X - X^{\mathrm{ss}}$ (see [**10**]), which agrees with the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ described above, as follows.

Each $\beta \in \mathcal{B}$ is rational in the sense that there is a natural number $m > 0$ such that $m\beta$ defines a 1-PS $\mathbb{C}^* \to H = T_{\mathbb{C}}$ whose restriction to $S^1 \to T$ has derivative at the identity

$$\mathbb{R} \cong 2\pi i\mathbb{R} \cong \operatorname{Lie} S^1 \to \mathfrak{t}$$

sending 1 to $m\beta$. For any rational $\beta \in \mathfrak{t}$, let $\lambda_\beta : \mathbb{C}^* \to H$ be the unique non-divisible 1-PS which is defined by $q\beta$ for some positive rational number q. Then if $\beta \in \mathcal{B} \setminus \{0\}$, we have

$$P_\beta = P(\lambda_\beta)$$

and λ_β is a 1-PS adapted to x.

3. *Quotients of the unstable strata*

Let $\beta \in \mathcal{B} \setminus \{0\}$ be a non-zero index for the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ and consider the projective completion

$$\hat{S_\beta} := G \times_{P_\beta} \overline{Y_\beta} \subset G \times_{P_\beta} X$$

of the stratum $S_\beta \cong G \times_{P_\beta} Y_\beta^{\mathrm{ss}}$, where $\overline{Y_\beta}$ is the closure of Y_β^{ss} in X.

Remark 7. It is always the case that

$$\overline{Y_\beta} \subseteq X \cap \{[x_0 : \cdots : x_n] : x_j = 0 \text{ if } \alpha_j \cdot \beta < \|\beta\|^2\}.$$

We often have equality here (for example when $X = \mathbb{P}^n$) but it might be the case, for example, that $X \cap \{[x_0 : \cdots x_n] : x_j = 0 \text{ if } \alpha_j \cdot \beta < \|\beta\|^2\}$ has connected components which do not meet Y_β.

It may also be the case that Z_β, Y_β and S_β are disconnected (cf. [**11**, § 5]), in which case we can, if we wish, refine the stratification by replacing Z_β with its connected components $Z_{\beta,j}$, say, and setting $S_{\beta,j} = GY_{\beta,j}^{\mathrm{ss}}$ where $Y_{\beta,j}^{\mathrm{ss}} = p_\beta^{-1}(Z_{\beta,j}^{\mathrm{ss}})$ and $Z_{\beta,j}^{\mathrm{ss}} = Z_{\beta,j} \cap Z_\beta^{\mathrm{ss}}$. Then each $S_{\beta,j}$ will be a connected component of S_β (so long as $Z_{\beta,j}^{\mathrm{ss}}$ is non-empty). In what follows, we will work with S_β for simplicity of notation, but we could also work with its connected components separately.

The action of P_β on X extends to an action of G on X so there is a natural isomorphism

$$G \times_{P_\beta} X \cong G/P_\beta \times X,$$
$$(g, x) \longmapsto (gP_\beta, g \cdot x).$$

In order to find new linearizations on $\hat{S_\beta}$ we can consider linearizations on $G \times_{P_\beta} X \cong G/P_\beta \times X$ and restrict them to $\hat{S_\beta}$. The quotient G/P_β is a partial flag variety and linearizations of the G-actions on such varieties are well understood.

3.1. *Line bundles on partial flag varieties G/P*

We review the construction of line bundles on partial flag varieties; for more detailed information, see [**13**–**16**].

For the moment, we assume that G is semisimple and simply connected. Fix sets of positive roots $\Phi_+ \subset \Phi$ and simple roots Π. Let ω_i denote the fundamental dominant weight associated to a simple root α_i. If $\lambda = \sum a_i \omega_i$ is a dominant weight, then define

$$\Pi_\lambda := \{\alpha_i \in \Pi : a_i = 0\} \subset \Pi.$$

Let λ also denote the corresponding one-parameter subgroup; then the parabolic subgroup $P(\lambda)$ associated to the 1-PS λ has associated simple roots

$$\Pi_{P(\lambda)} := \{\alpha_i \in \Pi : -\alpha_i \text{ is a root of } P(\lambda)\} \subset \Pi$$

and these sets agree, so that $\Pi_\lambda = \Pi_{P(\lambda)}$.

A character $\chi : H \to \mathbb{C}^*$ extends to $P(\lambda)$ if and only if $\chi \cdot \alpha^\vee = 0$ for all co-roots $\alpha^\vee$ such that $\alpha \in \Pi_{P(\lambda)}$. The weights naturally correspond to characters and the character defined by λ extends to $P(\lambda)$ since

$$\lambda \cdot \alpha_i^\vee = a_i = 0 \quad \text{for all } \alpha_i \in \Pi_{P(\lambda)}$$

by the definition of this set. We let λ also denote the associated character of $P(\lambda)$ and define a line bundle $\mathcal{L}(\lambda)$ on $G/P(\lambda)$ to be the line bundle associated to the character λ^{-1}; that is,

$$\begin{array}{c} \mathcal{L}(\lambda) := G \times_{P(\lambda)} \mathbb{C} \\ \downarrow \\ G/P(\lambda), \end{array}$$

where (g, z) and $(gp, \lambda(p)z)$ are identified for all $p \in P(\lambda)$. The sections of $\mathcal{L}(\lambda)$ are given by

$$H^0(G/P(\lambda), \mathcal{L}(\lambda)) = \{f : G \longrightarrow \mathbb{C} : f(gp) = \lambda(p)f(g) \text{ for all } g \in G, p \in P\}$$

and the natural left G-action gives this vector space a G-module structure. Let $V(\lambda)$ denote the representation of G of highest weight λ. By the Borel–Weil–Bott theorem [**3**], there is an isomorphism of G-modules

$$H^0(G/P(\lambda), \mathcal{L}(\lambda)) \cong V(\lambda)^*.$$

The line bundle $\mathcal{L}(\lambda)$ is very ample if and only if

$$\lambda \cdot \alpha_i^\vee = a_i > 0 \quad \text{for all } \alpha_i \notin \Pi_{P(\lambda)}$$

which is clearly the case by definition of $\Pi_\lambda = \Pi_{P(\lambda)}$. Thus, there is an embedding

$$G/P(\lambda) \hookrightarrow \mathbb{P}(H^0(G/P(\lambda), \mathcal{L}(\lambda))^*) \cong \mathbb{P}(V(\lambda))$$

which is the natural projective embedding of the partial flag variety $G/P(\lambda)$. More concretely, let $v_{\max}$ denote the highest weight vector in $V(\lambda)$, so that $v_{\max}$ is an eigenvector for the action of T with eigenvalue λ; then the embedding is given by the inclusion of the orbit $G \cdot v_{\max}$,

$$\begin{aligned} G/P(\lambda) &\hookrightarrow \mathbb{P}(V(\lambda)), \\ gP(\lambda) &\longmapsto [g \cdot v_{\max}]. \end{aligned}$$

REMARK 8. We will be primarily interested in the case when G is a subgroup of $\mathrm{GL}(n)$ and the weight λ is restricted from $\mathrm{GL}(n)$, and here we do not need to assume that G is simply connected or semisimple. For we can view the weight λ as an element of the dual of the Lie algebra of both $\mathrm{GL}(n)$ and $\mathrm{PGL}(n)$ or equivalently $\mathrm{SL}(n)$. There are associated parabolics $P(\lambda_{\mathrm{GL}})$ and $P(\lambda_{\mathrm{SL}})$, and the partial flag varieties for these two parabolics agree

$$\mathrm{GL}(n)/P(\lambda_{\mathrm{GL}}) = \mathrm{SL}(n)/P(\lambda_{\mathrm{SL}}).$$

Since $\mathrm{SL}(n)$ is semisimple and simply connected there is a projective embedding of this partial flag variety into $\mathbb{P}(V(\lambda_{\mathrm{SL}}))$ where $V(\lambda_{\mathrm{SL}})$ is the representation of $\mathrm{SL}(n)$ with highest weight λ_{SL}. We have $G \subset \mathrm{GL}(n)$ and $P(\lambda) = G \cap P(\lambda_{\mathrm{GL}})$ and so

$$G/P(\lambda) \subseteq \mathrm{GL}(n)/P(\lambda_{\mathrm{GL}}) = \mathrm{SL}(n)/P(\lambda_{\mathrm{SL}}).$$

Hence, we can use this inclusion and the embedding of $\mathrm{SL}(n)/P(\lambda_{\mathrm{SL}})$ described above to obtain a projective embedding of $G/P(\lambda)$.

3.2. *The canonical linearization on* $\hat{S}_\beta$

We have seen that given a one-parameter subgroup λ of G as above there is a natural ample linearization of the G-action on the partial flag variety $G/P(\lambda)$. We can apply this to the case when the parabolic subgroup is $P_\beta = P(\lambda_\beta)$. The natural embedding of the partial flag variety G/P_β is thus given by the very ample line bundle $\mathcal{L}(\lambda_\beta)$

$$G/P_\beta \hookrightarrow \mathbb{P}(H^0(G/P_\beta, \mathcal{L}(\lambda_\beta))^*) \cong \mathbb{P}(V(\beta)).$$

Let $\mathcal{L}_\beta$ denote the G-linearization on $G/P_\beta \times X$ given by the tensor product of the pullbacks of $\mathcal{L}(\lambda_{-\beta})$ on G/P_β and $\mathcal{L}$ on X to $G/P_\beta \times X$. We also let $\mathcal{L}_\beta$ denote the restriction of this linearization to $\hat{S}_\beta$ and call this the canonical linearization. There is also a canonical linearization $\mathcal{L}_\beta$ of the $\operatorname{Stab}\beta$-action on Z_β given by twisting the original linearization $\mathcal{L}$ by the character of $\operatorname{Stab}\beta$ corresponding to $-\beta$. Recall that Z_β^{ss} is defined to be the semistable subset for this linearization. The character of $\operatorname{Stab}\beta$ corresponding to $-\beta$ extends to a character of P_β and so there is also a canonical linearization $\mathcal{L}_\beta$ of the P_β-action (or the $\operatorname{Stab}\beta$-action) on Y_β given by twisting $\mathcal{L}$ by the character corresponding to $-\beta$. All of these linearizations are equal to the restriction of the canonical G-linearization $\mathcal{L}_\beta$ on $\hat{S}_\beta$ to the relevant subvarieties and subgroups. The following lemma explains why we call $\mathcal{L}_\beta$ the canonical linearization.

LEMMA 3.1. *We have isomorphisms of graded algebras*

$$\bigoplus_{r\geqslant 0} H^0(\hat{S}_\beta, \mathcal{L}_\beta^{\otimes r})^G \cong \bigoplus_{r\geqslant 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta} \cong \bigoplus_{r\geqslant 0} H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\operatorname{Stab}\beta}.$$

Proof. The first isomorphism follows from the fact that $\hat{S}_\beta = G \times_{P_\beta} \bar{Y}_\beta$ and the canonical G-linearization on $\hat{S}_\beta$ is equal to $G \times_{P_\beta} \mathcal{L}_\beta$ where here $\mathcal{L}_\beta$ is the canonical P_β-linearization on $\bar{Y}_\beta$.

Let $\lambda_\beta : \mathbb{C}^* \to G$ be the 1-PS determined by the rational weight β. Then $\lambda_\beta(\mathbb{C}^*) \subseteq P_\beta$ and so

$$\bigoplus_{r\geqslant 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta} \subseteq \bigoplus_{r\geqslant 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{\lambda_\beta(\mathbb{C}^*)}.$$

The torus $\lambda_\beta(\mathbb{C}^*)$ acts on $\bar{Y}_\beta$ with respect to the canonical linearization $\mathcal{L}_\beta$ with non-negative weights, and has zero weights exactly on Z_β. Hence

$$\bigoplus_{r\geqslant 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{\lambda_\beta(\mathbb{C}^*)} \cong \bigoplus_{r\geqslant 0} H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\lambda_\beta(\mathbb{C}^*)}$$

and so

$$\bigoplus_{r\geqslant 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta} \subseteq \bigoplus_{r\geqslant 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{\operatorname{Stab}\beta} \cong \bigoplus_{r\geqslant 0} H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\operatorname{Stab}\beta}.$$

Let $\sigma \in H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\operatorname{Stab}\beta}$ and consider $p_\beta^*\sigma \in H^0(Y_\beta, \mathcal{L}_\beta^{\otimes r})^{\operatorname{Stab}\beta}$ where $p_\beta : Y_\beta \to Z_\beta$ is the retraction defined by β. We have that $P_\beta = \operatorname{Stab}\beta U_\beta$ where U_β is the unipotent radical of P_β and there is a retraction $q_\beta : P_\beta \to \operatorname{Stab}\beta$ such that

$$p_\beta(p \cdot y) = q_\beta(p) \cdot p_\beta(y) \tag{3.1}$$

for all $y \in Y_\beta$ and $p \in P_\beta$. The action of P_β on $H^0(Y_\beta, \mathcal{L}_\beta^{\otimes r})$ is induced from its action on Y_β and $\mathcal{L}_\beta$, and so if $p \in P_\beta$ we have

$$p \cdot p_\beta^*\sigma = p_\beta^*(q_\beta(p) \cdot \sigma) = p_\beta^*\sigma$$

as σ is $\operatorname{Stab}\beta$ invariant. Therefore,

$$\bigoplus_{r\geqslant 0} H^0(\bar{Y}_\beta, \mathcal{L}_\beta^{\otimes r})^{P_\beta} \cong \bigoplus_{r\geqslant 0} H^0(Z_\beta, \mathcal{L}_\beta^{\otimes r})^{\operatorname{Stab}\beta}.$$

□

REMARK 9. Unfortunately if $\beta \neq 0$, then $\mathcal{L}(\lambda_{-\beta})$ is a non-ample linearization of the G-action on G/P_β, and the canonical G-linearization $\mathcal{L}_\beta$ on $\hat{S}_\beta$ is in general non-ample too, as the following example shows.

EXAMPLE 1. Consider $G = \mathrm{SL}(2, \mathbb{C})$ acting on the complex projective line $X = \mathbb{P}^1$ with respect to $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(1)$. The semistable set is empty and the action is transitive so there will be one non-zero index in the stratification of X. We choose a maximal torus $T = \{\mathrm{diag}(t, t^{-1}) : t \in S^1\}$; then the weights of T acting on $\mathbb{C}^2$ are $\alpha_0 = \alpha, \alpha_1 = \alpha^{-1}$ where

$$\alpha : T \longrightarrow S^1, \qquad \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \longmapsto t.$$

The Lie algebra of T is $\mathfrak{t} \cong \mathbb{R}$ and we pick the positive Weyl chamber $\mathfrak{t}_+$ which contains α. Then $\beta = \alpha$ is an index for the stratification of X and we have that

$$Z_\beta = Z_\beta^{\mathrm{ss}} = Y_\beta = Y_\beta^{\mathrm{ss}} = \{[1:0]\}$$

and $S_\beta = X$. The parabolic subgroup P_β is the Borel subgroup of upper triangular matrices and we have an isomorphism

$$G/P_\beta \cong \mathbb{P}^1, \qquad gP_\beta \longmapsto g \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

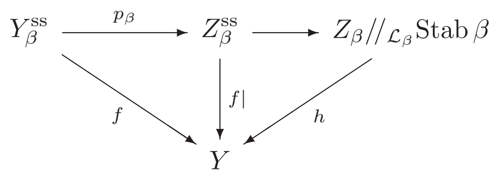
The very ample line bundle on G/P_β defined by β is $\mathcal{O}_{\mathbb{P}^1}(1)$ and the line bundle defined by $-\beta$ is $\mathcal{O}_{\mathbb{P}^1}(-1)$. The canonical linearization is given by restricting $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \cong G/P_\beta \times X$ to $\hat{S}_\beta \cong S_\beta = X$. The morphism $S_\beta \to \mathbb{P}^1 \times \mathbb{P}^1$ is the diagonal morphism and so the canonical linearization on $S_\beta = X$ is $\mathcal{L}_\beta = \mathcal{O}_{\mathbb{P}^1}$.

PROPOSITION 3.2. *The projective variety* $Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\,\beta$ *is a categorical quotient for the action of*

(i) $\mathrm{Stab}\,\beta$ *on* Z_β^{ss},
(ii) $\mathrm{Stab}\,\beta$ *on* Y_β^{ss},
(iii) P_β *on* Y_β^{ss},
(iv) G *on* S_β.

Proof. The natural morphism $Z_\beta^{\mathrm{ss}} \to Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\,\beta$ is a categorical quotient by classical GIT since $\mathcal{L}_\beta$ is ample on Z_β and $\mathrm{Stab}\,\beta$ is reductive, so (i) is proved.

There is a surjective morphism $Y_\beta^{\mathrm{ss}} \to Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\,\beta$ given by the composition of the retraction $p_\beta : Y_\beta^{\mathrm{ss}} \to Z_\beta^{\mathrm{ss}}$ with the categorical quotient $Z_\beta^{\mathrm{ss}} \to Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\,\beta$. Moreover, this surjective morphism $Y_\beta^{\mathrm{ss}} \to Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\,\beta$ is P_β-invariant by (3.1) and the $\mathrm{Stab}\,\beta$-invariance of $Z_\beta^{\mathrm{ss}} \to Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\,\beta$. Thus to prove (ii) and (iii) it suffices to show that any $\mathrm{Stab}\,\beta$-invariant morphism $f : Y_\beta^{\mathrm{ss}} \to Y$ factors through $Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\,\beta$. As f is $\mathrm{Stab}\,\beta$-invariant it is constant on orbit closures and so $f = f|_{Z_\beta^{\mathrm{ss}}} \circ p_\beta$. Since $f|_{Z_\beta^{\mathrm{ss}}} : Z_\beta^{\mathrm{ss}} \to Y$ is $\mathrm{Stab}\,\beta$-invariant, there is a morphism $h : Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\,\beta \to Y$ such that $f|_{Z_\beta^{\mathrm{ss}}}$ is the composition of h with the categorical quotient $Z_\beta^{\mathrm{ss}} \to Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\,\beta$ of the $\mathrm{Stab}\,\beta$-action on Z_β^{ss}. Then we have a commutative diagram



where $f| = f|_{Z_\beta^{\mathrm{ss}}}$ and the morphism f factors through $Z_\beta/\!/_{\mathcal{L}_\beta}\mathrm{Stab}\,\beta$ as required.

Thus (ii) and (iii) are proved, and (iv) now follows immediately from the fact that $S_\beta \cong G \times_{P_\beta} Y_\beta^{\mathrm{ss}}$. □

Remark 10. From Lemma 3.1 and Proposition 3.2, we see that $Z_\beta/\!/_{\mathcal{L}_\beta}\mathrm{Stab}\,\beta$ has properties we would like and expect for a GIT quotient of the actions of $\mathrm{Stab}\,\beta$ and P_β on $\bar{Y}_\beta$ and of G on $\hat{S}_\beta$ with respect to the linearization $\mathcal{L}_\beta$. The linearization $\mathcal{L}_\beta$ is ample on $\bar{Y}_\beta$ and the proofs above do indeed show that Y_β^{ss} is the semistable set for this linear action of $\mathrm{Stab}\,\beta$ and that the GIT quotient is $Z_\beta/\!/_{\mathcal{L}_\beta}\mathrm{Stab}\,\beta$. However, the parabolic subgroup P_β of G is not usually reductive and the linearization $\mathcal{L}_\beta$ is not in general ample on $\hat{S}_\beta$, so we cannot apply classical GIT to the actions of P_β on $\overline{Y_\beta}$ and G on $\hat{S}_\beta$ with respect to the linearization $\mathcal{L}$. For a linear action of a reductive group G on a variety X with respect to a non-ample line bundle, Mumford does define in [**19**] a notion of semistability and shows that the resulting semistable set X^{ss} has a categorical quotient; however, according to his definition for the linearization $\mathcal{L}_\beta$ on $\hat{S}_\beta$ we would not in general get $\hat{S}_\beta^{\mathrm{ss}} = S_\beta$ with the categorical quotient being $Z_\beta/\!/_{\mathcal{L}_\beta}\mathrm{Stab}\,\beta$. Indeed in Example 1 Mumford's semistable set and categorical quotient are empty.

Remark 11. The categorical quotient $S_\beta \to Z_\beta/\!/_{\mathcal{L}_\beta}\mathrm{Stab}\,\beta$ collapses more orbits than we might like, resulting in the GIT quotient having lower dimension than expected. This happens because if $y \in Y_\beta^{\mathrm{ss}}$, then $p_\beta(y) \in \overline{\mathrm{Stab}\,\beta \cdot y} \subseteq \overline{G \cdot y}$, and so in the quotient every point in Y_β^{ss} is identified with its projection to Z_β^{ss}.

3.3. *Perturbations of the canonical linearization*

To resolve the issue mentioned in Remark 11, we would like to perturb the canonical linearization $\mathcal{L}_\beta$ for the action of G on $\hat{S}_\beta$ or the action of P_β on $\overline{Y_\beta}$ and take a GIT quotient with respect to this perturbed linearization. Unfortunately, as we observed in Remark 10, on $\hat{S}_\beta$ the canonical G-linearization is not ample, whereas on $\bar{Y}_\beta$ it is ample, but P_β is not reductive, and so in each case applying GIT is delicate. On the other hand, $\mathrm{Stab}\,\beta$ is reductive and $\mathcal{L}_\beta$ is an ample $\mathrm{Stab}\,\beta$-linearization on $\bar{Y}_\beta$, so we can try perturbing this linearization.

Remark 12. Note that although $\bar{Y}_\beta/\!/_{\mathcal{L}_\beta}\mathrm{Stab}\,\beta \cong Z_\beta/\!/_{\mathcal{L}_\beta}\mathrm{Stab}\,\beta$ is a categorical quotient for the G-action on S_β by Proposition 3.2, after a perturbation we would no longer expect the GIT quotient $\bar{Y}_\beta/\!/\mathrm{Stab}\,\beta$ to give us a categorical quotient of the G-action on an open subset of S_β. Instead, if U is a $\mathrm{Stab}\,\beta$-invariant open subscheme of Y_β^{ss}, then a categorical quotient for the $\mathrm{Stab}\,\beta$-action on U will be a categorical quotient for the G-action on $G \times_{\mathrm{Stab}\,\beta} U$. Moreover, since $S_\beta \cong G \times_{P_\beta} Y_\beta^{\mathrm{ss}}$, we have a surjective morphism

$$\begin{aligned} G \times_{\mathrm{Stab}\,\beta} Y_\beta^{\mathrm{ss}} &\longrightarrow S_\beta, \\ [g, y] &\longmapsto g \cdot y \end{aligned}$$

with fibres isomorphic to $P_\beta/\mathrm{Stab}\,\beta \cong U_\beta$, the unipotent radical of P_β, which as an algebraic variety is isomorphic to an affine space.

Recall that the canonical $\mathrm{Stab}\,\beta$-linearization $\mathcal{L}_\beta$ on $\bar{Y}_\beta$ is ample and is equal to $\mathcal{L}$ twisted by the character of $\mathrm{Stab}\,\beta$ associated to $-\beta$. Therefore, to perturb this linearization, we can perturb the original linearization $\mathcal{L}$ and/or make a perturbation of the character by using $-(\beta+\epsilon\beta')$ rather than $-\beta$ where $\beta' \in \mathfrak{t}_+$ is a rational weight and ϵ is a small rational number.

The norm square of the moment map associated to the canonical $\mathrm{Stab}\,\beta$-linearization $\mathcal{L}_\beta$ on $\bar{Y}_\beta$ gives us a stratification

$$\bar{Y}_\beta = \bigsqcup_{\delta \in \hat{\mathcal{B}}_\beta} S_\delta^{\mathrm{can}}$$

of $\bar{Y}_\beta$ such that $S_0^{\mathrm{can}} = Y_\beta^{\mathrm{ss}}$. A perturbation of this linearization also has an associated moment map which gives us a new stratification

$$\bar{Y}_\beta = \bigsqcup_{\gamma \in \hat{\mathcal{B}}_\beta^{\mathrm{per}}} S_\gamma^{\mathrm{per}}$$

such that $S_0^{\mathrm{per}} \subseteq S_0^{\mathrm{can}} = Y_\beta^{\mathrm{ss}}$. The next proposition shows that provided the perturbation is sufficiently small, the second stratification is a refinement of the first stratification. In particular, this proposition shows that there is a subset

$$\mathcal{B}_\beta^{\mathrm{per}} \subset \hat{\mathcal{B}}_\beta^{\mathrm{per}}$$

such that

$$Y_\beta^{\mathrm{ss}} = \bigsqcup_{\gamma \in \mathcal{B}_\beta^{\mathrm{per}}} S_\gamma^{\mathrm{per}}.$$

Proposition 3.3. *Let X be a projective variety with a G-action and ample linearization $\mathcal{L}$ and let $\mathcal{L}^{\mathrm{per}}$ be an ample perturbation of this linearization. If μ (respectively, μ_{per}) denotes the moment map associated to $\mathcal{L}$ (respectively, $\mathcal{L}^{\mathrm{per}}$), then provided $\mathcal{L}^{\mathrm{per}}$ is a sufficiently small perturbation of $\mathcal{L}$ the stratification*

$$X = \bigsqcup_{\gamma \in \mathcal{B}^{\mathrm{per}}} S_\gamma^{\mathrm{per}}$$

associated to $\|\mu_{\mathrm{per}}\|^2$ is a refinement of the stratification

$$X = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

associated to $\|\mu\|^2$.

Proof. Fix a maximal torus $H = T_{\mathbb{C}} \subseteq G$ and consider its fixed point set X^H which has a finite number of connected components F_i for $i \in I$. Let α_i (respectively α_i^{per}) denote the weight with which T acts on $\mathcal{L}|_{F_i}$ (respectively, on $\mathcal{L}^{\mathrm{per}}|_{F_i}$). Then by definition $\mathcal{B}$ is the set of closest points to 0 of convex hulls of subsets of $\{\alpha_i : i \in I\}$ modulo the action of the Weyl group W. Similarly, $\mathcal{B}^{\mathrm{per}}$ is the set of closest points to 0 of convex hulls of subsets of $\{\alpha_i^{\mathrm{per}} : i \in I\}$ modulo the W-action. Fix $\gamma \in \mathfrak{t}$ representing a point of $\mathcal{B}_\beta^{\mathrm{per}}$, so that γ is the closest point to 0 of the convex hull of

$$\{\alpha_i^{\mathrm{per}} : i \in I \text{ and } \alpha_i^{\mathrm{per}} \cdot \gamma \geqslant \|\gamma\|^2\}$$

and we can list these weights as $\alpha_{i_0}^{\mathrm{per}}, \dots, \alpha_{i_k}^{\mathrm{per}}$, say. We define $\beta_\gamma \in \mathcal{B}$ to be the W-orbit of the closest point to zero of the convex hull of

$$\{\alpha_{i_0}, \dots, \alpha_{i_k}\}.$$

As the linearization $\mathcal{L}^{\rm per}$ becomes close to $\mathcal{L}$ the weight $\alpha_i^{\rm per}$ becomes close to α_i for each i and so γ approaches β_γ. We need to show that if this perturbation is sufficiently small, then

$$S_\beta = \bigsqcup_{\substack{\gamma\in\mathcal{B}^{\rm per}\\ \beta=\beta_\gamma}} S_\gamma^{\rm per}.$$

Since $\{S_\beta : \beta \in \mathcal{B}\}$ and $\{S_\gamma^{\rm per} : \gamma \in \mathcal{B}^{\rm per}\}$ are both stratifications of X, it suffices to show that

$$S_\gamma^{\rm per} \subseteq S_{\beta_\gamma}$$

for all $\gamma \in \mathcal{B}^{\rm per}$, and for this it is enough to show that

(i) $S_\gamma^{\rm per} \cap S_{\beta'} = \phi$ for all $\beta' \in \mathcal{B}$ such that $\|\beta'\| > \|\beta_\gamma\|$, and
(ii) $Y_\gamma^{\rm per} \subset \overline{Y_{\beta_\gamma}}$,

since then $S_\gamma^{\rm per} = GY_\gamma^{ss,{\rm per}} \subset G\overline{Y_{\beta_\gamma}} \setminus \bigcup_{\|\beta'\|>\|\beta_\gamma\|} S_{\beta'} = S_{\beta_\gamma}$ as required.

Firstly, we consider how small the perturbation must be for (i) and (ii) to hold. Let

$$\epsilon_0 := \min\{\|\beta\|^2 - \alpha_i \cdot \beta : \beta \in \mathcal{B}, i \in I \text{ such that } \|\beta\|^2 > \alpha_i \cdot \beta\}$$

and

$$\epsilon_1 := \min\{|\|\beta'\| - \|\beta\| \,| : \beta', \beta \in \mathcal{B} \text{ and } \|\beta'\| \neq \|\beta\|\}.$$

Then $\epsilon_0 > 0$ and $\epsilon_1 > 0$ depend only on the initial linearization $\mathcal{L}$ of the G-action on X. Since X is compact $M = \sup\{\|\mu(x)\| : x \in X\}$ exists and we can define

$$\epsilon := \min\left\{1, \frac{\epsilon_0}{4M+1}, \frac{\epsilon_1}{3}\right\} > 0.$$

If the perturbation $\mathcal{L}^{\rm per}$ is sufficiently small, then

(a) for all $\gamma \in \mathcal{B}^{\rm per}$, we have $\|\gamma - \beta_\gamma\| < \epsilon$, and
(b) for all $x \in X$, we have $\|\mu(x) - \mu_{\rm per}(x)\| < \epsilon$;

we will assume that these conditions are satisfied.

Proof of (i): Suppose that $\gamma \in \mathcal{B}^{\rm per}$ and $\beta' \in \mathcal{B}$ and $\|\beta'\| > \|\beta_\gamma\|$. If $y \in S_\gamma^{\rm per}$, then by (2.2) there exists $g \in G$ such that gy is arbitrarily close to some point x in $Z_\gamma^{\rm per} \cap \mu_{\rm per}^{-1}(\gamma)$, so there exists $g \in G$ such that

$$\|\mu_{\rm per}(gy)\| - \|\gamma\| < \epsilon.$$

Then (b) implies that $\|\mu(gy)\| - \|\mu_{\rm per}(gy)\| < \epsilon$ and (a) implies that $\|\gamma\| - \|\beta_\gamma\| < \epsilon$ so that $\|\mu(gy)\| < 3\epsilon + \|\beta_\gamma\|$. However, by the definition of ϵ, we know that $3\epsilon \leqslant \|\beta'\| - \|\beta_\gamma\|$, so we conclude that $\|\mu(gy)\| < \|\beta'\|$ which implies $gy \notin S_{\beta'}$, and so y does not belong to $S_{\beta'}$.

Proof of (ii): Let $y \in Y_\gamma^{\rm per}$ where $\gamma \in \mathcal{B}^{\rm per}$, and consider its gradient flow under the 1-PS associated to β_γ, which has limit point x, say. Then $x \in \overline{Y_\gamma^{\rm per}}$ since x is in the H-orbit closure of y and $Y_\gamma^{\rm per}$ is invariant under H, and hence

$$\mu_{\rm per}(x) \cdot \gamma \geqslant \|\gamma\|^2. \tag{3.2}$$

Note that

$$\mu_{\rm per}(x) \cdot \gamma - \mu(x) \cdot \beta_\gamma = (\mu_{\rm per}(x) - \mu(x)) \cdot \gamma + \mu(x) \cdot (\gamma - \beta_\gamma);$$

the assumption (b) implies that $|(\mu_{\rm per}(x) - \mu(x)) \cdot \gamma| < \epsilon\|\gamma\|$ and (a) together with the inequality $\|\mu(x)\| \leqslant M$ implies that $|\mu(x) \cdot (\gamma - \beta_\gamma)| < M\epsilon$, so that

$$|\mu_{\rm per}(x) \cdot \gamma - \mu(x) \cdot \beta_\gamma| < \epsilon(M + \|\gamma\|). \tag{3.3}$$

To prove that $y \in \overline{Y_{\beta_\gamma}}$ (at least interpreted as in Remark 7, which is sufficient for the purposes of this proof) it suffices to show that $\mu(x) \cdot \beta_\gamma > \alpha_i \cdot \beta_\gamma$ for all i such that $\alpha_i \cdot \beta_\gamma < \|\beta_\gamma\|^2$, so it is enough to show that

$$\mu(x)\cdot\beta_\gamma > \|\beta_\gamma\|^2 - \epsilon_0.$$

Combining (3.2) and (3.3) gives $\mu(x)\cdot\beta_\gamma > \|\gamma\|^2 - \epsilon(M + \|\gamma\|)$ and so by (a) we get the following inequality:

$$\mu(x)\cdot\beta_\gamma > \|\beta_\gamma\|^2 - \epsilon(M + \|\gamma\| + 2\|\beta_\gamma\|).$$

Again using (a) we have that $-\epsilon\|\gamma\| > -\epsilon\|\beta_\gamma\| - \epsilon^2$ and since $\|\beta_\gamma\| \leqslant M$ we see that

$$\mu(x)\cdot\beta_\gamma > \|\beta_\gamma\|^2 - (4M+\epsilon)\epsilon.$$

By the choice of ϵ we know that $(4M+\epsilon)\epsilon \leqslant (4M+1)\epsilon \leqslant \epsilon_0$ and so

$$\mu(x)\cdot\beta_\gamma > \|\beta_\gamma\|^2 - \epsilon_0$$

as required. This completes the proof of (ii) and hence of the proposition. □

4. *Extending to projective schemes*

In this section, we observe that the constructions in the previous sections for non-singular projective varieties can be extended to the case when X is any projective scheme with an ample G-linearization $\mathcal{L}$. For this, it is enough to deal with the case when $\mathcal{L}$ is very ample and check that the resulting constructions do not change when $\mathcal{L}$ is replaced with $\mathcal{L}^{\otimes m}$ for any positive integer m.

Thus, let us assume that X is a closed subscheme of $\mathbb{P}^n$ and the action of G on X is given by a linear representation $G \to \mathrm{GL}(n+1)$. For the G-action on the ambient projective space $\mathbb{P}^n$ we can define the subvarieties Z_β^{ss} and Y_β^{ss} as before. We can also define the closed subvariety $\bar{Y}_\beta$ of $\mathbb{P}^n$ and use the scheme structure on $\mathbb{P}^n$ to give this the reduced induced closed scheme structure as in [6, II Example 3.2.6]. This gives $\hat{S}_\beta := G \times_{P_\beta} \bar{Y}_\beta$ its scheme structure. Then the open subsets $S_\beta \subset \hat{S}_\beta$ and $Y_\beta^{\mathrm{ss}} \subset \bar{Y}_\beta$ get an induced scheme structure as open subsets of schemes. We have a stratification

$$\mathbb{P}^n = \bigsqcup_{\beta\in\mathcal{B}} S_\beta$$

into G-invariant locally closed subschemes and the morphism

$$G \times_{P_\beta} Y_\beta^{\mathrm{ss}} \longrightarrow \mathbb{P}^n$$

induced by the group action

$$G \times_{P_\beta} \mathbb{P}^n \longrightarrow \mathbb{P}^n$$

is an isomorphism onto S_β.

To go from the stratification of the ambient projective space $\mathbb{P}^n$ to a stratification of X we intersect the above stratification by taking fibre products. For any subscheme S of $\mathbb{P}^n$ we let

$$S(X) := S \times_{\mathbb{P}^n} X$$

be the fibre product of X and S over $\mathbb{P}^n$. Then $\bar{Y}_\beta(X)$ is a closed subscheme of X and $\hat{S}_\beta(X) = G \times_{P_\beta} \bar{Y}_\beta(X)$ is a projective completion of $S_\beta(X)$. The morphism

$$G \times_{P_\beta} Y_\beta^{\mathrm{ss}}(X) \longrightarrow X$$

is an isomorphism onto $S_\beta(X)$ by using the universal property of the fibre product $S_\beta(X)$ and the fact that $G \times_{P_\beta} Y_\beta^{\mathrm{ss}} \cong S_\beta$ for the ambient projective space. We have a stratification

$$X = \bigsqcup_{\beta\in\mathcal{B}} S_\beta(X)$$

into G-invariant locally closed subschemes (although for some indices β the stratum $S_\beta(X)$ may be empty). We note at this point that this stratification can be refined by taking connected components of $Z_\beta^{\mathrm{ss}}(X)$ in the same way as it can for varieties (cf. Remark 3.1).

We can also define the canonical linearization on $\hat{S}_\beta$ in exactly the same way as we do for varieties and this can be restricted to $\hat{S}_\beta(X)$. In this situation, it is still true that the GIT quotient

$$Z_\beta(X)/\!/_{\mathcal{L}_\beta} \operatorname{Stab}\beta$$

is a categorical quotient of the G-action on $S_\beta(X)$.

Finally, we observe that the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $\mathbb{P}^n$ is unchanged (except for a minor modification of its labelling) if we replace $\mathcal{O}_{\mathbb{P}^n}(1)$ with $\mathcal{O}_{\mathbb{P}^n}(m)$ for any $m > 0$, and if we regard $\mathbb{P}^n$ as a G-invariant linear subspace of a bigger projective space $\mathbb{P}^N$ on which G acts linearly. Thus, we obtain well-defined constructions for any projective scheme X with an ample G-linearization $\mathcal{L}$, which are unaffected by replacing $\mathcal{L}$ with $\mathcal{L}^{\otimes m}$ for any $m > 0$. Moreover, in the case when X is a non-singular projective variety these constructions agree with those in §§ 2 and 3 (cf. Remark 2.3).

5. *Simpson's construction of moduli of semistable sheaves*

Let W be a complex projective scheme with ample invertible sheaf $\mathcal{O}(1)$. We consider the moduli problem of classifying pure coherent algebraic sheaves on W up to isomorphism. From now on we will use the term sheaf to mean coherent algebraic sheaf and unless otherwise specified sheaves will be on W. Gieseker introduced a notion of semistability for sheaves in [**5**] and constructed coarse moduli spaces of semistable torsion-free sheaves in the case when W is a smooth projective variety of dimension at most 2. Maruyama [**17**, **18**] generalized this to torsion-free sheaves over integral projective schemes. Later Simpson [**21**] constructed coarse moduli spaces of semistable pure sheaves on an arbitrary complex projective scheme W. We follow the more general construction of Simpson where the moduli space of semistable pure sheaves on W of fixed dimension and Hilbert polynomial is constructed as a GIT quotient of a subscheme Q of a quot scheme by the action of a special linear group G. The linearization is given by using Grothendieck's embedding of the quot scheme into a Grassmannian and then using the Plücker embedding of the Grassmannian into projective space.

We fix a rational polynomial $P \in \mathbb{Q}[x]$ of degree e which takes integer values when x is integral. Recall that a sheaf is pure of dimension e if its support has dimension e and all non-zero subsheaves have support of dimension e. Then we consider pure sheaves of dimension e with Hilbert polynomial P calculated with respect to $\mathcal{O}(1)$.

Definition 1. Let $\mathcal{F}$ be a pure sheaf of dimension e over W. We define the *multiplicity* of $\mathcal{F}$ to be $r(\mathcal{F}) = e!a_e$ where a_e is the leading coefficient in the Hilbert polynomial of e. If $\mathcal{F}$ is torsion-free this is just the rank of $\mathcal{F}$. The *reduced Hilbert polynomial* of $\mathcal{F}$ is defined to be the quotient $P(\mathcal{F})/r(\mathcal{F})$.

Definition 2. A sheaf $\mathcal{F}$ is *semistable* if it is pure and every non-zero subsheaf $\mathcal{F}' \subset \mathcal{F}$ satisfies

$$\frac{P(\mathcal{F}')}{r(\mathcal{F}')} \leqslant \frac{P(\mathcal{F})}{r(\mathcal{F})}$$

where the ordering on polynomials is given by lexicographic ordering of their coefficients. The sheaf is *stable* if the above inequality is strict for every proper non-zero subsheaf. A semistable sheaf $\mathcal{F}$ has a Jordan–Hölder filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_s = \mathcal{F},$$

where $\mathcal{F}_i/\mathcal{F}_{i-1}$ is stable with reduced Hilbert polynomial $P(\mathcal{F})/r(\mathcal{F})$ for $1 \leqslant i \leqslant s$. This filtration is not in general canonical but the associated graded sheaf

$$\mathrm{Gr}^{JH}(\mathcal{F}) = \bigoplus_{i=1}^{s} \mathcal{F}_i/\mathcal{F}_{i-1}$$

is canonically associated to $\mathcal{F}$ (up to isomorphism). Two semistable sheaves $\mathcal{F}$ and $\mathcal{G}$ over W are *S-equivalent* if $\mathrm{Gr}^{JH}(\mathcal{F})$ and $\mathrm{Gr}^{JH}(\mathcal{G})$ are isomorphic.

Simpson [**21**, Theorem 1.1] shows that the semistable sheaves with Hilbert polynomial P are bounded, and hence we can choose $n \gg 0$ so that all such sheaves are n-regular. In particular, this means that for any such sheaf $\mathcal{F}$ the evaluation map $H^0(\mathcal{F}(n)) \otimes \mathcal{O}(-n) \to \mathcal{F}$ is surjective and the higher cohomology of $\mathcal{F}(n)$ vanishes, that is,

$$H^i(\mathcal{F}(n)) = 0 \quad \text{for } i > 0,$$

so that $P(n) = P(\mathcal{F}, n) = \dim H^0(\mathcal{F}(n))$.

Let V be a vector space of dimension $P(n)$. Then the evaluation map for $\mathcal{F}$ and a choice of isomorphism $H^0(\mathcal{F}(n)) \cong V$ determine a point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in the quot scheme

$$\mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$$

of quotients with Hilbert polynomial P of the sheaf $V \otimes \mathcal{O}(-n)$ on W. We consider the open subscheme $Q \subset \mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$ consisting of quotients $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ such that $\mathcal{F}$ is pure of dimension e and the map on sections $H^0(\rho(n)) : V \to H^0(\mathcal{F}(n))$ induced by ρ tensored with the identity on $\mathcal{O}(n)$ is an isomorphism. The group $G := \mathrm{SL}(V)$ acts on this quot scheme by acting on the vector space V, so that $g \cdot \rho$ is the composition

$$g \cdot \rho : V \otimes \mathcal{O}(-n) \xrightarrow{g^{-1}\cdot} V \otimes \mathcal{O}(-n) \xrightarrow{\quad\rho\quad} \mathcal{F}$$

for $g \in G$ and $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in the quot scheme. The subscheme Q is preserved by this action and the G-orbits correspond to isomorphism classes of sheaves.

Simpson considers a linearization of this action given by an equivariant embedding of the quot scheme $\mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$ into a Grassmannian. Grothendieck showed that for $m \gg n$ the morphism

$$\begin{aligned} \mathrm{Quot}(V \otimes \mathcal{O}(-n), P) &\longrightarrow \mathrm{Gr}(V \otimes H^0(\mathcal{O}(m-n)), P(m)), \\ \rho : V \otimes \mathcal{O}(-n) \longrightarrow \mathcal{F} &\longmapsto H^0(\rho(m)) : V \otimes H \longrightarrow H^0(\mathcal{F}(m)) \end{aligned}$$

is an embedding, where $H := H^0(\mathcal{O}(m-n))$ and $\mathrm{Gr}(V \otimes H^0(\mathcal{O}(m-n)), P(m))$ is the Grassmannian of $P(m)$-dimensional quotients of the vector space $V \otimes H$. The Plücker embedding

$$\begin{aligned} \mathrm{Gr}(V \otimes H, P(m)) &\hookrightarrow \mathbb{P}((\wedge^{P(m)}(V \otimes H))^*), \\ H^0(\rho(m)) &\longmapsto \wedge^{P(m)} H^0(\rho(m)) \end{aligned}$$

then gives an embedding of $\mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$ in the projective space $\mathbb{P}((\wedge^{P(m)}(V \otimes H))^*)$. Let $\bar{Q}$ denote the closure of Q in the quot scheme $\mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$, let $\mathcal{U}$ be the restriction to $\bar{Q} \times W$ of the universal quotient sheaf on the product of the quot scheme and W, and let $\pi_{\bar{Q}}$ and π_W be the projections from $\bar{Q} \times W$ to $\bar{Q}$ and W. Then since $m \gg n$ the higher cohomology groups $H^i(\mathcal{F}(m))$ for $i > 0$ all vanish for $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in $\mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$ and

$$\mathcal{L} = \det(\pi_{\bar{Q}*}(\mathcal{U} \otimes \pi_W^* \mathcal{O}(m))) \tag{5.1}$$

is the ample invertible sheaf corresponding to the embedding of $\bar{Q}$ into the projective space $\mathbb{P}((\wedge^{P(m)}(V \otimes H))^*)$ above. There is a natural lift of the G-action on $\bar{Q}$ to the universal quotient $\mathcal{U}$ and this gives an action of G on $\mathcal{L}$; by abuse of notation, we let $\mathcal{L}$ denote this linearization as well as the line bundle underlying it. We assume n and m are both chosen sufficiently large (for details see [**21**]).

THEOREM 5.1 [**21**, Theorem 1.21]. *Let W be a projective scheme, $e \leqslant \dim(W)$ a positive integer and P a Hilbert polynomial of degree e. Then if $m \gg n \gg 0$ the GIT quotient $\bar{Q}/\!/_{\mathcal{L}}G$ defined as above is a coarse moduli space for semistable sheaves of pure dimension e with Hilbert polynomial P up to S-equivalence.*

5.1. *Calculating the Hilbert–Mumford function*

The Hilbert–Mumford criterion (see [**19**, Theorem 2.1]) gives a way to test the (semi)stability of a point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ of Q in terms of one-parameter subgroups of G. If λ is a 1-PS, then $\lim_{t\to 0} \lambda(t) \cdot \rho \in \bar{Q}$ is a fixed point for the $\mathbb{C}^*$-action induced by λ, and so the group $\mathbb{C}^*$ acts on the fibre of $\mathcal{L}$ over this fixed point by some character of $\mathbb{C}^*$, say $t \mapsto t^w$ for some integer w. The Hilbert–Mumford function of $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ evaluated at λ is defined as

$$\mu^{\mathcal{L}}(\rho, \lambda) := w.$$

Let

$$M^{\mathcal{L}}(\rho) = \inf \frac{\mu^{\mathcal{L}}(\rho, \lambda)}{\|\lambda\|},$$

where the infimum is taken over all non-trivial one-parameter subgroups λ of G, and as before the norm is determined by an invariant inner product on the Lie algebra of the maximal compact subgroup $\mathrm{SU}(V)$ of $G = \mathrm{SL}(V)$. Then the Hilbert–Mumford criterion states that ρ is semistable with respect to $\mathcal{L}$ if and only if $\mu^{\mathcal{L}}(\rho, \lambda) \geqslant 0$ for every 1-PS λ of G, or equivalently $M^{\mathcal{L}}(\rho) \geqslant 0$. If ρ is unstable with respect to $\mathcal{L}$, then $M^{\mathcal{L}}(\rho)$ is negative and a non-divisible 1-PS achieving this value is said to be adapted to ρ (cf. Remark 6). In this section, we will calculate the Hilbert–Mumford function $\mu^{\mathcal{L}}(\rho, \lambda)$ for any 1-PS λ of G.

First of all, we make use of the fact that any 1-PS induces a decomposition of V as a direct sum of weight spaces:

$$\{\text{1-PSs of } \mathrm{SL}(V)\} \longleftrightarrow \left\{ \begin{matrix} \text{decompositions } V = \bigoplus_{k\in\mathbb{Z}} V_k \\ \text{such that } \sum k \dim V_k = 0 \end{matrix} \right\},$$
$$\lambda \longmapsto V_k := \{v \in V : \lambda(t) \cdot v = t^k v\}.$$

The relation $\sum k \dim V_k = 0$ ensures that we obtain a 1-PS of the special linear group as opposed to the general linear group. Such a decomposition determines a filtration of V given by

$$\cdots \subseteq V_{\geqslant k+1} \subseteq V_{\geqslant k} \subseteq V_{\geqslant k-1} \subseteq \cdots$$

where $V_{\geqslant k} := \bigoplus_{l \geqslant k} V_l$. There are only finitely many integers k such that $V_k \neq 0$, say

$$k_1 > \cdots > k_s;$$

let $V^{(i)} = V_{\geqslant k_i}$ for $i = 1, \ldots, s$. Then we obtain a map

$$\left\{ \begin{matrix} \text{1-PSs of } \mathrm{SL}(V) \\ \lambda \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} 0 = V^{(0)} \subset V^{(1)} \subset \cdots \subset V^{(s)} = V \\ \text{filtrations of } V \text{and integers } k_1 > \cdots > k_s \\ \text{such that } \sum k_i \dim V^{(i)}/V^{(i-1)} = 0 \end{matrix} \right\}$$

Let λ be a 1-PS of $G = \mathrm{SL}(V)$ and let $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ be a point in $\bar{Q}$. Then the filtration of V determined by λ induces a filtration of $\mathcal{F}$ given by

$$0 = \mathcal{F}^{(0)} \subset \mathcal{F}^{(1)} \subset \cdots \subset \mathcal{F}^{(s)} = \mathcal{F},$$

where $\mathcal{F}^{(i)} = \rho(V^{(i)} \otimes \mathcal{O}(-n))$. Let $\mathcal{F}_{\geqslant k}$ denote the image of $V_{\geqslant k} \otimes \mathcal{O}(-n)$ under ρ for any integer k. Then ρ induces

$$\rho_k : V_k \otimes \mathcal{O}(-n) \longrightarrow \mathcal{F}_k = \mathcal{F}_{\geqslant k}/\mathcal{F}_{\geqslant k+1}$$

for each integer k; here, $\mathcal{F}_k$ and ρ_k can only be non-zero if $k = k_i$ for some i with $1 \leqslant i \leqslant s$. We define

$$\bar{\rho} = \bigoplus_{k\in\mathbb{Z}} \rho_k : \bigoplus_{k\in\mathbb{Z}} V_k \otimes \mathcal{O}(-n) \longrightarrow \bar{\mathcal{F}} = \bigoplus_{k\in\mathbb{Z}} \mathcal{F}_k \tag{5.2}$$

(cf. [8, § 4.4]). We now have a formula for the Hilbert–Mumford function.

Lemma 5.2. *The Hilbert–Mumford function evaluated at a one-parameter subgroup λ of $G = \mathrm{SL}(V)$ for a point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in $\bar{Q}$ is given by*

$$\mu^{\mathcal{L}}(\rho, \lambda) = \sum_{i=1}^{s-1} (k_i - k_{i+1}) \left(P(\mathcal{F}^{(i)}, m) - \dim V^{(i)} \frac{P(\mathcal{F}, m)}{P(\mathcal{F}, n)} \right),$$

where $V^{(i)}$ and $\mathcal{F}^{(i)}$ are defined as above.

Proof. By [8, Lemma 4.4.3] the fixed point $\lim_{t\to 0} \lambda(t)\cdot\rho$ in $\bar{Q}$ is equal to $\bar{\rho}$. To calculate the value of the Hilbert–Mumford function we need to calculate the weight of the $\mathbb{C}^*$-action on the fibre at $\bar{\rho}$ of the line bundle $\mathcal{L}$ defined at (5.1). For this, we follow the argument of Huybrechts and Lehn [8, Lemma 4.4.4], though using a left action as opposed to a right action. Since $m \gg n \gg 0$, we have $H^i(\bar{\mathcal{F}}(m)) = 0$ for $i > 0$ and the line bundle

$$\mathcal{L} = \det(\pi_{\bar{Q}*}(\mathcal{U} \otimes \pi_W^* \mathcal{O}(m)))$$

has fibre

$$\det(H^0(\overline{\mathcal{F}}(m))) = \wedge^{P(m)} H^0(\overline{\mathcal{F}}(m))$$

at $\bar{\rho}$. The $\mathbb{C}^*$-action induced by λ on ρ_k has weight $-k$ because $\lambda(t)\cdot\rho_k$ is the composition

$$V_k \otimes \mathcal{O}(-n) \xrightarrow{\lambda^{-1}(t)} V_k \otimes \mathcal{O}(-n) \xrightarrow{\quad\rho_k\quad} \mathcal{F}_k$$

and

$$\lambda^{-1}(t)\cdot v_k = t^{-k} v_k \quad \text{for all } v_k \in V_k.$$

Therefore, the weight of the $\mathbb{C}^*$-action on $\det H^0(\mathcal{F}_k(m))$ is equal to k times the dimension of $H^0(\mathcal{F}_k(m))$, which is the value $P(\mathcal{F}_k, m)$ at m of the Hilbert polynomial $P(\mathcal{F}_k)$. The weight of the $\mathbb{C}^*$-action on the fibre of $\mathcal{L}$ over $\bar{\rho}$ is the sum of the weights of the $\mathbb{C}^*$-action on $\det H^0(\mathcal{F}_k(m))$, and so

$$\mu^{\mathcal{L}}(\rho, \lambda) = \sum_{k\in\mathbb{Z}} k P(\mathcal{F}_k, m) = \sum_{i=1}^{s} k_i P(\mathcal{F}_{k_i}, m).$$

Since λ is a 1-PS of the special linear group $G = \mathrm{SL}(V)$ we have $\sum_{i=1}^{s} k_i \dim V_{k_i} = 0$, so we may write this as

$$\begin{aligned}
\mu^{\mathcal{L}}(\rho,\lambda) &= \sum_{i=1}^{s} k_i \left(P(\mathcal{F}_{k_i}, m) - \dim V_{k_i} \frac{P(\mathcal{F},m)}{P(\mathcal{F},n)} \right) \\
&= \sum_{i=1}^{s} k_i \Bigg(P(\mathcal{F}^{(i)}, m) - P(\mathcal{F}^{(i-1)}, m) - \dim V^{(i)} \frac{P(\mathcal{F},m)}{P(\mathcal{F},n)} \\
&\qquad + \dim V^{(i-1)} \frac{P(\mathcal{F},m)}{P(\mathcal{F},n)} \Bigg) \\
&= k_s \left(P(\mathcal{F}, m) - \dim V \frac{P(\mathcal{F},m)}{P(\mathcal{F},n)} \right) \\
&\qquad + \sum_{i=1}^{s-1} (k_i - k_{i+1}) \left(P(\mathcal{F}^{(i)}, m) - \dim V^{(i)} \frac{P(\mathcal{F},m)}{P(\mathcal{F},n)} \right)
\end{aligned}$$

which gives the required result since $\dim V = P(\mathcal{F}, n)$. □

6. *The stratification of the closure of* Q

We consider the group $G = \mathrm{SL}(V)$ acting on the subscheme $\bar{Q}$ of the quot scheme $\mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$ with respect to the linearization $\mathcal{L}$ defined at (5.1), for which the GIT quotient $\bar{Q}/\!/_{\mathcal{L}} G$ is a coarse moduli space for semistable sheaves on W with Hilbert polynomial P. The linearization $\mathcal{L}$ defines a G-equivariant embedding of $\bar{Q}$ in the projective space $\mathbb{P}((\wedge^{P(m)}(V \otimes H))^*)$ and we can choose a Kähler structure on $\mathbb{P}((\wedge^{P(m)}(V \otimes H))^*)$ which is invariant under the maximal compact subgroup $\mathrm{SU}(V)$ of $G = \mathrm{SL}(V)$. There is a stratification of this ambient projective space associated to this action and by intersecting this with $\bar{Q}$ we obtain a stratification

$$\bar{Q} = \bigsqcup_{\beta \in \mathcal{B}} S_\beta$$

into G-invariant locally closed subschemes as in § 4. The aim of this section is to prove Proposition 6.8 which relates the stratum S_β containing a point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ of Q to the Harder–Narasimhan type of the sheaf $\mathcal{F}$. Versions of this result have been well known for a long time (cf. [**2**, **12**, **20**] for the case when W is a non-singular projective curve) but we provide a proof for the sake of completeness.

Fix a basis of V and pick the maximal torus $T \subset \mathrm{SU}(V)$ consisting of diagonal matrices of determinant 1 with entries in S^1. Then the Lie algebra of T consists of purely imaginary trace-free diagonal matrices. We choose a positive Weyl chamber given by

$$\mathfrak{t}_+ = \left\{ i \,\mathrm{diag}(a_1, \ldots, a_{\dim(V)}) : \begin{array}{c} a_i \in \mathbb{R} \text{ such that } \sum a_i = 0, \\ \text{and } a_1 \geqslant a_2 \geqslant \cdots \geqslant a_{\dim(V)}. \end{array} \right\}$$

The indexing set $\mathcal{B}$ for the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ is a finite set of points in $\mathfrak{t}_+$. We note at this point that the strata S_β may not be connected and so may be stratified further into their connected components (cf. Remark 7).

6.1. *The refined stratum associated to a fixed Harder–Narasimhan type*

Any sheaf of pure dimension e over W has a canonical filtration by subsheaves whose successive quotients are semistable with decreasing reduced Hilbert polynomials, known as the Harder–Narasimhan filtration.

Definition 3. Let $\mathcal{F}$ be a pure sheaf; then its *Harder–Narasimhan filtration* is a filtration

$$0 = \mathcal{F}^{(0)} \subsetneq \mathcal{F}^{(1)} \subsetneq \cdots \subsetneq \mathcal{F}^{(s)} = \mathcal{F}$$

such that the successive quotients $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ are semistable with decreasing reduced Hilbert polynomials

$$\frac{P(\mathcal{F}^{(1)})}{r(\mathcal{F}^{(1)})} > \frac{P(\mathcal{F}^{(2)}/\mathcal{F}^{(1)})}{r(\mathcal{F}^{(2)}/\mathcal{F}^{(1)})} > \cdots > \frac{P(\mathcal{F}^{(s)}/\mathcal{F}^{(s-1)})}{r(\mathcal{F}^{(s)}/\mathcal{F}^{(s-1)})}.$$

We will denote by $\mathrm{Gr}^{HN}(\mathcal{F})$ the associated graded sheaf

$$\mathrm{Gr}^{HN}(\mathcal{F}) = \bigoplus_{i=1}^{s} \mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}.$$

We call the first sheaf $\mathcal{F}^{(1)}$ appearing in the Harder–Narasimhan filtration the *maximal destabilizing subsheaf*. The Harder–Narasimhan type of $\mathcal{F}$ is specified by the vector of Hilbert polynomials of the successive quotients,

$$HN(\mathcal{F}) := (P(\mathcal{F}^{(1)}), P(\mathcal{F}^{(2)}/\mathcal{F}^{(1)}), \ldots, P(\mathcal{F}^{(s)}/\mathcal{F}^{(s-1)})).$$

For each point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in Q we define the Harder–Narasimhan type of ρ to be the Harder–Narasimhan type of $\mathcal{F}$.

The different types of Harder–Narasimhan filtrations allow us to decompose Q into subsets of fixed Harder–Narasimhan type.

Definition 4. If τ is a Harder–Narasimhan type, let $R_\tau \subseteq Q$ be the set of points $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ such that $\mathcal{F}$ has Harder–Narasimhan type τ. Then we can write Q as

$$Q = \bigsqcup_{\tau} R_\tau.$$

Let $\tau_0 = (P)$ denote the trivial Harder–Narasimhan type; then R_{τ_0} parameterizes semistable sheaves and so is equal to the stratum S_0 by [**21**, Theorem 1.21] (cf. Theorem 5.1).

For the rest of this section, we fix a non-trivial Harder–Narasimhan type $\tau = (P_1, \ldots, P_s)$, where $P_1, \ldots, P_s$ are polynomials of degree e such that $P_1 + \cdots + P_s = P$, and we assume that there is a sheaf of pure dimension e over W with this Harder–Narasimhan type. The following lemma shows that if n is sufficiently large, then R_τ parameterizes all sheaves with Harder–Narasimhan type τ.

Lemma 6.1. *The set of sheaves of pure dimension e with Hilbert polynomial P and Harder–Narasimhan type τ is bounded.*

Proof. This follows from a result of Simpson (see [**21**, Theorem 1.1]) that a set of sheaves on W of pure dimension e and Hilbert polynomial P is bounded if the slopes of their subsheaves are bounded above by a fixed constant, where the slope of a sheaf is (up to multiplication by a positive constant) the second to top coefficient of its reduced Hilbert polynomial. Any sheaf $\mathcal{F}$ with Harder–Narasimhan type τ has a maximal destabilizing subsheaf $\mathcal{F}^{(1)}$ with Hilbert polynomial P_1, and all subsheaves of $\mathcal{F}$ have reduced Hilbert polynomial less than or equal to the reduced Hilbert polynomial of $\mathcal{F}^{(1)}$. Let μ_1 denote the slope of $\mathcal{F}^{(1)}$, which depends only on the polynomial P_1; then any subsheaf $\mathcal{F}'$ of $\mathcal{F}$ has slope less than or equal to μ_1 and this proves the result. □

This boundedness result means that we may assume n is chosen so that all pure sheaves with Hilbert polynomial P and Harder–Narasimhan type τ are n-regular, and therefore are parameterized by Q. We may also assume that all sheaves with Harder–Narasimhan type $(P_{i_1}, \dots, P_{i_k})$ for any $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant s$ are n-regular; in particular, the sheaves $\mathcal{F}^{(i)}$ occurring in the Harder–Narasimhan filtration of any sheaf $\mathcal{F}$ of Harder–Narasimhan type τ are n-regular.

We want to show that the subset R_τ indexed by a fixed Harder–Narasimhan type is contained in a stratum $S_{\beta(\tau)}$ occurring in the stratification $\{S_\beta : \beta \in \mathcal{B}\}$. In order to do this, we look for a candidate for $\beta = \beta(\tau)$ depending only on the information coming from the Harder–Narasimhan type τ. The definitions of Z_β and Y_β, Z_β^{ss} and Y_β^{ss} are valid for any $\beta \in \mathfrak{t}$ and do not require β to belong to the indexing set $\mathcal{B}$, but Y_β^{ss} will only be non-empty when $\beta \in \mathcal{B}$. Therefore, we can look for a candidate $\beta \in \mathfrak{t}_+$, and if Y_β^{ss} is non-empty, then this will imply that β belongs to $\mathcal{B}$.

We fix a point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in R_τ and let

$$0 = \mathcal{F}^{(0)} \subsetneq \mathcal{F}^{(1)} \subsetneq \cdots \subsetneq \mathcal{F}^{(s)} = \mathcal{F}$$

denote the Harder–Narasimhan filtration of $\mathcal{F}$. We want to find β such that ρ belongs to Y_β, so first we look for a 1-PS λ of $G = \mathrm{SL}(V)$ which is adapted to ρ (cf. Remark 6). We have seen that all 1-PSs give rise to filtrations of $\mathcal{F}$ and it is reasonable to expect that a 1-PS adapted to ρ will give rise to the filtration of $\mathcal{F}$ which is most responsible for its instability, namely its Harder–Narasimhan filtration. With this in mind, we let

$$V^{(i)} := H^0(\rho(n))^{-1}(H^0(\mathcal{F}^{(i)}(n)))$$

and choose a basis of V (and corresponding maximal torus of $G = \mathrm{SL}(V)$) by first taking a basis of $V^{(1)}$, then extending to $V^{(2)}$ and so on. This gives us a decomposition

$$V = V_1 \oplus \cdots \oplus V_s$$

of V such that $V^{(i)} = V_1 \oplus \cdots \oplus V_i$ and so $V^{(i)}/V^{(i-1)} \cong V_i$. Then we consider 1-PSs in $G = \mathrm{SL}(V)$ of the form

$$\lambda(t) = \begin{pmatrix} t^{\beta_1} I_{V_1} & & & \\ & t^{\beta_2} I_{V_2} & & \\ & & \ddots & \\ & & & t^{\beta_s} I_{V_s} \end{pmatrix}$$

where $\beta_1, \dots, \beta_s$ are integers such that $\beta_1 > \cdots > \beta_s$ and $\sum \beta_i P(\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}, n) = 0$.

Remark 13. Since we are assuming that each $\mathcal{F}^{(i)}$ and $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ is n-regular, we have that

$$P(\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}, n) = \dim V^{(i)}/V^{(i-1)} = \dim V_i$$

for each i.

Recall that a non-trivial 1-PS λ of G is adapted to ρ if

$$\frac{\mu^{\mathcal{L}}(\rho,\lambda)}{\|\lambda\|}$$

is minimal among non-trivial 1-PSs of G. Therefore, let us choose the integers $(\beta_1,\dots,\beta_s)$ to minimize the function

$$f(\beta_1,\dots,\beta_s) := \frac{\sum_{i=1}^{s-1}(\beta_i-\beta_{i+1})(P(\mathcal{F}^{(i)},m)-P(\mathcal{F}^{(i)},n)(P(\mathcal{F},m)/P(\mathcal{F},n)))}{(\sum_{i=1}^{s}\beta_i^2 P(\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)},n))^{1/2}}$$

subject to the condition that $g(\beta_1,\dots,\beta_s) := \sum_{i=1}^{s}\beta_i P(\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)},n)=0$. We introduce a Lagrangian multiplier η and define

$$\Lambda(\beta_1,\dots,\beta_s,\eta) := f(\beta_1,\dots,\beta_s)-\eta g(\beta_1,\dots,\beta_s);$$

then we look for solutions to

$$\frac{\partial}{\partial\beta_j}\Lambda(\beta_1,\dots,\beta_s,\eta)=0 \quad \text{for } j=1,\dots,s \quad \text{and} \quad \frac{\partial}{\partial\eta}\Lambda(\beta_1,\dots,\beta_s,\eta)=0. \tag{6.1}$$

Note that for any $a\in\mathbb{R}_{>0}$, we have $f(a\beta_1,\dots,a\beta_s)=f(\beta_1,\dots,\beta_s)$ and $g(a\beta_1,\dots,a\beta_s)=0$ is equivalent to $g(\beta_1,\dots,\beta_s)=0$. It is easy to check that

$$(\beta_1,\dots,\beta_s,\eta)=\left(\frac{P(\mathcal{F},m)}{P(\mathcal{F},n)}-\frac{P_1(m)}{P_1(n)},\dots,\frac{P(\mathcal{F},m)}{P(\mathcal{F},n)}-\frac{P_s(m)}{P_s(n)},0\right)$$

provides a solution to equation (6.1). Note that $\beta_1>\cdots>\beta_s$ where

$$\beta_i=\frac{P(\mathcal{F},m)}{P(\mathcal{F},n)}-\frac{P_i(m)}{P_i(n)}$$

because the reduced Hilbert polynomial of P_i is strictly greater than that of P_{i+1}. Now consider

$$\beta = i\,\mathrm{diag}(\beta_1,\dots,\beta_1,\beta_2,\dots,\beta_2,\dots,\beta_s\cdots\beta_s)\in\mathfrak{t}_+, \tag{6.2}$$

where β_i appears $P_i(n)$ times.

REMARK 14. This β depends on the Harder–Narasimhan type τ (as well as on n and m) and will be written as $\beta=\beta(\tau)$ if it is necessary to make this dependence explicit. We note that for two distinct Harder–Narasimhan types τ and τ', for all n and m sufficiently large the associated weights $\beta(\tau)$ and $\beta(\tau')$ will also be distinct.

Consider the subschemes Z_β and Y_β of $\bar{Q}$ defined as in (2.1) and (2.3).

LEMMA 6.2. *Suppose $n\gg 0$, then the point $\rho: V\otimes\mathcal{O}(-n)\to\mathcal{F}$ in R_τ belongs to Y_β where $\beta=\beta(\tau)$.*

Proof. Our assumptions on n imply that $\mathcal{F}$ and all the subquotients appearing in its Harder–Narasimhan filtration are n-regular. The point ρ belongs to Y_β if and only if the limit point

$$\lim_{t\to 0}\lambda_\beta(t)\cdot\rho=\bar{\rho}$$

of its path of steepest descent under the function $\mu\cdot\beta$ belongs to Z_β. By [8, Lemma 4.4.3] this limit point is

$$\bar{\rho}: V\otimes\mathcal{O}(-n)\longrightarrow \mathrm{Gr}^{HN}(\mathcal{F})=\bigoplus_{i=1}^{s}\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}.$$

The weight of λ_β acting on a point lying over $\bar{\rho}$ is given by

$$-\mu^{\mathcal{L}}(\rho, \lambda_\beta) = \sum \frac{P_i(m)^2}{P_i(n)} - \frac{P(m)^2}{P(n)}$$

which is equal to $\|\lambda_\beta\|^2 = \|\beta\|^2$, and so $\bar{\rho} \in Z_\beta$ as required. □

Recall that we have a decomposition $V = V_1 \oplus \cdots \oplus V_s$ into weight spaces for the 1-PS λ_β.

LEMMA 6.3. *Let $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ be a point in $\bar{Q}$. Then ρ is fixed by the 1-PS λ_β if and only if $\mathcal{F}$ has a decomposition*

$$\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_s$$

and we also have a decomposition

$$\rho = \rho_1 \oplus \cdots \oplus \rho_s,$$

where $\rho_i : V_i \otimes \mathcal{O}(-n) \to \mathcal{F}_i$ lies in the quot scheme $\mathrm{Quot}(V_i \otimes \mathcal{O}(-n), P(\mathcal{F}_i))$.

The fixed point locus of $\lambda_\beta(\mathbb{C}^*)$ acting on $\bar{Q}$ decomposes into components indexed by the tuple of Hilbert polynomials of the direct summands. Let $Q_i := \mathrm{Quot}(V_i \otimes \mathcal{O}(-n), P_i)$ and consider

$$F = \left\{ q \in \mathrm{Quot}(V \otimes \mathcal{O}(-n), P) : q = \bigoplus_{i=1}^{s} q_i \text{ such that } q_i \in Q_i \right\} \cong Q_1 \times \cdots \times Q_s.$$

COROLLARY 6.4. *The scheme $F \cap \bar{Q}$ is a union of connected components of Z_β.*

Proof. Clearly, F is a union of connected components of the fixed point locus of the one-parameter subgroup λ_β. By definition Z_β is the connected components of the fixed point locus in $\bar{Q}$ on which λ_β acts with weight $\|\beta\|^2$. Let $q = \bigoplus_{i=1}^s q_i$ be a point in F where $q_i : V_i \otimes \mathcal{O}(-n) \to \mathcal{E}_i$ is a quotient sheaf in Q_i. The Hilbert–Mumford function $\mu^{\mathcal{L}}(q, \lambda_\beta)$ is equal to minus the weight of the action of λ_β on a point lying over q. By direct calculation, we have

$$\|\beta\|^2 = \sum_{i=1}^{s} \beta_i^2 P_i(n) = \sum_{i=1}^{s} \frac{P_i(m)^2}{P_i(n)} - \frac{P(m)^2}{P(n)}$$

and

$$\mu^{\mathcal{L}}(q, \lambda_\beta) = \sum_{i=1}^{s} \beta_i P(\mathcal{E}_i, m) = \sum_{i=1}^{s} \beta_i P_i(m) = \frac{P(m)^2}{P(n)} - \sum_{i=1}^{s} \frac{P_i(m)^2}{P_i(n)}$$

so that $F \cap \bar{Q}$ is a union of connected components of Z_β. □

REMARK 15. Recall from Remark 7 that from the decomposition $Z_\beta = \sqcup Z_{(\tau')}$ into disjoint closed subsets we obtain similar decompositions $Y_\beta = \sqcup Y_{(\tau')}$ and $Y_\beta^{\mathrm{ss}} = \sqcup Y_{(\tau')}^{\mathrm{ss}}$ and

$$S_\beta = \sqcup G Y_{(\tau')}^{\mathrm{ss}} \cong \sqcup G \times_{P_\beta} Y_{(\tau')}^{\mathrm{ss}},$$

where $Y_{(\tau')} = p_\beta^{-1}(Z_{(\tau')}) \subseteq Y_\beta$ and $Y_{(\tau')}^{\mathrm{ss}} = p_\beta^{-1}(Z_{(\tau')}^{\mathrm{ss}})$. Thus $GY_{(\tau)}^{\mathrm{ss}} \cong \sqcup G \times_{P_\beta} Y_{(\tau')}^{\mathrm{ss}}$ is a union of connected components of S_β.

We want to show that ρ belongs to Y_β^{ss}, which is equivalent to showing that $\bar\rho \in Z_\beta^{\mathrm{ss}}$. Recall that the subscheme Z_β is invariant under the subgroup of $\mathrm{SL}(V)$ which stabilizes β,

$$\mathrm{Stab}\,\beta = \left(\prod_{i=1}^{s} \mathrm{GL}(V_i)\right) \cap \mathrm{SL}(V).$$

The original linearization $\mathcal{L}$ restricts to a $\mathrm{Stab}\,\beta$ linearization on Z_β which we also denote by $\mathcal{L}$. Associated to $-\beta$ is a character

$$\chi_{-\beta} : \mathrm{Stab}\,\beta \longrightarrow \mathbb{C}^*,$$
$$(g_1,\dots,g_s) \longmapsto \Pi_{i=1}^s \det g_i^{-\beta_i},$$

which we can use to twist the linearization $\mathcal{L}$; we let $\mathcal{L}^{\chi_{-\beta}}$ denote this twisted linearization on Z_β. By definition,

$$Z_\beta^{\mathrm{ss}} := Z_\beta^{\mathrm{Stab}\,\beta-\mathrm{ss}}(\mathcal{L}^{\chi_{-\beta}})$$

is the open subscheme of Z_β whose geometric points are semistable for this $\mathrm{Stab}\,\beta$ action.

Note that the centre of $\mathrm{Stab}\,\beta$ is

$$Z(\mathrm{Stab}\,\beta) = \left\{(t_1,\dots,t_s) \in (\mathbb{C}^*)^s : \prod_{i=1}^{s} t_i^{P_i(n)} = 1\right\}.$$

Consider the subgroup

$$G' = \prod_{i=1}^{s} \mathrm{SL}(V_i)$$

of $\mathrm{Stab}\,\beta$.

LEMMA 6.5. *There is an isomorphism* $\mathrm{Stab}\,\beta \cong (G' \times Z(\mathrm{Stab}\,\beta))/(\prod_{i=1}^s \mathbb{Z}/P_i(n)\mathbb{Z})$. *Furthermore, the semistable subscheme* $F^{\mathrm{ss}} := F^{\mathrm{Stab}\,\beta-\mathrm{ss}}(\mathcal{L}^{\chi_{-\beta}})$ *for the* $\mathrm{Stab}\,\beta$ *action on* F *with respect to* $\mathcal{L}^{\chi_{-\beta}}$ *is equal to the semistable subset for the* G'*-action on* F *with respect to* $\mathcal{L}$.

Proof. The stabilizer of β is

$$\mathrm{Stab}\,\beta = \left(\prod_{i=1}^{s} \mathrm{GL}(V_i)\right) \cap \mathrm{SL}(V)$$

and there is a surjection

$$G' \times Z(\mathrm{Stab}\,\beta) \longrightarrow \mathrm{Stab}\,\beta,$$
$$((g_1',\dots,g_m'),(t_1,\dots,t_s)) \longmapsto (t_1 g_1',\dots,t_s g_s')$$

with kernel $\prod_{i=1}^s \mathbb{Z}/P_i(n)\mathbb{Z}$. Hence $\mathrm{Stab}\,\beta$ is the quotient of the product $G' \times Z(\mathrm{Stab}\,\beta)$ by this product of the finite cyclic groups of order $P_i(n)$. However, finite groups do not make any difference to GIT semistability, so we can just consider the action of $G' \times Z(\mathrm{Stab}\,\beta)$.

The centre $Z(\mathrm{Stab}\,\beta)$ fixes each point $q = \oplus q_i$ in F and acts on the fibre of $\mathcal{L}$ at q as multiplication by a character χ. Since q_i is multiplied by t_i^{-1}, $\det H^0(q_i(m))$ is multiplied by $t_i^{-P_i(m)}$, and we find that $\chi(t_1,\dots,t_s) = \prod_{i=1}^s t_i^{P_i(m)}$. Since $\prod t_i^{P_i(n)} = 1$ we may rewrite this as

$$\chi(t_1,\dots,t_s) = \prod_{i=1}^{s} t_i^{-(P_i(n)(P(n)/P(m)) - P_i(m))} = \prod_{i=1}^{s} t_i^{-\beta_i P_i(n)}.$$

The centre acts on $\mathcal{L}_q$ via the character $\chi_{-\beta}$ and so it acts trivially on the fibre over the modified linearization $\mathcal{L}^{\chi_{-\beta}}$. In particular, the semistable set for the action of $\operatorname{Stab}\beta = G'Z(\operatorname{Stab}\beta)$ with respect to $\mathcal{L}^{\chi_{-\beta}}$ is equal to the semistable set for the G' action with respect to $\mathcal{L}$. $\square$

Recall the following standard result:

LEMMA 6.6. *Let $X_1, \ldots, X_k$ be complex projective schemes and suppose G_i is a reductive group acting on X_i for $1 \leqslant i \leqslant k$. Let $\mathcal{L}_i$ be an ample linearization of the G_i action on X_i. Then*

$$\left(\prod_{i=1}^k X_i\right)^{\prod G_i-\mathrm{ss}} \left(\bigotimes_{i=1}^k \pi_i^* \mathcal{L}_i\right) = \prod_{i=1}^k X_i^{G_i-ss}(\mathcal{L}_i),$$

where $\pi_j : \prod_{i=1}^k X_i \to X_j$ is the projection map.

Recall that

$$F \cong Q_1 \times \cdots \times Q_s,$$

where $Q_i = \operatorname{Quot}(V_i \otimes \mathcal{O}(-n), P_i)$. Consider the linearization $\mathcal{L}_i = \det(\pi_{Q_i *}(\mathcal{U}_i \otimes \pi_W^* \mathcal{O}(m)))$ of the $\mathrm{SL}(V_i)$-action on Q_i where $\mathcal{U}_i$ is the universal quotient sheaf on this quot scheme. By [**21**, Theorem 1.19], provided n and m are sufficiently large, the points of the semistable subscheme

$$Q_i^{\mathrm{ss}} := Q_i^{\mathrm{SL}(V_i)-\mathrm{ss}}(\mathcal{L}_i)$$

are quotient sheaves $q_i : V_i \otimes \mathcal{O}(-n) \to \mathcal{E}_i$ where $\mathcal{E}_i$ is Gieseker semistable.

PROPOSITION 6.7. *Under the isomorphism $F \cong Q_1 \times \cdots \times Q_s$ the semistable part of F with respect to $\mathcal{L}^{\chi_{-\beta}}$ is isomorphic to the product of the GIT semistable subschemes Q_i^{ss} :*

$$F^{\mathrm{ss}} \cong Q_1^{\mathrm{ss}} \times \cdots \times Q_s^{\mathrm{ss}}.$$

Furthermore, for n and m sufficiently large the limit point $\bar{\rho} \in F^{\mathrm{ss}} \cap Q \subset Z_\beta^{\mathrm{ss}}$ and so β is an index in the stratification of $\bar{Q}$.

Proof. By Lemma 6.5, we have that $F^{\mathrm{ss}} := F^{\operatorname{Stab}\beta-\mathrm{ss}}(\mathcal{L}^{\chi_{-\beta}}) = F^{G'-\mathrm{ss}}(\mathcal{L})$ where $G' = \prod_{i=1}^s \mathrm{SL}(V_i)$. If we can show that $\mathcal{L}|_F \cong \otimes \pi_i^* \mathcal{L}_i$, then, by Lemma 6.6,

$$F^{G'-\mathrm{ss}}(\mathcal{L}) = F^{G'-\mathrm{ss}}(\otimes \pi_i^* \mathcal{L}_i) = Q_1^{\mathrm{ss}} \times \cdots \times Q_s^{\mathrm{ss}},$$

where $\pi_i : F \cong \prod_{j=1}^s Q_j \to Q_i$ is the ith projection map. Let $i : F \hookrightarrow \operatorname{Quot}(V \otimes \mathcal{O}(-n), P)$ and $j : F \times W \hookrightarrow \operatorname{Quot}(V \otimes \mathcal{O}(-n), P) \times W$ denote the inclusions. Then

$$\begin{aligned}\mathcal{L}|_F &= i^* \det(\pi_*(\mathcal{U} \otimes \pi_W^* \mathcal{O}(n))) \\ &\cong \det \pi_{F*} j^*(\mathcal{U} \otimes \pi_W^* \mathcal{O}(n)),\end{aligned}$$

since the determinant commutes with pullbacks and i is flat. The universal family $\mathcal{U}$ pulls back via the morphism $j : F \times W \hookrightarrow \operatorname{Quot}(V \otimes \mathcal{O}(-n), P) \times W$ to the family $\bigoplus_{i=1}^s p_i^* \mathcal{U}_i$ parameterized by F, where $p_i : F \times W \cong (\prod_{j=1}^s Q_j) \times W \to Q_i \times W$ is the obvious projection map. Thus,

$$\begin{aligned}\mathcal{L}|_F &\cong \det\left(\bigoplus_{i=1}^{s}\pi_{F*}(p_i^*\mathcal{U}_i\otimes(\pi_W^{F\times W})^*\mathcal{O}(n))\right)\\ &\cong \bigotimes_{i=1}^{s}\det\pi_{F*}p_i^*(\mathcal{U}_i\otimes(\pi_W^{Q_i\times W})^*\mathcal{O}(n))\\ &\cong \bigotimes_{i=1}^{s}\det\pi_i^*\pi_{Q_i*}(\mathcal{U}_i\otimes(\pi_W^{Q_i\times W})^*\mathcal{O}(n))\\ &\cong \bigotimes_{i=1}^{s}\pi_i^*\mathcal{L}_i.\end{aligned}$$

We have $\bar{\rho}=\oplus\rho_i$ where $\rho_i : V_i\otimes\mathcal{O}(-n)\to\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ is a quotient of $V_i\otimes\mathcal{O}(-n)$ such that $H^0(\rho_i(n))$ is an isomorphism and $\mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ is a semistable sheaf. We pick n and then m sufficiently large as in [**21**] so that GIT semistability of points in Q_i with respect to $\mathcal{L}_i$ is equivalent to Gieseker semistability of the associated sheaves. Then

$$Q_i^{\mathrm{ss}} := Q_i^{\mathrm{SL}(V_i)-\mathrm{ss}}(\mathcal{L}_i)$$

is the open subset of quotients parameterizing semistable sheaves. By definition of the Harder–Narasimhan filtration $\rho_i\in Q_i^{\mathrm{ss}}$ and so $\bar{\rho}\in Q\cap F^{\mathrm{ss}}\subset Z_\beta^{\mathrm{ss}}$. In particular, S_β is non-empty and so β is an index for the stratification of $\bar{Q}$. □

Proposition 6.8. *Choose an ordered basis of V and a positive Weyl chamber $\mathfrak{t}_+$ in the Lie algebra of the associated maximal torus of $G=\mathrm{SL}(V)$. Let $\tau=(P_1,\dots,P_s)$ be a Harder–Narasimhan type and let $\beta=\beta(\tau)=\beta(\tau,n,m)\in\mathfrak{t}_+$ be as at (6.2). If n and m are sufficiently large, then we can give R_τ a scheme structure such that every connected component of R_τ is a connected component of S_β.*

Proof. Let n and m be chosen as in Proposition 6.7. Let R_i be the open subscheme of Q_i consisting of quotient sheaves $q_i : V_i\otimes\mathcal{O}(-n)\to\mathcal{E}_i$ which are pure of dimension e and such that $H^0(q_i(n))$ is an isomorphism. Let R_i^{ss} denote the semistable subscheme for the $\mathrm{SL}(V_i)$-action on R_i. Then consider the subschemes

$$Z_{(\tau)}^{\mathrm{ss}}=\left\{q=\bigoplus_{i=1}^{s}q_i : (q_i : V_i\otimes\mathcal{O}(-n)\longrightarrow\mathcal{E}_i)\in R_i^{\mathrm{ss}}\right\}$$

of Z_β^{ss} and $Y_{(\tau)}^{\mathrm{ss}}=p_\beta^{-1}(Z_{(\tau)}^{\mathrm{ss}})$ of Y_β^{ss}.

Any quotient sheaf $q : V\otimes\mathcal{O}(-n)\to\mathcal{F}$ in $Y_{(\tau)}^{\mathrm{ss}}$ has a filtration and associated graded object $\bar{q} : V\otimes\mathcal{O}(-n)\to\mathcal{F}$ for which the successive quotients are semistable with Hilbert polynomials $P_1,\dots,P_s$; that is, $\mathcal{F}$ has Harder–Narasimhan type τ. As R_τ is G-invariant it follows immediately that every point in $GY_{(\tau)}^{\mathrm{ss}}$ is a point in R_τ. Conversely, let $\rho : V\otimes\mathcal{O}(-n)\to\mathcal{E}$ be any point in R_τ; then the Harder–Narasimhan filtration of $\mathcal{E}$ gives rise to a filtration of V by subspaces $W^{(i)}=H^0(q(n))^{-1}(H^0(\mathcal{E}^{(i)}(n)))$. We choose $g\in G=\mathrm{SL}(V)$ to be a change of basis matrix sending $W^{(i)}$ to $V^{(i)}$ for each i, which is possible since $\dim W^{(i)}=\dim V^{(i)}=\sum_{j\leqslant i}P_i(n)$. Then $g\cdot q\in Y_{(\tau)}^{\mathrm{ss}}$ by Proposition 6.7, so

$$R_\tau=GY_{(\tau)}^{\mathrm{ss}}\cong G\times_{P_\beta}Y_{(\tau)}^{\mathrm{ss}}$$

and this gives the set R_τ its scheme structure.

Since $\bar{R}_i^{\mathrm{ss}} = R_i^{\mathrm{ss}}$ (cf. [**21**, Theorem 1.19]) the subscheme $Z_{(\tau)}^{\mathrm{ss}}$ is closed in $F^{\mathrm{ss}} \cap \bar{Q}$ and is thus a union of connected components of Z_β^{ss} by Corollary 6.4. It follows that $R_\tau = GY_{(\tau)}^{\mathrm{ss}}$ is a union of connected components of S_β by Remark 15. □

7. *n-rigidified sheaves of fixed Harder–Narasimhan type*

As in the previous section, we let $\tau = (P_1, \dots, P_s)$ be a Harder–Narasimhan type and let $\beta = \beta(\tau) = \beta(\tau, n, m) \in \mathfrak{t}_+$ be the associated rational weight given at (6.2). In § 8, we consider the action of $\mathrm{Stab}\,\beta$ on the closure $\bar{Y}_{(\tau)}$ in the quot scheme $\mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$ of the subscheme $Y_{(\tau)}^{\mathrm{ss}}$ defined in the proof of Proposition 6.8. We know that the P_β-orbits in $Y_{(\tau)}^{\mathrm{ss}}$ correspond to G-orbits in $R_\tau \cong G \times_{P_\beta} Y_{(\tau)}^{\mathrm{ss}}$ and thus to isomorphism classes of sheaves of Harder–Narasimhan type τ, and so in this section we study the objects parametrized by the $\mathrm{Stab}\,\beta$-orbits in $Y_{(\tau)}^{\mathrm{ss}}$.

Definition 5. Let n be a positive integer and $\mathcal{F}$ be a sheaf with Harder–Narasimhan type τ. Let $0 \subset \mathcal{F}^{(1)} \subset \cdots \subset \mathcal{F}^{(s)} = \mathcal{F}$ denote the Harder–Narasimhan filtration of $\mathcal{F}$ and $\mathcal{F}_i := \mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$ denote the successive quotients. Then an n-rigidification for $\mathcal{F}$ is an isomorphism

$$H^0(\mathcal{F}(n)) \cong \bigoplus_{i=1}^{s} H^0(\mathcal{F}_i(n))$$

which is compatible with the inclusion morphisms $j^{(i)} : \mathcal{F}^{(i)} \hookrightarrow \mathcal{F}$ and projection morphisms $\pi^{(i)} : \mathcal{F}^{(i)} \to \mathcal{F}_i$; that is, for each i we have a commutative triangle

$$\begin{array}{ccc} H^0(\mathcal{F}^{(i)}(n)) & \xrightarrow{j^{(i)}_*} & H^0(\mathcal{F}(n)) \\ \downarrow{\scriptstyle \pi^{(i)}_*} & \swarrow & \\ H^0(\mathcal{F}_i(n)) & & \end{array}$$

where the unlabelled arrow is the given isomorphism $H^0(\mathcal{F}(n)) \cong \bigoplus_{i=1}^{s} H^0(\mathcal{F}_i(n))$ followed by the ith projection. An isomorphism of two n-rigidified sheaves $\mathcal{E}$ and $\mathcal{F}$ is an isomorphism of sheaves $\phi : \mathcal{E} \cong \mathcal{F}$ such that for each i the induced isomorphisms $H^0(\mathcal{E}^{(i)}(n)) \cong H^0(\mathcal{F}^{(i)}(n))$ are compatible with the n-rigidifications; that is, we have a commutative square of isomorphisms

$$\begin{array}{ccc} H^0(\mathcal{E}(n)) & \longrightarrow & H^0(\mathcal{F}(n)) \\ \downarrow & & \downarrow \\ \oplus_{i=1}^{s} H^0(\mathcal{E}_i(n)) & \longrightarrow & \oplus_{i=1}^{s} H^0(\mathcal{F}_i(n)) \end{array}$$

where the horizontal morphisms are induced by the isomorphism ϕ and the vertical morphisms are the given n-rigidifications for each sheaf.

Remark 16. Any sheaf $\mathcal{F}$ with Harder–Narasimhan type τ has an n-rigidification for $n \gg 0$ where n is sufficiently large so the higher cohomology of $\mathcal{F}(n)$ and $\mathcal{F}_i(n)$ vanish. In fact, if we pick n as required for Proposition 6.8, then the quotient sheaf $q : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ has a natural n-rigidification coming from the eigenspace decomposition $V = \bigoplus_{i=1}^{s} V_i$ of V for $\lambda_\beta(\mathbb{C}^*)$ and the isomorphisms $V \cong H^0(\mathcal{F}(n))$ and $V_i \cong H^0(\mathcal{F}_i(n))$ induced by q.

Lemma 7.1. *Consider the n-rigidified sheaves represented by points $q : V \otimes \mathcal{O}(-n) \to \mathcal{E}$ and $q' : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in $Y_{(\tau)}^{\mathrm{ss}}$ as in Remark 16. These n-rigidified sheaves are isomorphic if and only if there is some $g \in \Pi_{i=1}^s \mathrm{GL}(V_i)$ such that $g \cdot q = q'$.*

Proof. If $\mathcal{E}$ and $\mathcal{F}$ are isomorphic as n-rigidified sheaves then, in particular, they are isomorphic as sheaves and so there is some $g \in \mathrm{GL}(V)$ such that $g \cdot q = q'$. As q and q' are both in Y_β^{ss} and $GY_\beta^{\mathrm{ss}} \cong G \times_{P_\beta} Y_\beta^{\mathrm{ss}}$, we know that $g \in P_\beta$ is block upper triangular with respect to the blocks for β. Then as the isomorphism is compatible with the n-rigidifications, we see that g must be block diagonal; that is, g is an element of $\mathrm{Stab}\,\beta = \Pi_{i=1}^s \mathrm{GL}(V_i)$.

Conversely, if there is a $g \in \Pi_{i=1}^s \mathrm{GL}(V_i)$ such that $g \cdot q = q'$, then this induces a sheaf isomorphism $\mathcal{E} \cong \mathcal{F}$. The fact that g is block diagonal with respect to the blocks for β means this isomorphism is an isomorphism of n-rigidified sheaves. □

8. *Moduli spaces of rigidified unstable sheaves*

In this final section, we construct moduli spaces of n-rigidified sheaves of fixed Harder–Narasimhan type τ as GIT quotients $\overline{Y_{(\tau)}}//\mathrm{Stab}\,\beta$, where $\beta = \beta(\tau)$, with respect to perturbations of the canonical linearization $\mathcal{L}_\beta$ for the $\mathrm{Stab}\,\beta$-action on $\overline{Y_{(\tau)}}$.

Remark 17. We would like to construct moduli spaces of sheaves of fixed Harder–Narasimhan type τ as GIT quotients $\overline{Y_{(\tau)}}//P_\beta$ or $G \times_{P_\beta} \overline{Y_{(\tau)}}//G$ for suitable perturbations of the linearization $\mathcal{L}_\beta$. However, there are difficulties here since in general the group P_β is not reductive and the linearization $\mathcal{L}_\beta$ on $G \times_{P_\beta} \overline{Y_{(\tau)}}$ is not ample.

Remark 18. Moduli spaces of unstable bundles of rank 2 on the projective plane have been constructed by Strømme [**22**] and this has been generalized to sheaves with Harder–Narasimhan filtrations of length 2 over smooth projective varieties by Drézet in [**4**].

We will define a notion of θ-(semi)stability for sheaves over W of a fixed Harder–Narasimhan type τ corresponding to a sequence of Hilbert polynomials $(P_1, \ldots, P_s)$ and a moduli functor of θ-semistable n-rigidified sheaves of Harder–Narasimhan type τ over W. This notion of θ-(semi) stability depends on a parameter $\theta \in \mathbb{Q}^s$ (see Definition 7), and we will show that if $m \gg n \gg 0$, then θ determines for us a perturbed $\mathrm{Stab}\,\beta$-linearization on the closure $\bar{Y}_{(\tau)}$ of $Y_{(\tau)}$ as in § 3.3 with the following properties:

- (i) any $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in Y_τ^{ss} is GIT semistable for the perturbed linearization associated to θ if and only if the sheaf $\mathcal{F}$ of Harder–Narasimhan type τ is θ-semistable (Theorem 8.8), and
- (ii) the associated GIT quotient is a projective scheme which corepresents the moduli functor of θ-semistable n-rigidified sheaves of Harder–Narasimhan type τ over W (Theorem 8.9).

Fix a Harder–Narasimhan type $\tau = (P_1, \ldots, P_s)$ and let $P = \sum_i P_i$; then, by Proposition 6.8, for n and m sufficiently large the subvariety $R_\tau = GY_{(\tau)}^{\mathrm{ss}} \cong G \times_{P_\beta} Y_{(\tau)}^{\mathrm{ss}}$ of Q parametrizing sheaves of Harder–Narasimhan type τ is a union of connected components of a stratum $S_{\beta(\tau)}$ in the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $\bar{Q}$ given by

$$\beta(\tau) = i\,\mathrm{diag}(\beta_1, \ldots, \beta_1, \ldots, \beta_s, \ldots, \beta_s) \in \mathfrak{t}_+,$$

where

$$\beta_i = \frac{P(m)}{P(n)} - \frac{P_i(m)}{P_i(n)}$$

appears $P_i(n)$ times. The stratum S_β for $\beta = \beta(\tau)$ is isomorphic to $G \times_{P_\beta} Y_\beta^{\mathrm{ss}}$ and as in § 3 we consider linearizations of the G-action on the projective completion

$$\hat{S_\beta} := G \times_{P_\beta} \bar{Y}_\beta,$$

where $\bar{Y}_\beta$ is the closure of Y_β^{ss} in $\bar{Q}$. Since $R_\tau \cong G \times_{P_\beta} Y_{(\tau)}^{\mathrm{ss}}$ where $Y_{(\tau)}^{\mathrm{ss}}$ is a union of connected components of Y_β^{ss} we let

$$\hat{R_\tau} = G \times_{P_\beta} \bar{Y}_{(\tau)},$$

where $\bar{Y}_{(\tau)}$ is the closure of $Y_{(\tau)}^{\mathrm{ss}}$ in $\bar{Q}$; this is the closure of R_τ in $\hat{S_\beta}$ and is a projective completion of R_τ.

Let $\mathcal{L}_\beta$ denote the canonical linearization on $\hat{S_\beta}$ as defined in § 3.2 and let $\mathcal{L}_\beta$ also denote its restriction to $\hat{R_\tau}$. As was noted in § 3, S_β and R_τ have categorical quotients

$$S_\beta \to Z_\beta /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\beta$$

and

$$R_\tau \to Z_{(\tau)} /\!/_{\mathcal{L}_\beta} \mathrm{Stab}\beta$$

but these are far from orbit spaces: the map $p_\beta : Y_{(\tau)}^{\mathrm{ss}} \to Z_{(\tau)}^{\mathrm{ss}}$ sends a point y to the graded object associated to its Harder–Narasimhan filtration and since $p_\beta(y)$ is contained in the orbit closure of y these points are S-equivalent, in the sense that they represent the same points in the categorical quotient. In fact two sheaves $\mathcal{F}$ and $\mathcal{G}$ with Harder–Narasimhan type τ are S-equivalent in this context if and only if the graded objects associated to their Jordan–Hölder filtrations are isomorphic. We would like a finer notion of equivalence.

As was noted in § 3.3, one possible approach to avoiding this problem is to perturb the canonical linearization, but applying GIT to either the canonical G-linearization on $\hat{S_\beta}$ or the canonical P_β-linearization on $\bar{Y}_\beta$ is delicate. So instead we will consider perturbations of the canonical $\mathrm{Stab}\,\beta$-linearization $\mathcal{L}_\beta$ on $\bar{Y}_\beta$ given by making a small perturbation to the character $\chi_{-\beta} : \mathrm{Stab}\,\beta \to \mathbb{C}^*$ used to twist $\mathcal{L}$.

8.1. *Semistability*

We will choose a perturbation of the canonical $\mathrm{Stab}\,\beta$-linearization $\mathcal{L}_\beta$ on $\bar{Y}_{(\tau)}$ which depends on a parameter $\theta = (\theta_1, \dots, \theta_s) \in \mathbb{Q}^s$. A notion of (semi)stability with respect to this parameter θ will be defined for all sheaves over W with Harder–Narasimhan type τ. Before stating the definition, we first need an easy lemma which enables us to write down the Harder–Narasimhan filtration of a direct sum of pure sheaves $\mathcal{E} \oplus \mathcal{F}$ in terms of the Harder–Narasimhan filtrations of $\mathcal{E}$ and $\mathcal{F}$.

LEMMA 8.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be pure sheaves of dimension e with Harder–Narasimhan filtrations*

$$0 \subset \mathcal{E}^{(1)} \subset \cdots \subset \mathcal{E}^{(N)} = \mathcal{E}$$

and

$$0 \subset \mathcal{F}^{(1)} \subset \cdots \subset \mathcal{F}^{(M)} = \mathcal{F}.$$

Then the maximal destabilizing subsheaf of $\mathcal{E} \oplus \mathcal{F}$ is

(i) $\mathcal{E}^{(1)}$ *if* $P(\mathcal{E}^{(1)})r(\mathcal{F}^{(1)}) > P(\mathcal{F}^{(1)})r(\mathcal{E}^{(1)})$,
(ii) $\mathcal{F}^{(1)}$ *if* $P(\mathcal{E}^{(1)})r(\mathcal{F}^{(1)}) < P(\mathcal{F}^{(1)})r(\mathcal{E}^{(1)})$,

(iii) $\mathcal{E}^{(1)} \oplus \mathcal{F}^{(1)}$ *if* $P(\mathcal{E}^{(1)})r(\mathcal{F}^{(1)}) = P(\mathcal{F}^{(1)})r(\mathcal{E}^{(1)})$.

Proof. Suppose $P(\mathcal{E}^{(1)})r(\mathcal{F}^{(1)}) > P(\mathcal{F}^{(1)})r(\mathcal{E}^{(1)})$; then we need to show $\mathcal{E}^{(1)}$ is the maximal destabilizing subsheaf of $\mathcal{E} \oplus \mathcal{F}$. We know $\mathcal{E}^{(1)}$ is semistable and we also claim that there is no sheaf $\mathcal{G} \subset \mathcal{E} \oplus \mathcal{F}$ with reduced Hilbert polynomial greater than $\mathcal{E}^{(1)}$. To prove this suppose such a sheaf $\mathcal{G}$ exists; then, we may assume, without loss of generality, that $\mathcal{G}$ is semistable. As $\mathrm{Hom}(\mathcal{G}, \mathcal{E}) = 0$, the composition

$$\mathcal{G} \hookrightarrow \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{E}$$

is zero and so $\mathcal{G}$ is contained completely in $\mathcal{F}$. This contradicts the fact that $\mathcal{F}^{(1)}$ is the maximal destabilizing subsheaf in $\mathcal{F}$.

Now suppose there is $\mathcal{E}^{(1)} \subsetneq \mathcal{G} \subset \mathcal{E} \oplus \mathcal{F}$ such that $\mathcal{G}$ and $\mathcal{E}^{(1)}$ have the same reduced Hilbert polynomial. Then the composition

$$\mathcal{G} \hookrightarrow \mathcal{E} \oplus \mathcal{F} \longrightarrow \mathcal{F}$$

is zero and so $\mathcal{G}$ is contained in $\mathcal{E}$ which contradicts the fact that $\mathcal{E}^{(1)}$ is the maximal destabilizing subsheaf in $\mathcal{E}$. Therefore, $\mathcal{E}^{(1)}$ is the maximal destabilizing subsheaf of the direct sum.

The other cases follow from similar standard arguments and will be omitted. □

DEFINITION 6. We say a sheaf $\mathcal{F}$ is τ-compatible if it has a filtration

$$0 \subseteq \mathcal{F}^{(1)} \subseteq \cdots \subseteq \mathcal{F}^{(s)} = \mathcal{F}$$

such that $\mathcal{F}_i = \mathcal{F}^{(i)}/\mathcal{F}^{(i-1)}$, if non-zero, is semistable with reduced Hilbert polynomial P_i/r_i where $\tau = (P_1, \ldots, P_s)$. We call such a filtration a generalized Harder–Narasimhan filtration of $\mathcal{F}$; it is the same as the Harder–Narasimhan filtration of $\mathcal{F}$ except that we may have $\mathcal{F}^{(i)} = \mathcal{F}^{(i-1)}$ for some i. Note that the generalized Harder–Narasimhan filtration of a τ-compatible sheaf $\mathcal{F}$ is uniquely determined by $\mathcal{F}$ and τ.

Of course any sheaf of Harder–Narasimhan type τ is τ-compatible.

DEFINITION 7. A τ-compatible sheaf $\mathcal{F}$ is θ-semistable if for all proper non-zero τ-compatible subsheaves $\mathcal{F}' \subset \mathcal{F}$ for which $\mathcal{F}/\mathcal{F}'$ is also τ-compatible we have

$$\frac{\sum_{i=1}^{s} \theta_i P(\mathcal{F}'_i)}{P(\mathcal{F}')} \geqslant \frac{\sum_{i=1}^{s} \theta_i P(\mathcal{F}_i)}{P(\mathcal{F})},$$

where $\mathcal{F}'_i$ and $\mathcal{F}_i$ denote the successive quotients appearing in the generalized Harder–Narasimhan filtrations of $\mathcal{F}'$ and $\mathcal{F}$. We say $\mathcal{F}$ is θ-stable if this inequality is strict for all such subsheaves.

REMARK 19. To obtain a non-trivial notion of semistability, we will always assume that the θ_i are not all equal to each other. In addition, we will usually assume that for all $m \gg n \gg 0$

$$\frac{\sum \theta_i P_i(n)}{P(n)} \geqslant \frac{\sum \theta_i P_i(m)}{P(m)}. \tag{8.1}$$

If (8.1) does not hold, we can still define θ-(semi)stability but there will be no θ-semistable sheaves with Harder–Narasimhan type τ.

8.2. *Families and the moduli functor*

Let S be a complex scheme, and recall that a flat family of sheaves over W parametrized by S is a sheaf $\mathcal{V}$ over $W \times S$ which is flat over S. We say this is a flat family of semistable sheaves which are pure of dimension e with Hilbert polynomial P if for each point $s \in S$ the sheaf $\mathcal{V}_s := \mathcal{V}|_{W\times\{s\}}$ is a semistable pure sheaf of dimension e with Hilbert polynomial P. We say two flat families $\mathcal{V}$ and $\mathcal{W}$ over W parametrized by S are isomorphic if there is a line bundle L on S such that $\mathcal{V} \cong \mathcal{W} \otimes \pi_S^* L$ where $\pi_S : W \times S \to S$ is the projection. Given a morphism $f : T \to S$, we can pull back a family on S to a family on T in the standard way.

Definition 8. Let $\tau = (P_1, \ldots, P_s)$ be a Harder–Narasimhan type of a pure sheaf of dimension e. A flat family $\mathcal{V}$ of sheaves over W parametrized by S has Harder–Narasimhan type τ if $\mathcal{V}$ is a family of pure sheaves of dimension e with Hilbert polynomial $\sum_{i=1}^m P_i$ and there is a filtration by subsheaves

$$0 \subsetneq \mathcal{V}^{(1)} \subsetneq \cdots \subsetneq \mathcal{V}^{(s)} = \mathcal{V}$$

such that $\mathcal{V}_i = \mathcal{V}^{(i)}/\mathcal{V}^{(i-1)}$ is a flat family of semistable sheaves of pure of dimension e with Hilbert polynomial P_i.

Let n be a positive integer. A flat family $\mathcal{V}$ of n-rigidified sheaves of Harder–Narasimhan type τ over W parametrized by S is a flat family $\mathcal{V}$ of sheaves of Harder–Narasimhan type τ parametrized by S which has an n-rigidification; that is, an isomorphism

$$H^0(\mathcal{V}(n)) \cong \bigoplus_{i=1}^{s} H^0(\mathcal{V}_i(n))$$

which is compatible with the inclusion morphisms $\mathcal{V}^{(i)} \hookrightarrow \mathcal{V}$ and projection morphisms $\mathcal{V}^{(i)} \to \mathcal{V}_i$ in the sense of Definition 5.

Finally, we say such a family is θ-semistable if for each $s \in S$ the sheaf $\mathcal{V}_s$ is θ-semistable.

Lemma 8.2. *There exists a flat family $\mathcal{V}$ of n-rigidified sheaves of Harder–Narasimhan type τ over W parametrized by $Y_{(\tau)}^{\mathrm{ss}}$ which is given by restricting the universal quotient sheaf $\mathcal{U}$ on $\mathrm{Quot}(V \otimes \mathcal{O}(-n), P) \times W$ to $Y_{(\tau)}^{\mathrm{ss}} \times W$.*

Proof. We use the vector space filtration $0 \subset V^{(1)} \subset \cdots \subset V^{(s)} = V$ corresponding to $\beta = \beta(\tau)$, defined as in § 7, to induce a universal Harder–Narasimhan filtration for $\mathcal{V}$. Then a universal n-rigidification comes from the eigenspace decomposition $V = \bigoplus_{i=1}^{s} V_i$ for β. □

Definition 9. The moduli functor of θ-semistable n-rigidified sheaves over W of Harder–Narasimhan type τ is the contravariant functor $\mathcal{M}^{\theta-\mathrm{ss}}(W, \tau, n)$ from complex schemes to sets such that if S is a scheme over $\mathbb{C}$, then $\mathcal{M}^{\theta-\mathrm{ss}}(W, \tau, n)(S)$ is the set of isomorphism classes of families of θ-semistable n-rigidified sheaves over W parametrized by S with Harder–Narasimhan type τ.

8.3. *Boundedness*

By Lemma 6.1, if n is sufficiently large, then all sheaves with Hilbert polynomial P and Harder–Narasimhan type τ are n-regular and the successive quotients appearing in their Harder–Narasimhan filtrations are n-regular. A similar argument gives us the following:

Lemma 8.3. *Fix a Harder–Narasimhan type $\tau = (P_1, \ldots, P_s)$. Then for n sufficiently large every τ-compatible subsheaf $\mathcal{F}' \subset \mathcal{F}$ of a sheaf with Harder–Narasimhan type τ is n-regular.*

Moreover, the successive quotients $\mathcal{F}'_i$ appearing in the generalized Harder–Narasimhan filtration of $\mathcal{F}'$ are also n-regular.

We also have the following:

LEMMA 8.4. *Fix a Harder–Narasimhan type $\tau = (P_1, \dots, P_s)$. If n is sufficiently large, then for any τ-compatible subsheaf $\mathcal{F}' \subset \mathcal{F}$ of a sheaf with Harder–Narasimhan type τ the following inequalities are equivalent*:

$$\frac{\sum \theta_i P(\mathcal{F}'_i)}{P(\mathcal{F}')} \geqslant \frac{\sum \theta_i P(\mathcal{F}_i)}{P(\mathcal{F})} \iff \frac{\sum \theta_i P(\mathcal{F}'_i, n)}{P(\mathcal{F}', n)} \geqslant \frac{\sum \theta_i P(\mathcal{F}_i, n)}{P(\mathcal{F}, n)},$$

where $\mathcal{F}'_i$ and $\mathcal{F}_i$ are the successive quotients in the generalized Harder–Narasimhan filtrations of $\mathcal{F}'$ and $\mathcal{F}$.

Proof. The Hilbert polynomials of $\mathcal{F}$ and $\mathcal{F}_i$ are fixed, and the successive quotients $\mathcal{F}'_i$ are semistable with reduced Hilbert polynomial

$$\frac{P(\mathcal{F}'_i)}{r'_i} = \frac{P_i}{r_i},$$

where r'_i denotes the multiplicity of $\mathcal{F}'_i$, so since there are only a finite number of possibilities for r'_i, there are only a finite number of possible Hilbert polynomials for $\mathcal{F}'_i$. Thus, the inequalities are equivalent for all sufficiently large n. □

8.4. *The choice of perturbed linearization*

Let $\theta = (\theta_1, \dots, \theta_s) \in \mathbb{Q}^s$ be a stability parameter satisfying the condition (8.1) of Remark 19. Then θ defines a perturbation of the canonical linearization $\mathcal{L}_\beta$ in the following way. For any natural number n, we can define

$$\beta'_i := \theta_i - \frac{\sum_{j=1}^s \theta_j P_j(n)}{P(n)} \tag{8.2}$$

and let $\beta' := i\,\mathrm{diag}(\beta'_1, \dots \beta'_1, \dots, \beta'_s, \dots, \beta'_s) \in \mathfrak{t}$ where β'_i appears $P_i(n)$ times. Then

$$\sum_{i=1}^s \beta'_i P_i(n) = 0$$

and the assumption (8.1) on θ means that

$$\beta' \cdot \beta = \sum_{i=1}^s \beta'_i \beta_i P_i(n) \geqslant 0.$$

For any small positive rational number ϵ consider the perturbation $\mathcal{L}_\beta^{\mathrm{per}}$ of the canonical $\mathrm{Stab}\,\beta$-linearization $\mathcal{L}_\beta$ on $\bar{Y}_{(\tau)}$ given by twisting the original ample linearization $\mathcal{L}$ on $\bar{Q}$ by the character $\chi_{-(\beta+\epsilon\beta')} : \mathrm{Stab}\,\beta \to \mathbb{C}^*$ corresponding to the rational weight $-(\beta+\epsilon\beta')$. By Proposition 3.3 if $\epsilon > 0$ is sufficiently small, then the stratification associated to the Stab β-action on $\bar{Y}_{(\tau)}$ with respect to $\mathcal{L}_\beta^{\mathrm{per}}$ is a refinement of the stratification associated to the $\mathrm{Stab}\,\beta$-action on $\bar{Y}_{(\tau)}$ with respect to $\mathcal{L}_\beta$. We assume that $\epsilon > 0$ is sufficiently small for this to be the case, and then since $\bar{Y}_{(\tau)}^{\mathrm{Stab}\,\beta-\mathrm{ss}}(\mathcal{L}_\beta) = Y_{(\tau)}^{\mathrm{ss}}$ it follows that

$$Y_{(\tau)}^{\mathrm{ss}} = \bigsqcup_{\gamma \in \mathcal{C}} S_\gamma^{(\beta)}$$

where $S_\gamma^{(\beta)}$ is a stratum appearing in the stratification for the perturbed linearization, and we have, for each $\gamma \in \mathcal{C}$,

$$S_\gamma^{(\beta)} = GY_\gamma^{(\beta)-\mathrm{ss}} \cong G \times_{P_\beta} Y_\gamma^{(\beta)-\mathrm{ss}}$$

where $Y_\gamma^{(\beta)-\mathrm{ss}} = (p_\gamma^{(\beta)})^{-1}(Z_\gamma^{(\beta)-\mathrm{ss}})$ and $Y_\gamma^{(\beta)-\mathrm{ss}}$ and $Z_\gamma^{(\beta)-\mathrm{ss}}$ are the subschemes of $Y_{(\tau)}^{\mathrm{ss}}$ defined following (2.1) and (2.3).

A 1-PS $\lambda : \mathbb{C}^* \to \operatorname{Stab} \beta \cong \mathrm{SL}(V) \cap \Pi \mathrm{GL}(V_i)$ of $\operatorname{Stab} \beta$ is given by 1-PSs $\lambda_i : \mathbb{C} \to \mathrm{GL}(V_i)$ for $i = 1, \dots, s$ such that

$$\prod_{i=1}^{s} \det \lambda_i(t) = 1$$

for all $t \in \mathbb{C}^*$. As in § 5.1, we can diagonalize each 1-PS simultaneously to get weights $k_1 > \cdots > k_r$ and for each i a decomposition $V_i = V_i^1 \oplus \cdots \oplus V_i^r$ into weight spaces and a filtration

$$0 \subset V_i^{[1]} \subset \cdots \subset V_i^{[r]} = V_i$$

of V_i where $V_i^{[j]} := \bigoplus_{l \leqslant j} V_i^l$ such that

$$\sum_{i=1}^{s} \sum_{j=1}^{r} k_j \dim V_i^j = 0.$$

There is an associated filtration

$$0 \subset V^{[1]} \subset \cdots \subset V^{[r]} = V$$

of V where $V^{[j]} := \bigoplus_{i=1}^{s} V_i^{[j]}$ and we let $V^j := V^{[j]}/V^{[j-1]}$.

Now suppose $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ is a point in $Y_{(\tau)}^{\mathrm{ss}}$ such that the limit $\bar{\rho} := \lim_{t\to 0} \lambda(t) \cdot \rho$ is also in $Y_{(\tau)}^{\mathrm{ss}}$. Then the 1-PS λ determines a filtration

$$0 \subset \mathcal{F}^{[1]} \subset \cdots \subset \mathcal{F}^{[r]} = \mathcal{F}$$

where $H^0(\mathcal{F}^{[j]}(n)) = V^{[j]}$ and $\bar{\rho} = \bigoplus_{j=1}^{r} \rho^j$ where $\bar{\rho}^j : V^j \otimes \mathcal{O}(-n) \to \mathcal{F}^j := \mathcal{F}^{[j]}/\mathcal{F}^{[j-1]}$. As $\bar{\rho}$ is also a point in $Y_{(\tau)}^{\mathrm{ss}}$, the sheaf $\bar{\mathcal{F}} := \bigoplus_{j=1}^{r} \mathcal{F}^j$ has Harder–Narasimhan type τ and the filtration $0 \subset V^{(1)} \subset \cdots \subset V^{(s)} = V$ induces this filtration. In particular, each direct summand $\mathcal{F}^j$ is τ-compatible (see Lemma 8.1) and has generalized Harder–Narasimhan filtration

$$0 \subseteq \mathcal{F}^j_{(1)} \subseteq \cdots \subseteq \mathcal{F}^j_{(s)} = \mathcal{F}^j.$$

We let $\mathcal{F}_i^j$ denote the successive quotients in this generalized Harder–Narasimhan filtration.

LEMMA 8.5. *Suppose $m \gg n \gg 0$ and let λ be a 1-PS of* $\operatorname{Stab} \beta$ *and $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ be a point in $Y_{(\tau)}^{\mathrm{ss}}$. If the limit $\bar{\rho} := \lim_{t\to 0} \lambda(t) \cdot \rho$ is also in $Y_{(\tau)}^{\mathrm{ss}}$, then using the above notation, we have*

- (i) *for $0 \leqslant l < j \leqslant r$ the quotient sheaf $\mathcal{F}^{[j]}/\mathcal{F}^{[l]}$ is τ-compatible with generalized Harder–Narasimhan filtration induced by that of $\mathcal{F}$;*
- (ii) *the Hilbert–Mumford function is given by*

$$\mu^{\mathcal{L}_\beta^{\mathrm{per}}}(\rho, \lambda) = \epsilon \sum_{j=1}^{r} \sum_{i=1}^{s} k_j \beta_i' P(\mathcal{F}_i^j, n).$$

Proof. Let $n \gg 0$ so that Lemma 8.3 holds. Let $0 \subset \bar{\mathcal{F}}^{(1)} \subset \cdots \subset \bar{\mathcal{F}}^{(s)} = \bar{\mathcal{F}}$ denote the Harder–Narasimhan filtration of $\bar{\mathcal{F}}$ where $V^{(i)} \cong H^0(\bar{\mathcal{F}}^{(i)}(n))$. By Lemma 8.1, the direct summands $\mathcal{F}^j$ have generalized Harder–Narasimhan filtrations

$$0 \subset \mathcal{F}^j_{(1)} \subset \cdots \subset \mathcal{F}^j_{(s)} = \mathcal{F}^j,$$

where

$$\mathcal{F}^j_{(i)} := \mathcal{F}^j \cap \bar{\mathcal{F}}^{(i)} = \bar{\rho}(V^{(i)} \otimes \mathcal{O}(-n)) \cap \bar{\rho}(V^j \otimes \mathcal{O}(-n)).$$

Since $\bar{\rho}$ is a direct sum of maps which send $V^j \otimes \mathcal{O}(-n)$ to $\mathcal{F}^j$ and $H^0(\bar{\rho}(n))$ is an isomorphism this is equal to

$$\mathcal{F}^j_{(i)} = \bar{\rho}((V^{(i)} \cap V^j) \otimes \mathcal{O}(-n)) = \frac{\rho((V^{(i)} \cap V^{[j]}) \otimes \mathcal{O}(-n))}{\rho((V^{(i)} \cap V^{[j-1]}) \otimes \mathcal{O}(-n))}.$$

Let $\mathcal{F}^{[j]}_{(i)} := \rho((V^{(i)} \cap V^{[j]}) \otimes \mathcal{O}(-n))$; then these sheaves define a filtration

$$0 \subset \mathcal{F}^{[j]}_{(1)} \subset \cdots \subset \mathcal{F}^{[j]}_{(s)} = \mathcal{F}^{[j]}$$

of $\mathcal{F}^{[j]}$. We claim that this filtration is a generalized Harder–Narasimhan filtration for $\mathcal{F}^{[j]}$ and thus that $\mathcal{F}^{[j]}$ is τ-compatible. It is enough to show that $\mathcal{F}^{[j]}_i := \mathcal{F}^{[j]}_{(i)}/\mathcal{F}^{[j]}_{(i-1)}$ is semistable with reduced Hilbert polynomial P_i/r_i if it is non-zero. We prove this by induction on j. For $j = 1$ it is clear as $\mathcal{F}^{[1]}$ is τ-compatible so suppose we know this is true for $j-1$. We have a diagram of short exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{F}^{[j-1]}_{(i-1)} & \longrightarrow & \mathcal{F}^{[j-1]}_{(i)} & \longrightarrow & \mathcal{F}^{[j-1]}_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}^{[j]}_{(i-1)} & \longrightarrow & \mathcal{F}^{[j]}_{(i)} & \longrightarrow & \mathcal{F}^{[j]}_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{F}^{j}_{(i-1)} & \longrightarrow & \mathcal{F}^{j}_{(i)} & \longrightarrow & \mathcal{F}^{j}_i & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

and so $\mathcal{F}^{[j-1]}_{(i-1)} = \mathcal{F}^{[j-1]}_{(i)} \cap \mathcal{F}^{[j]}_{(i-1)}$ from which it follows that the third column is also a short exact sequence. As the outer sheaves in this short exact sequence are both semistable with reduced Hilbert polynomial P_i/r_i, so is the middle sheaf. This completes the induction and shows that $\mathcal{F}^{[j]}/\mathcal{F}^{[l]}$ is also τ-compatible.

Recall that $\mathcal{L}^{\mathrm{per}}_\beta$ was constructed by twisting the original linearization $\mathcal{L}$ on $\bar{Y}_{(\tau)}$ by the character of $\operatorname{Stab}\beta$ corresponding to $-(\beta + \epsilon\beta')$; therefore,

$$\mu^{\mathcal{L}^{\mathrm{per}}_\beta}(\rho, \lambda) = \mu^{\mathcal{L}}(\rho, \lambda) + (\beta + \epsilon\beta') \cdot \lambda$$

where $\cdot$ denotes the natural pairing between characters and 1-PSs of $\operatorname{Stab}\beta$. We have calculated

$$\mu^{\mathcal{L}}(\rho, \lambda) = \sum_{j=1}^{r} k_j P(\mathcal{F}^j, m)$$

(see Lemma 5.2) and

$$(\beta + \epsilon\beta') \cdot \lambda = \sum_{i=1}^{s} \sum_{j=1}^{r} k_j (\beta_i + \epsilon\beta'_i) v_{i,j},$$

where $v_{i,j}$ is the dimension of $(V^j \cap V^{(i)}/V^j \cap V^{(i-1)})$. Observe that $v_{i,j} = P(\mathcal{F}_i^j, n)$ where $\mathcal{F}_i^j = \mathcal{F}_{(i)}^j / \mathcal{F}_{(i-1)}^j$ as $H^0(\bar{\rho}(n))$ is an isomorphism, so that $V^j \cap V^{(i)} \cong H^0(\mathcal{F}_{(i)}^j(n))$ and the $\mathcal{F}_i^j$ are all n-regular. Then since $\mathcal{F}^j$ is τ-compatible this means $\mathcal{F}_i^j$, if non-zero, has reduced Hilbert polynomial equal to P_i/r_i so

$$\sum_{i=1}^{s} \frac{P_i(m)}{P_i(n)} v_{i,j} = \sum_{i=1}^{s} P(\mathcal{F}_i^j, m) = P(\mathcal{F}^j, m).$$

Thus

$$\begin{aligned} \mu^{\mathcal{L}_\beta^{\mathrm{per}}}(\rho, \lambda) &= \sum_{j=1}^{r} k_j \left(P(\mathcal{F}^j, m) + \sum_{i=1}^{s} \left(\epsilon\beta'_i - \frac{P_i(m)}{P_i(n)} \right) v_{i,j} \right) \\ &= \epsilon \sum_{j=1}^{r} \sum_{i=1}^{s} k_j \beta'_i P(\mathcal{F}_i^j, n) \end{aligned}$$

and the proof is complete. □

We can use this lemma to study the indices $\gamma \in \mathfrak{t}_+$ of the stratification $\{S_\gamma^{(\beta)} : \gamma \in \mathcal{C}\}$ of $Y_{(\tau)}^{\mathrm{ss}}$. Recall that γ determines a 1-PS λ_γ of $\operatorname{Stab}\beta$, and as above this determines a decomposition $V = V^1 \oplus \cdots \oplus V^r$ of V into weight spaces and an associated filtration $0 \subset V^{[1]} \subset \cdots \subset V^{[r]} = V$ where $V^{[j]} = \bigoplus_{l \leqslant j} V^l$, together with a sequence of rational numbers $\gamma_1 > \cdots > \gamma_r$ such that $\sum \gamma_j \dim V^j = 0$.

Proposition 8.6. *Suppose that $m \gg n \gg 0$ and that γ is a non-zero index in the stratification $\{S_\gamma^{(\beta)} : \gamma \in \mathcal{C}\}$ of $Y_{(\tau)}^{\mathrm{ss}}$. If $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ belongs to the subscheme $Y_\gamma^{(\beta)-\mathrm{ss}}$ of $Y_{(\tau)}^{\mathrm{ss}}$, then $\bar{\rho} = p_\gamma^{(\beta)}(\rho) \in Z_\gamma^{(\beta)-\mathrm{ss}}$ is given by $\bar{\rho} = \bigoplus_{j=1}^r \rho^j : \bigoplus_{j=1}^r V^j \otimes \mathcal{O}(-n) \to \bigoplus_{j=1}^r \mathcal{F}^j$ where $\mathcal{F}^{[j]} = \rho(V^{[j]} \otimes \mathcal{O}(-n))$ and $\mathcal{F}^j = \mathcal{F}^{[j]}/\mathcal{F}^{[j-1]}$. In particular, the $\mathcal{F}^j$ are τ-compatible and so have generalized Harder–Narasimhan filtrations*

$$0 \subsetneq \mathcal{F}_{(1)}^j \subsetneq \cdots \subsetneq \mathcal{F}_{(s)}^j = \mathcal{F}^j.$$

Let $\mathcal{F}_i^j := \mathcal{F}_{(i)}^j / \mathcal{F}_{(i-1)}^j$; then

$$\gamma_j = -\frac{\epsilon \sum_{i=1}^{s} \beta'_i P(\mathcal{F}_i^j, n)}{P(\mathcal{F}^j, n)}.$$

Proof. We assume $m \gg n \gg 0$ so that the statements of Proposition 6.8 and Lemma 8.3 hold. We have seen that

$$\bar{\rho} = \bigoplus_{j=1}^{r} \rho^j : \bigoplus_{j=1}^{r} V^j \otimes \mathcal{O}(-n) \longrightarrow \bigoplus_{j=1}^{r} \mathcal{F}^j$$

is the graded object associated to the filtration $0 = \mathcal{F}^{[0]} \subset \mathcal{F}^{[1]} \subset \cdots \subset \mathcal{F}^{[r]} = \mathcal{F}$ of $\mathcal{F}$ given by $\mathcal{F}^{[j]} = \rho(V^{[j]} \otimes \mathcal{O}(-n))$, by [8, Lemma 4.4.3]. In particular $\bar{\rho} \in Y_{(\tau)}^{\mathrm{ss}}$, so by Lemma 8.5 the $\mathcal{F}^j$ are τ-compatible and

$$\mu^{\mathcal{L}_\beta^{\mathrm{per}}}(\rho, \lambda_\gamma) = \epsilon \sum_{j=1}^{r} \sum_{i=1}^{s} \gamma_j \beta'_i P(\mathcal{F}_i^j, n).$$

Since $\rho \in Y_\gamma^{(\beta)-\mathrm{ss}}$ the associated 1-PS λ_γ is adapted to ρ, and so

$$\frac{\mu^{\mathcal{L}_\beta^{\mathrm{per}}}(\rho, \lambda)}{\|\lambda\|}$$

takes its minimum value for λ a non-trivial 1-PS of $\mathrm{Stab}\,\beta$ when $\lambda = \lambda_\gamma$; this will enable us to determine the values of γ_j for $1 \leqslant j \leqslant r$. If we minimize the quantity

$$\frac{\mu^{\mathcal{L}_\beta^{\mathrm{per}}}(\rho, \lambda_\gamma)}{\|\lambda_\gamma\|}$$

subject to $\sum_{i=1}^r \gamma_j P(\mathcal{F}^j, n) = 0$, we see that

$$\gamma_j = -\frac{\epsilon \sum_{i=1}^s \beta_i' P(\mathcal{F}_i^j, n)}{P(\mathcal{F}^j, n))}.$$

The γ_j have been scaled so that $\mu^{\mathcal{L}_\beta^{\mathrm{per}}}(\rho, \lambda_\gamma) = -\|\gamma\|^2$, which ensures that $\bar{\rho}$ is a point in $Z_\gamma^{(\beta)}$. □

From this description, we can write down the strata inductively, starting with the highest stratum. In particular, we know the GIT semistable set, corresponding to the open stratum $S_0^{(\beta)}$, is the complement of the (closures of the) higher strata.

PROPOSITION 8.7. *Suppose $m \gg n \gg 0$ and $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ is a point in $Y_{(\tau)}^{\mathrm{ss}}$. Then ρ is semistable with respect to $\mathcal{L}_\beta^{\mathrm{per}}$ if and only if for all proper non-zero τ-compatible subsheaves $\mathcal{F}' \subset \mathcal{F}$ for which $\mathcal{F}/\mathcal{F}'$ is τ-compatible we have*

$$\sum_{i=1}^s \beta_i' P(\mathcal{F}_i', n) \geqslant 0,$$

where since $\mathcal{F}'$ is τ-compatible it has a generalized Harder–Narasimhan filtration

$$0 \subseteq \mathcal{F}'_{(1)} \subseteq \cdots \subseteq \mathcal{F}'_{(s)} = \mathcal{F}'$$

and $\mathcal{F}_i' := \mathcal{F}'_{(i)}/\mathcal{F}'_{(i-1)}$.

Proof. We suppose $m \gg n \gg 0$ are chosen as at the beginning of Proposition 8.6. Suppose ρ is semistable with respect to $\mathcal{L}_\beta^{\mathrm{per}}$ and let $\mathcal{F}' \subset \mathcal{F}$ be a τ-compatible subsheaf such that $\mathcal{F}/\mathcal{F}'$ is τ-compatible. Let $V' = H^0(\rho(n))^{-1}(H^0(\mathcal{F}'(n))) \subset V$ and let v'' be a complement to V' in V. Consider the 1-PS

$$\lambda(t) = \begin{pmatrix} t^{v-v'} I_{V'} & 0 \\ 0 & t^{-v'} I_{v''} \end{pmatrix},$$

where v' and v denote the dimension of V' and V, respectively. Then

$$\bar{\rho} := \left(\lim_{t\to 0} \lambda(t) \cdot \rho\right) : (V' \oplus v'') \otimes \mathcal{O}(-n) \longrightarrow \bar{\mathcal{F}},$$

where $\bar{\mathcal{F}} = \mathcal{F}' \oplus \mathcal{F}/\mathcal{F}'$ has Harder–Narasimhan type τ. Since ρ is semistable

$$\mu^{\mathcal{L}_\beta^{\mathrm{per}}}(\rho, \lambda) \geqslant 0,$$

but by Lemma 8.5

$$\mu^{\mathcal{L}_\beta^{\mathrm{per}}}(\rho, \lambda) = v\epsilon \sum_{i=1}^s \beta_i' P(\mathcal{F}_i', n),$$

where $v\epsilon > 0$, so $\sum_{i=1}^s \beta_i' P(\mathcal{F}_i', n) \geqslant 0$.

Now suppose ρ is unstable with respect to $\mathcal{L}_\beta^{\mathrm{per}}$. Then there is a non-zero $\gamma \in \mathcal{C}$ such that ρ belongs to $S_\gamma^{(\beta)}$, and in fact by conjugating γ by an element of $\operatorname{Stab}\beta$ we may assume $\rho \in Y_\gamma^{(\beta)-\mathrm{ss}}$. Then γ determines a filtration $0 \subset V^{[1]} \subset \cdots \subset V^{[r]} = V$ and sequence of rational numbers $\gamma_1 > \cdots > \gamma_r$, and by Proposition 8.6

$$\gamma_j = -\frac{\epsilon \sum_{i=1}^s \beta_i' P(\mathcal{F}_i^j, n)}{P(\mathcal{F}^j, n)}.$$

We claim for $j = 2, \dots, r$ that

$$\frac{\sum_{i=1}^s \beta_i' P(\mathcal{F}_i^{[1]}, n)}{P(\mathcal{F}^{[1]}, n)} < \frac{\sum_{i=1}^s \beta_i' P(\mathcal{F}_i^{[j]}, n)}{P(\mathcal{F}^{[j]}, n)}.$$

For $j = 2$ this is equivalent to the inequality $\gamma_1 > \gamma_2$. Then we proceed by induction as combining the above inequality with $\gamma_1 > \gamma_{j+1}$ gives the inequality for $j+1$. In particular, if $j = r$, then

$$\frac{\sum_{i=1}^s \beta_i' P(\mathcal{F}_i^{[1]}, n)}{P(\mathcal{F}^{[1]}, n)} < \frac{\sum_{i=1}^s \beta_i' P(\mathcal{F}_i, n)}{P(\mathcal{F}, n)} = 0$$

by construction of β'. Let $\mathcal{F}' = \mathcal{F}^{[1]}$; then by Lemma 8.5 both $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}'$ are τ-compatible and we have shown that

$$\sum_{i=1}^s \beta_i' P(\mathcal{F}_i', n) < 0.$$

□

8.5. *Moduli of θ-semistable n-rigidified sheaves of fixed Harder–Narasimhan type*

As before let W be a complex projective scheme and let $\tau = (P_1, \dots, P_s)$ be a fixed Harder–Narasimhan type. Let $P = \sum_{i=1}^s P_i$ and for $n \gg 0$ let V be a vector space of dimension $P(n)$. Recall that Q is the open subscheme of $\mathrm{Quot}(V \otimes \mathcal{O}(-n), P)$ representing quotient sheaves $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ which are pure of dimension e and such that $H^0(\rho(n))$ is an isomorphism. We defined in § 6.1, a subscheme $R_\tau = GY_{(\tau)}^{\mathrm{ss}}$ of Q consisting of the quotient sheaves $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ which have Harder–Narasimhan type τ. Let $\beta = \beta(\tau)$ be the corresponding index of the stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of $\bar{Q}$ as defined in § 6, and recall from Proposition 6.8 that for $m \gg n \gg 0$ the subscheme R_τ is a union of connected components of S_β. A choice of $\theta \in \mathbb{Q}^s$ defines a notion of (semi)stability for sheaves of Harder–Narasimhan type τ (see Definition 7) and an ample $\operatorname{Stab}\beta$-linearization $\mathcal{L}_\beta^{\mathrm{per}}$ on a projective completion $\bar{Y}_{(\tau)}$ of $Y_{(\tau)}^{\mathrm{ss}}$ in terms of

$$\beta' = i \operatorname{diag}(\beta_1', \cdots \beta_1', \dots, \beta_s', \dots, \beta_s') \in \mathfrak{t}$$

defined as at (8.2) where

$$\beta_i' = \theta_i - \frac{\sum_{j=1}^s \theta_j P_j(n)}{P(n)}$$

appears $P_i(n)$ times (see § 8.4).

Theorem 8.8. *Suppose n is sufficiently large and for fixed n that m is sufficiently large. Then $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in $Y_{(\tau)}^{\mathrm{ss}}$ is GIT semistable for the action of* $\operatorname{Stab}\beta$ *on $\overline{Y_{(\tau)}}$ with respect to $\mathcal{L}_\beta^{\mathrm{per}}$ if and only if $\mathcal{F}$ is θ-semistable.*

Proof. We pick n sufficiently large so that the statements of Lemmas 8.3 and 8.4 hold. Then pick m as in [**21**] so that GIT semistability of points in Q with respect to $\mathcal{L}$ is equivalent to semistability of the corresponding sheaf. We also assume that n and m are chosen large enough for Proposition 6.8 to hold.

Suppose $\mathcal{F}$ is θ-semistable and consider a τ-compatible subsheaf $\mathcal{F}'$ of $\mathcal{F}$ such that the quotient $\mathcal{F}/\mathcal{F}'$ is also τ-compatible. Then by θ-semistability, we have an inequality

$$\frac{\sum \theta_i P(\mathcal{F}'_i, n)}{P(\mathcal{F}', n)} \geqslant \frac{\sum \theta_i P(\mathcal{F}_i, n)}{P(\mathcal{F}, n)}$$

which by the definition of β' is equivalent to $\sum \beta'_i P(\mathcal{F}'_i, n) \geqslant 0$, and so by Proposition 8.7 we conclude that ρ is GIT semistable with respect to $\mathcal{L}_\beta^{\mathrm{per}}$.

Now suppose ρ is GIT semistable with respect to $\mathcal{L}_\beta^{\mathrm{per}}$ and take a τ-compatible subsheaf $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}/\mathcal{F}'$ is τ-compatible. Then $\sum \beta'_i P(\mathcal{F}'_i, n) \geqslant 0$ by Proposition 8.7, or equivalently

$$\frac{\sum \theta_i P(\mathcal{F}'_i, n)}{P(\mathcal{F}', n)} \geqslant \frac{\sum \theta_i P(\mathcal{F}_i, n)}{P(\mathcal{F}, n)}.$$

We have chosen n so that we can apply the results of Lemma 8.4 and conclude that $\mathcal{F}$ is θ-semistable. □

Remark 20. It is straightforward to modify the proof of this theorem to show that, under the same assumptions on n and m, a point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in $Y_{(\tau)}^{\mathrm{ss}}$ is GIT stable with respect to $\mathcal{L}_\beta^{\mathrm{per}}$ if and only if the sheaf $\mathcal{F}$ of Harder–Narasimhan type τ is θ-stable in the sense of Definition 7.

Remark 21. Our aim is to take a GIT quotient of $\bar{Y}_{(\tau)}$ by the action of $\operatorname{Stab}\beta$, so we need to examine semistability here. If a point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in $\bar{Y}_{(\tau)}$ is θ-semistable, then the sheaf $\mathcal{F}$ is τ-compatible, and since $\mathcal{F}$ also has Hilbert polynomial P it must have Harder–Narasimhan type τ, so that ρ actually belongs to $Y_{(\tau)}^{\mathrm{ss}}$. Let

$$Y_{(\tau)}^{\theta-\mathrm{ss}} := \bar{Y}_{(\tau)}^{\theta-\mathrm{ss}}$$

be the set of θ-semistable sheaves in $\bar{Y}_{(\tau)}$; then as we saw above this set is contained in $Y_{(\tau)}^{\mathrm{ss}}$. We are assuming that $\epsilon > 0$ is sufficiently small that the perturbation $\mathcal{L}_\beta^{\mathrm{per}}$ of $\mathcal{L}_\beta$ satisfies Proposition 3.3. Therefore, it follows from Theorem 8.8 that on $\bar{Y}_{(\tau)}$ GIT (semi)stability with respect to $\mathcal{L}_\beta^{\mathrm{per}}$ of a point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ is equivalent to θ-(semi)stability of the quotient sheaf $\mathcal{F}$ for n and m sufficiently large.

Definition 10. Let $\mathcal{F}$ be a θ-semistable n-rigidified sheaf of Harder–Narasimhan type τ. A Jordan–Hölder filtration of $\mathcal{F}$ with respect to θ is a filtration

$$0 \subset \mathcal{F}^{\{1\}} \subset \cdots \subset \mathcal{F}^{\{r\}} = \mathcal{F}$$

such that the following hold.

(1) The successive quotients $\mathcal{F}^j := \mathcal{F}^{\{j\}}/\mathcal{F}^{\{j-1\}}$ are τ-compatible and θ-stable with

$$\frac{\sum_{i=1}^s \theta_i P(\mathcal{F}_i^j)}{P(\mathcal{F}^j)} = \frac{\sum \theta_i P(\mathcal{F}_i)}{P(\mathcal{F})}.$$

(ii) The n-rigidification for $\mathcal{F}$ induces generalized n-rigidifications for each $\mathcal{F}^j$; that is, an isomorphism $H^0(\mathcal{F}^j(n)) \cong \bigoplus_{i=1}^s H^0(\mathcal{F}_i^j(n))$ with the usual compatibilities.

The associated graded sheaf $\bigoplus_{j=1}^r \mathcal{F}^j$ thus has an n-rigidification and is of Harder–Narasimhan type τ. Moreover, this sheaf is θ-polystable; that is, a direct sum of θ-stable sheaves. Standard arguments show that the n-rigidified sheaf $\bigoplus_{j=1}^r \mathcal{F}^j$ is uniquely determined up to isomorphism by $\mathcal{F}$. Finally, we say two θ-semistable n-rigidified sheaves $\mathcal{F}$ and $\mathcal{G}$ of Harder–Narasimhan type τ are S-equivalent if they have Jordan–Hölder filtrations such that the associated graded sheaves are isomorphic as n-rigidified sheaves.

REMARK 22. In exactly the same way as in the original proofs for S-equivalence of semistable sheaves, we see that the Stab β-orbit closures in $Y_{(\tau)}^{\theta-\text{ss}}$ of two n-rigidified sheaves $\mathcal{F}$ and $\mathcal{G}$ in $Y_{(\tau)}^{\theta-\text{ss}}$ intersect if and only if $\mathcal{F}$ and $\mathcal{G}$ are S-equivalent, and that the Stab β-orbit of a point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in $Y_{(\tau)}^{\theta-\text{ss}}$ is closed if and only if $\mathcal{F}$ is polystable. We briefly recap the argument here. From the general theory of GIT, we know that the closure in $Y_{(\tau)}^{\theta-\text{ss}}$ of any Stab β-orbit contains a unique closed Stab β-orbit. For any $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in $Y_{(\tau)}^{\theta-\text{ss}}$, we can choose a 1-PS whose limit as t tends to zero is the graded object $\bar{\rho} : V \otimes \mathcal{O}(-n) \to \bar{\mathcal{F}}$ associated to a Jordan–Hölder filtration of $\mathcal{F}$, so that $\bar{\mathcal{F}}$ is polystable and $\bar{\rho}$ is in the orbit closure of ρ. Now suppose that $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ is a point in $Y_{(\tau)}^{\theta-\text{ss}}$ such that $\mathcal{F}$ is a polystable sheaf $\mathcal{F} = \oplus \mathcal{F}_i$ and suppose that $\rho' : V \otimes \mathcal{O}(-n) \to \mathcal{F}'$ in $Y_{(\tau)}^{\theta-\text{ss}}$ lies in the orbit closure of ρ. Then there is a family $\mathcal{V}$ of θ-semistable sheaves parameterized by a curve C such that $\mathcal{V}_{c_0} = \mathcal{F}'$ for some $c_0 \in C$ and for $c \neq c_o$ the corresponding sheaf is $\mathcal{V}_c = \mathcal{F}$. By semicontinuity

$$\hom(\mathcal{F}_i, \mathcal{F}') \geqslant \hom(\mathcal{F}_i, \mathcal{F})$$

and by θ-stability of $\mathcal{F}_i$ and θ-semistability of $\mathcal{F}'$ we see that each non-zero morphism $\mathcal{F}_i \to \mathcal{F}'$ must be injective. From this, we can conclude that $\mathcal{F}' \cong \oplus \mathcal{F}_i = \mathcal{F}$ and that ρ' lies in the same Stab β-orbit as ρ, so the Stab β-orbit of ρ is closed. Thus the unique closed Stab β-orbit in the Stab β-orbit closure in $Y_{(\tau)}^{\theta-\text{ss}}$ of any point $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ in $Y_{(\tau)}^{\theta-\text{ss}}$ is the orbit of the graded object $\bar{\rho} : V \otimes \mathcal{O}(-n) \to \bar{\mathcal{F}}$ associated to a Jordan–Hölder filtration of $\mathcal{F}$.

Just as for moduli of semistable sheaves over a projective scheme W (cf. [**21**, Theorem 1.21]), we obtain a projective scheme which corepresents the moduli functor of θ-semistable n-rigidified sheaves of Harder–Narasimhan type τ over W, in the sense of [**21**, § 1] or [**1**, Definition 4.6].

THEOREM 8.9. *Let W be a projective scheme over $\mathbb{C}$ and $\tau = (P_1, \ldots, P_s)$ be a fixed Harder–Narasimhan type. For $\theta \in \mathbb{Q}^s$ and $n \gg 0$ there is a projective scheme $M^{\theta-\text{ss}}(W, \tau, n)$ which corepresents the moduli functor $\mathcal{M}^{\theta-\text{ss}}(W, \tau, n)$ of θ-semistable n-rigidified sheaves of Harder–Narasimhan type τ over W. The points of $M^{\theta-\text{ss}}(W, \tau, n)$ correspond to S-equivalence classes of θ-semistable n-rigidified sheaves with Harder–Narasimhan type τ.*

Proof. The proof is based on that of [**21**, Theorem 1.21] (see also [**1**, § 4]). Pick n and m as in the beginning of Theorem 8.8. For a complex scheme R let $\underline{R} = \text{Hom}(-, R)$ denote its functor of points, and if R has a G-action, then let $\underline{R}/\underline{G}$ denote the quotient functor.

Let $\bar{Y}_{(\tau)}$ be the closure of $Y_{(\tau)}^{\text{ss}}$ as at the beginning of § 8 and $\mathcal{L}_\beta^{\text{per}}$ the linearization defined in § 8.4; then, let

$$M^{\theta-\text{ss}}(W, \tau, n) := \bar{Y}_{(\tau)} /\!/_{\mathcal{L}_\beta^{\text{per}}} \text{Stab}\, \beta.$$

By Theorem 8.8 and Remark 21 the projective scheme $M^{\theta-\text{ss}}(W, \tau, n)$ is a categorical quotient of the open subset $Y_{(\tau)}^{\theta-\text{ss}} \subseteq \bar{Y}_{(\tau)}$ parameterizing points $\rho : V \otimes \mathcal{O}(-n) \to \mathcal{F}$ of $\bar{Y}_{(\tau)}$ such that $\mathcal{F}$ is θ-semistable for the action of Stab β, or equivalently by the action of $H := \Pi_{i=1}^s \text{GL}(V_i)$ since the central 1-PS $\mathbb{C}^* \subset \text{GL}(V)$ acts trivially on $\bar{Y}_{(\tau)}$. The quotient map $Y_{(\tau)}^{\theta-\text{ss}} \to M^{\theta-\text{ss}}(W, \tau, n)$ is H-invariant and so induces a natural transformation

$$\varphi_1 : \underline{Y_{(\tau)}}^{\theta-\text{ss}} / \underline{H} \longrightarrow \underline{M^{\theta-\text{ss}}(W, \tau, n)},$$

and as $M^{\theta-\mathrm{ss}}(W,\tau,n)$ is a categorical quotient it corepresents the quotient functor $\underline{Y_{(\tau)}}^{\theta-\mathrm{ss}}/\underline{H}$.

Let $\mathcal{V}$ denote the restriction to $Y_{(\tau)}^{\theta-\mathrm{ss}}$ of the family of θ-semistable n-rigidified sheaves of Harder–Narasimhan type τ parameterized by $Y_{(\tau)}^{\mathrm{ss}}$ (cf. Lemma 8.2). Then this family defines a natural transformation

$$\phi : \underline{Y_{(\tau)}}^{\theta-\mathrm{ss}} \longrightarrow \mathcal{M}^{\theta-\mathrm{ss}}(X,\tau,n)$$

by sending a morphism $f : S \to Y_{(\tau)}^{\theta-\mathrm{ss}}$ to the family $f^*\mathcal{V}$ for any scheme S. Following Lemma 7.1 two elements of $\underline{Y_{(\tau)}}^{\theta-\mathrm{ss}}(S)$ define isomorphic families if and only if locally on S they are related by an element of $\underline{H}(S)$, this descends to a local isomorphism (in the sense of [**21**, § 1] or [**1**, Definition 4.3])

$$\tilde{\phi} : \underline{Y_{(\tau)}}^{\theta-\mathrm{ss}}/\underline{H} \longrightarrow \mathcal{M}^{\theta-\mathrm{ss}}(X,\tau,n).$$

Since local isomorphism means isomorphism after sheafification and $M^{\theta-\mathrm{ss}}(W,\tau,n)$ corepresents $\underline{Y_{(\tau)}}^{\theta-\mathrm{ss}}/\underline{H}$, it also corepresents $\mathcal{M}^{\theta-\mathrm{ss}}(X,\tau,n)$ (cf. [**1**, Lemma 4.7]). Finally the fact that the points of $M^{\theta-\mathrm{ss}}(W,\tau,n)$ correspond to S-equivalence classes follows from Remark 22. □

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