

8. EINSTEIN'S EQUATION

8.1. Newtonian gravity. Newton formulated his description of gravity in terms of a force between a pair of objects separated by a finite distance, causing them to accelerate toward each other. However, the principle of equivalence shows that the concept of absolute acceleration can be misleading.

In order to make the transition to a relativistic theory of gravity, we need to reformulate Newton's theory in two ways. It should be expressed in terms of local fields satisfying differential equations, and we should only be concerned with relative motion, not absolute motion.

The Newtonian theory models space as Euclidean \mathbb{R}^3 . Newtonian gravity can be expressed in terms of the gravitational field, \mathbf{g} . The motion of a free-falling particle is described by

$$\frac{d^2\mathbf{r}}{dt^2} = \mathbf{g}$$

where \mathbf{g} is evaluated at the point $\mathbf{r}(t)$. Now, imagine another free-falling particle with trajectory $\mathbf{r}(t) + \varepsilon\mathbf{u}(t)$, where ε is infinitesimal.

The separation between these particles satisfies the differential equation,

$$\frac{d^2\mathbf{u}^i}{dt^2} = g_{ij}^i\mathbf{u}^j,$$

which suggests defining the *tidal tensor*, $\Psi_j^i := -g_{ij}^i$. This describes how the varying gravitational force stretches and squeezes objects, such as the Earth.

The gravitational field of a point particle of mass M and located at the origin $\mathbf{r} = 0$ is

$$\mathbf{g} = -GM\frac{\mathbf{r}}{r^3}, \quad (8.1)$$

where G is Newton's gravitational constant. The field of an arbitrary distribution of matter is (in principle) obtained by adding up terms of this form. This happens to be the gradient of GMr^{-1} , therefore the gravitational field of an arbitrary distribution of matter is minus the gradient of a scalar potential, and Ψ is the second derivative. This has two consequences; Ψ is symmetric

$$\Psi_{ij} = \Psi_{ji} \quad (8.2a)$$

and (since space is flat) its derivative is symmetric

$$\Psi_{j|k}^i = \Psi_{k|j}^i. \quad (8.2b)$$

These conditions imply that Ψ is the second derivative of a scalar.

The gravitational field (8.1) has vanishing divergence $\nabla \cdot \mathbf{g} = 0$ except at the origin. From this, Gauss' theorem, and a simple calculation, we see that for any compact region $V \subset \mathbb{R}^3$, the flux integral $\oint_{\partial V} \mathbf{g} \cdot d\mathbf{S}$ equals $-4\pi GM$ if the point mass is located in V and 0 otherwise. This shows that for a general gravitational field $\oint_{\partial V} \mathbf{g} \cdot d\mathbf{S}$ is $-4\pi G$ times the mass contained in V . If the mass is smoothly distributed with density μ , then

$$\oint_{\partial V} \mathbf{g} \cdot d\mathbf{S} = -4\pi G \int_V \mu dV$$

and by Gauss' theorem, $\nabla \cdot \mathbf{g} = -4\pi G \mu$. In terms of the tidal tensor, this is

$$\Psi^i{}_i = 4\pi G \mu. \quad (8.2c)$$

The tidal tensor in Newtonian gravity is completely determined by these three equations and the boundary condition that Ψ goes to 0 at infinity. This reformulation makes exactly the same predictions as Newtonian gravity, but without referring to absolute acceleration.

8.2. Einstein's equation. General relativity postulates that the world line of a free-falling object is actually just a geodesic in spacetime. To compare this with Newtonian gravity, we need to consider an approximation in which everything is moving slowly (compared to the speed of light) and pressures are much smaller than densities. Of course, motion is relative, so we need to clarify, "Moving slowly relative to what?"

Let $V \in \mathcal{X}(M)$ be a timelike unit vector field ($V^\mu V_\mu = 1$) on a spacetime M . For the purpose of the Newtonian approximation, we define "slow" relative to V .

Consider a free-falling test particle with velocity $v \approx V$ and parametrized by proper time, τ . The infinitesimal separation εu with another free-falling test particle is governed by the geodesic deviation equation:

$$\nabla_{\frac{d}{d\tau}}^2 u^\mu = -R^\mu{}_{\nu\eta\lambda} v^\nu u^\eta v^\lambda \approx -R^\mu{}_{\nu\eta\lambda} V^\nu u^\eta V^\lambda.$$

This suggests that the analogue of the tidal tensor is $\Psi_\nu^\mu := R^\mu{}_{\eta\nu\lambda} V^\eta V^\lambda$. Because of the symmetries of the Riemann tensor, this is automatically symmetric $\Psi_{\mu\nu} = \Psi_{\nu\mu}$ and spatial in the sense that $V^\nu \Psi_\nu^\mu = 0$. This immediately suggests a field equation,

$$4\pi G \mu = \Psi^\lambda{}_\lambda = R^\mu{}_{\eta\mu\lambda} V^\eta V^\lambda = R_{\mu\nu} V^\mu V^\nu.$$

But what is μ ? This should be the mass density according to an observer with velocity V .

The obvious answer is $\mu = T_{\mu\nu} V^\mu V^\nu$. This implies $R_{\mu\nu} V^\mu V^\nu = 4\pi G T_{\mu\nu} V^\mu V^\nu$, and since this should be true independently of V , $R_{\mu\nu} = 4\pi G T_{\mu\nu}$. However, this cannot be. The Ricci tensor satisfies the contracted Bianchi identity and the energy tensor satisfies the conservation equation. Together, these would imply that $R = 4\pi G T^\lambda{}_\lambda$ is constant throughout the universe. It is not⁸.

We must cast a wider net. Assuming that μ can be constructed from $T^{\mu\nu}$ and V , it should be a linear function of $T^{\mu\nu}$. Any nonlinear term would involve a dimensionful coefficient, but gravitational observations within the solar system do not reveal any fundamental length or energy scale. In order for the field equation to be independent of V , μ must also be expressible as a quadratic function of V . There are only two possible expressions satisfying these conditions, therefore μ must be of the form

$$\mu = a T_{\mu\nu} V^\mu V^\nu + b T^\lambda{}_\lambda = (a T_{\mu\nu} + b g_{\mu\nu} T^\lambda{}_\lambda) V^\mu V^\nu,$$

for some numbers a and b . If the field equation is true for any V , then we must have

$$R_{\mu\nu} = 4\pi G (a T_{\mu\nu} + b g_{\mu\nu} T^\lambda{}_\lambda).$$

The trace of this gives $R = 4\pi G (a + 4b) T^\lambda{}_\lambda$. Two conditions determine a and b .

⁸Among other things, this would imply that the speed of sound is always $c/\sqrt{3}$.

The contracted Bianchi identity and conservation equation imply that

$$\begin{aligned} R_{|\mu} &= 4\pi G(\alpha + 4b)T_{\lambda|\mu}^\lambda \\ &= 2R^\nu{}_{\mu|\nu} = 8\pi GbT_{\lambda|\mu}^\lambda. \end{aligned}$$

Therefore $\alpha + 2b = 0$.

In order to fit with Newtonian gravity, we also need that $\mu \approx T_{\mu\nu}V^\mu V^\nu$. Assuming that (as for ordinary matter) the pressure is very small compared to the density, we have $T^\lambda{}_\lambda \approx T_{\mu\nu}V^\mu V^\nu$. Therefore $\alpha + b = 1$.

These conditions give that $\alpha = 2$ and $b = -1$. The resulting field equation can be written in terms of the Einstein tensor $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ as

$$G^{\mu\nu} = 8\pi GT^{\mu\nu}. \quad (8.3)$$

This is Einstein's equation, the field equation for the geometry of spacetime in general relativity.

8.3. From Einstein back to Newton. Does Einstein's equation (8.3) really imply equations (8.2) as an approximation?

First, note that Minkowski space is flat, therefore the Riemann and Ricci tensors vanish and this is a solution of Einstein's equation for a vacuum. In fact, there is a very strong result (Witten's positive energy theorem) which shows that Minkowski space is the ground state of general relativity. This suggests that for a small amount of matter at low density, spacetime may approximately resemble Minkowski space. This justifies the approximation of treating space as flat.

We have already seen that the symmetries of the Riemann tensor imply that the tidal tensor is symmetric $\Psi_{\mu\nu} = \Psi_{\nu\mu}$. In this sense, general relativity implies eq. (8.2a).

The Bianchi identity gives,

$$0 = (R^\mu{}_{\lambda\nu\eta|\epsilon} - R^\mu{}_{\lambda\epsilon\eta|\nu})V^\lambda V^\eta + V^\lambda \nabla_\nu R^\mu{}_{\lambda\epsilon\nu}$$

The last term can be ignored if we assume that the ∇_V (time) derivative of the Riemann tensor is small. If we also assume that the spatial (orthogonal to V) derivative of V is small, then this implies that the *spatial part* of

$$\Psi^\mu{}_{\nu|\lambda}$$

is (approximately) symmetric. In this sense, the Bianchi identity and the Newtonian approximation imply eq. (8.2b), which then implies the existence of a gravitational potential.

In the Newtonian approximation to general relativity, the gravitational mass density is $\mu = 2T^{\mu\nu}V_\mu V_\nu - T^\lambda{}_\lambda$. For a perfect fluid with velocity v , this is

$$\mu = (2[v^\mu V_\mu]^2 - 1)\rho + (2[v^\mu V_\mu]^2 + 1)p.$$

So, pressure actually increases the gravitational mass. If the fluid is moving slowly (relative to V) then $v \approx V$ and $v^\mu V_\mu \approx 1$, so

$$\mu \approx \rho + 3p.$$

Finally, if the pressure is much smaller than the density, then $\mu \approx \rho$, which conforms to Newtonian gravity.

8.4. The cosmological constant. Although gravitational physics within the solar system does not exhibit any fundamental length scale, such a thing has been observed in the universe at large. This is the cosmological constant Λ . Einstein's equation can be modified very slightly to

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = 8\pi G T^{\mu\nu}.$$

This does not effect the Newtonian approximation within the solar system, provided that Λ is sufficiently small.

This term was introduced by Einstein to explain why the universe is not expanding. He came to regret this decision after it was discovered that the universe *is* expanding.

Until a decade ago, measurements of the expansion of the universe indicated that $\Lambda = 0$. However, more recent observations suggest that $\Lambda \approx 10^{-35} \text{s}^{-2}$. The effects of Λ are significant for gravitational interaction over distances on the order of 10 billion light years. These effects were overlooked in the laboratory experiments of Newton's time.

It may be more appropriate to view the cosmological constant as an additional term in the energy tensor. It is equivalent to a perfect fluid with density $\rho = \frac{\Lambda}{8\pi G}$ and negative pressure $p = -\rho$.

The energy conservation equation and Bianchi identity imply that Λ is indeed constant. However, scalar fields such as the Higgs' field in the standard model (of particle physics) can give an energy tensor which effectively looks like a cosmological constant. This effective value of Λ can change through the phase transition of spontaneous symmetry breaking.

9. KOMAR INTEGRALS

9.1. General formula. The first derivative $\xi_{\mu|\nu}$ of a Killing vector is analogous to the electromagnetic field $F_{\mu\nu}$ in several ways. It is antisymmetric, so it can be viewed as a 2-form. As a 2-form it is exact, $\xi_{\mu|\nu} = -\frac{1}{2}(d\xi)_{\mu\nu}$, so it automatically satisfies one of Maxwell's equations, and $-\frac{1}{2}\xi_{,\mu}$ is analogous to the electromagnetic potential. The analogue of Maxwell's other equation, $F_{|\nu}^{\mu\nu} = J^\mu$ is the contracted form (6.12) of Killing's second equation,

$$\xi^{\mu|\nu}{}_{,\nu} = -R^\mu{}_\nu \xi^\nu.$$

The right hand side depends upon the energy tensor because of Einstein's equation. In terms of differential forms, this is $d*d\xi = 2*(R_{\mu\nu}\xi^\mu dx^\nu)$. Following the analogy with Maxwell's equations, This can be written as an integral equation. For any hypersurface Σ

$$\oint_{\partial\Sigma} *d\xi = 2 \int_{\Sigma} *(R_{\mu\nu}\xi^\mu dx^\nu). \quad (9.1)$$

According to Einstein's equation (without cosmological constant), the Ricci tensor vanishes in regions of spacetime containing no matter⁹, so this integral depends upon the matter contained in Σ , but not on the precise shape of the boundary $\partial\Sigma$.

⁹Here, "matter" is anything which contributes to the energy tensor. This includes the electromagnetic field.

Generally, for a closed 2-surface S ,

$$\oint_S *d\xi$$

is called a Komar integral. Its precise physical significance depends on what kind of Killing vector ξ is.

9.2. The energy integral.

9.2.1. *Asymptotic flatness.* Suppose that M is a stationary spacetime, i.e., there exists a timelike, future-pointing Killing vector ξ . An object is stationary if its world line is tangent to ξ .

Energy is rather subjective in relativity. In special relativity, it must be defined according to some reference frame. In general relativity, conserved energy of a particle is defined relative to some timelike Killing vector, but this depends upon how the Killing vector is normalized.

Suppose that a stationary object deep in a gravitational field emits a photon with energy E . When that photon reaches a distant stationary observer, its energy will be less than E because of gravitational redshift.

If we want to discuss the total energy in a spacetime M , then we must choose some observer to define energy. The only natural choice is a stationary observer “at infinity”. This idea only makes sense if M is *asymptotically flat*, i.e., the gravitational field diminishes with distance, and the geometry of M far from the gravitational source is approximately that of Minkowski space.

With this assumption, ξ “near infinity” should look approximately like a Killing vector in Minkowski space. In Minkowski space, a Killing vector is only timelike near infinity if it is timelike everywhere, i.e., it is a time-translation Killing vector. This means that on M , $\xi^\mu \xi_\mu$ should converge to a constant as the distance from the gravitational source goes to infinity. We can normalize this limit to be 1, so that ξ converges to the velocity of a stationary observer at infinity.

In the end, it will not always be appropriate to assume that ξ is timelike everywhere. It will be sufficient to assume that it is timelike near infinity.

9.2.2. *Interior interpretation.* In the Newtonian approximation, we can choose a hypersurface Σ such that the (future-pointing) unit normal vector n^μ is approximately ξ^μ .

The energy density is (by definition) $T^{\mu\nu}n_\mu n_\nu$. In the Newtonian approximation, the energy density is (on average) much greater than the pressure, so $T^\lambda_\lambda \approx T^{\mu\nu}n_\mu n_\nu$.

Let ϵ_Σ be the volume form of Σ . The integral on the right hand side of (9.1) is

$$\begin{aligned} \int_\Sigma * (R_{\mu\nu} \xi^\mu dx^\nu) &= \int_\Sigma R_{\mu\nu} \xi^\mu n^\nu \epsilon_\Sigma \\ &\approx \int_\Sigma R_{\mu\nu} n^\mu n^\nu \epsilon_\Sigma = 8\pi G \int_\Sigma (T^{\mu\nu} n_\mu n_\nu - \frac{1}{2} T^\lambda_\lambda) \epsilon_\Sigma \\ &\approx 4\pi G \int_\Sigma T^{\mu\nu} n_\mu n_\nu \epsilon_\Sigma. \end{aligned}$$

In the Newtonian approximation, this is the mass. Since the Komar integral is well defined in general relativity (if ξ exists), this suggests that

$$M = \frac{1}{8\pi G} \oint_S *d\xi \quad (9.2)$$

is the gravitational mass enclosed by the 2-surface S , even if S is not a boundary. This is valid provided that, in the limit of approaching spatial infinity, $\xi^\mu \xi_{,\mu} \rightarrow 1$ and ξ is future-pointing.

9.2.3. Boundary interpretation. According to general relativity, free-falling objects do not accelerate toward a gravitational source. Instead, it is stationary objects which are accelerating *away* from the gravitational source to remain stationary.

If $\xi^\mu \xi_{,\mu} \approx 1$, then ξ is approximately the velocity of stationary observers. The acceleration of a stationary observer is approximately $n^\nu \xi^\mu_{;\nu}$. This is minus the Newtonian “acceleration due to gravity”, \mathbf{g} . If $\xi^{\mu\nu}$ is analogous to an electromagnetic field, then $-\mathbf{g}$ is analogous to the electric field vector.

In the Newtonian approximation,

$$\frac{1}{8\pi G} \oint_S *d\xi \approx -\frac{1}{4\pi G} \oint_{\partial\Sigma} \mathbf{g} \cdot d\mathbf{S}.$$

The right hand side is the Newtonian mass integral.

9.3. Angular momentum. Now consider the opposite extreme where ζ is a rotation. That is, the flow of ζ returns to the identity after 2π .

Suppose that Σ is a hypersurface *tangent* to ζ . This means that $\zeta^\mu n_\mu = 0$. The Komar integral is *exactly*

$$\begin{aligned} \oint_{\partial\Sigma} *d\zeta &= 2 \int_\Sigma * (R_{\mu\nu} \zeta^\mu dx^\nu) = 2 \int_\Sigma R_{\mu\nu} \zeta^\mu n^\nu \epsilon_\Sigma \\ &= 16\pi G \int_\Sigma T^{\mu\nu} \zeta_\mu n_\nu \epsilon_\Sigma = 16\pi G \int_\Sigma * (T_{\mu\nu} \zeta^\mu dx^\nu). \end{aligned}$$

The last integral is simply the flux through Σ of the current $T^{\mu\nu} \zeta_\mu$ of (angular) momentum in the direction of ζ .

This suggests that we can generally define the (ζ -direction) angular momentum contained within a 2-surface S to be given by the Komar integral,

$$J = \frac{1}{16\pi G} \oint_S *d\zeta. \quad (9.3)$$

Note that the mass and angular momentum formulas differ by a factor of 2.

The main lesson of this is that in general relativity — unlike Newtonian gravity — angular momentum has an effect on the gravitational field.

There is no Komar integral for translational momentum. If the matter distribution is translationally symmetric, then it cannot be compact and we cannot enclose it with a 2-surface.

From now on, we will set $G = 1$. This means that we identify units of energy with units of time in the same way that setting $c = 1$ identifies time with distance. It would actually make more sense to set $G = \frac{1}{4\pi}$, but $G = 1$ is the standard convention.