Workshop on Noncommutative Manifolds
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**Dirac operator on** $SU_q(2)$

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"The Dirac operator on $SU_q(2)$"
Introduction

• Noncommutative Geometry vs. Quantum Groups

• Construct $q$-version of spin geometry on $SU(2)$:
  - Homogeneous space:
    \[
    SU(2) = \frac{\text{Spin}(4)}{\text{Spin}(3)} = \frac{SU(2) \times SU(2)}{SU(2)}
    \]
    with $\text{Spin}(3)$ the diagonal $SU(2)$ subgroup of $\text{Spin}(4)$.
    Quotient map: $(p, q) \mapsto pq^{-1}$
  - Action of $\text{Spin}(4) = SU(2) \times SU(2)$ on $SU(2)$:
    \[
    (p, q) \cdot x = pxq^{-1}
    \]
Let $q$ be a positive real number, $q \neq 1$.

**Definition.** Define the algebra $A := A(SU_q(2))$ of polynomials on $SU_q(2)$ to be the $\ast$-algebra generated by $a$ and $b$, subject to the following relations:

\[
ba = qab, \quad b^*a = qab^*, \quad bb^* = b^*b \\
a^*a + q^2b^*b = 1, \quad aa^* + bb^* = 1.
\]

As a consequence, $a^*b = qba^*$ and $a^*b^* = qb^*a^*$.

*Correspondence with [Kl-Schm],[Chakr-Pal],[Con]:* $a \leftrightarrow a^*$, $b \leftrightarrow -b$. 
This becomes a Hopf $\ast$-algebra with

- the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by

\[
\Delta a := a \otimes a - q b \otimes b^*, \\
\Delta b := b \otimes a^* + a \otimes b,
\]

- the counit $\varepsilon : \mathcal{A} \to \mathbb{C}$ defined by $\varepsilon(a) = 1$ and $\varepsilon(b) = 0$,

- the antipode $S : \mathcal{A} \to \mathcal{A}$ defined as an antilinear map by

\[
Sa = a^*, \quad Sb = -q b, \\
Sb^* = -q^{-1} b^*, \quad Sa^* = a.
\]
Definition. The $\ast$-algebra $\mathcal{U} := \mathcal{U}_q(\mathfrak{su}(2))$ is generated by elements $e, f, k$, with $k$ invertible, satisfying the relations

\[
e k = q k e, \quad k f = q f k, \quad k^2 - k^{-2} = (q - q^{-1})(f e - e f)\]

Correspondence with [Kl-Schm]: $q \leftrightarrow q^{-1}$, or, equivalently: $e \leftrightarrow f$.

Hopf $\ast$-algebra structure given by: coproduct $\Delta$:

\[
\Delta k = k \otimes k, \quad \Delta e = e \otimes k + k^{-1} \otimes e, \quad \Delta f = f \otimes k + k^{-1} \otimes f,
\]

counit $\epsilon(k) = 1, \epsilon(f) = \epsilon(e) = 0$, antipode $S$,

\[
S k = k^{-1}, \quad S f = -q f, \quad S e = -q^{-1} e,
\]

and star structure: $k^\ast = k, f^\ast = e$. 
Representation theory of $\mathcal{U}_q(su(2))$

The irreducible finite dimensional representations $\sigma_l$ of $\mathcal{U}_q(su(2))$ are labelled by nonnegative half-integers $l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$, and given by

$$
\sigma_l(k) |lm\rangle = q^m |lm\rangle,
\sigma_l(f) |lm\rangle = \sqrt{[l-m][l+m+1]} |l,m+1\rangle,
\sigma_l(e) |lm\rangle = \sqrt{[l-m+1][l+m]} |l,m-1\rangle,
$$
on the irreducible $\mathcal{U}$-modules $V_l = \text{Span}\{|lm\rangle\}_{m=-l,\ldots,l}$.

The brackets denote $q$-integers: 
$$
[n] = \frac{q^n-q^{-n}}{q-q^{-1}}
$$
provided $q \neq 1$. 

**Action of** $\mathcal{U}_q(su(2))$ **on** $\mathcal{A}(SU_q(2))$

Dual pairing of $\langle ., . \rangle : \mathcal{U} \times \mathcal{A} \to \mathbb{C}$ induces left and right action of $h \in \mathcal{U}_q(su(2))$ on $x \in \mathcal{A}(SU_q(2))$:

\[
h \triangleright x := x_1 \langle h, x_2 \rangle \quad \quad x \triangleleft h := \langle h, x_1 \rangle x_2,
\]

where we use Sweedler’s notation for the coproduct in $\mathcal{A}(SU_q(2))$:

\[
\Delta x = x_1 \otimes x_2, \quad (x \in \mathcal{A})
\]

Using the antipode, the right action can be transformed into a left action, which we will denote by $h \cdot x$. 

7
We establish the left regular representation of $\mathcal{A}$ as an equivariant representation with respect to two copies of $\mathcal{U}$ acting via $\cdot$ and $\triangleright$ on the left.

**Definition.** Let $\lambda$ and $\rho$ be mutually commuting representations of the Hopf algebra $\mathcal{U}$ on a vector space $V$. A representation $\pi$ of the algebra $\mathcal{A}$ on $V$ is $(\lambda, \rho)$-equivariant if the following compatibility relations hold:

$$\lambda(h) \pi(x) \xi = \pi(h_{(1)} \cdot x) \lambda(h_{(2)}) \xi,$$

$$\rho(h) \pi(x) \xi = \pi(h_{(1)} \triangleright x) \rho(h_{(2)}) \xi,$$

for all $h \in \mathcal{U}$, $x \in \mathcal{A}$ and $\xi \in V$. 

**Left regular representation of $\mathcal{A}(SU_q(2))$**
Equivariant representation of $A(SU_q(2))$

Representation space:

$$V := \bigoplus_{2l=0}^{\infty} V_l \otimes V_l$$

Two copies of $\mathcal{U}_q(su(2))$ act via the irreducible representations $\sigma$ on the first and the second leg of the tensor product, respectively:

$$\lambda(h) = \sigma_l(h) \otimes \text{id}, \quad \rho(h) = \text{id} \otimes \sigma_l(h) \quad \text{on } V_l \otimes V_l.$$ 

We abbreviate $|lmn\rangle := |lm\rangle \otimes |ln\rangle$, for $m, n = -l, \ldots, l$. 

Proposition. A \((\lambda, \rho)\)-equivariant *-representation \(\pi\) of \(A(SU_q(2))\) on \(V\) is necessarily given by the left regular representation. Explicitly:

\[
\pi(a) |l\,m\,n\rangle = A_{l\,m\,n}^+ |l + \frac{1}{2}, m + \frac{1}{2}, n + \frac{1}{2}\rangle + A_{l\,m\,n}^- |l - \frac{1}{2}, m + \frac{1}{2}, n + \frac{1}{2}\rangle,
\]

\[
\pi(b) |l\,m\,n\rangle = B_{l\,m\,n}^+ |l + \frac{1}{2}, m + \frac{1}{2}, n - \frac{1}{2}\rangle + B_{l\,m\,n}^- |l - \frac{1}{2}, m + \frac{1}{2}, n - \frac{1}{2}\rangle,
\]

where for example the constants \(A_{l\,m\,n}^\pm\) are given by

\[
A_{l\,m\,n}^+ = q^{(-2l+m+n-1)/2} \left( \frac{[l + m + 1][l + n + 1]}{[2l + 1][2l + 2]} \right)^{\frac{1}{2}},
\]

\[
A_{l\,m\,n}^- = q^{(2l+m+n+1)/2} \left( \frac{[l - m][l - n]}{[2l][2l + 1]} \right)^{\frac{1}{2}}.
\]
Spinor representation

We amplify representation $\pi$ of $A$ to the spinor representation defined by $\pi' = \pi \otimes \text{id}$ on $V \otimes \mathbb{C}^2$, and set $\rho' = \rho \otimes \text{id}$, but $\lambda'$ as the tensor product of the representations $\lambda$ on $V$ and $\sigma_{\frac{1}{2}}$ on $V_{\frac{1}{2}} = \mathbb{C}^2$:

$$\lambda'(h) := (\lambda \otimes \sigma_{\frac{1}{2}})(\Delta h) = \lambda(h_{(1)}) \otimes \sigma_{\frac{1}{2}}(h_{(2)}).$$

**Proposition.** The representation $\pi'$ of $A$ is ($\lambda'$, $\rho'$)-equivariant.

**Clebsch-Gordan decomposition:**

$$V \otimes \mathbb{C}^2 = \left( \bigoplus_{2l=0}^{\infty} V_l \otimes V_l \right) \otimes V_{\frac{1}{2}} \simeq V_{\frac{1}{2}} \oplus \bigoplus_{2j=1}^{\infty} (V_{j+\frac{1}{2}} \otimes V_j) \oplus (V_{j-\frac{1}{2}} \otimes V_j).$$
Basis vectors \((l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots)\):

\[
|j \mu n \uparrow\rangle := C_{j+1, \mu} |j + \frac{1}{2}, \mu - \frac{1}{2}, n\rangle \otimes |\frac{1}{2}, +\frac{1}{2}\rangle - S_{j+1, \mu} |j + \frac{1}{2}, \mu + \frac{1}{2}, n\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle,
\]

where \(\mu = -j, \ldots, j\) and \(n = -(j + \frac{1}{2}), \ldots, j + \frac{1}{2}\)

\[
|j \mu n \downarrow\rangle := S_{j \mu} |j - \frac{1}{2}, \mu - \frac{1}{2}, n\rangle \otimes |\frac{1}{2}, +\frac{1}{2}\rangle + C_{j \mu} |j - \frac{1}{2} \mu + \frac{1}{2} n\rangle \otimes |\frac{1}{2}, -\frac{1}{2}\rangle,
\]

where \(\mu = -j, \ldots, j\) and \(n = -(j - \frac{1}{2}), \ldots, j - \frac{1}{2}\), and the \(q\)-Clebsch-Gordan coefficients come from the well-known representation theory of \(U_q(su(2))\):

\[
C_{j \mu} := q^{-(j+\mu)/2} \frac{[j - \mu] \frac{1}{2}}{[2j] \frac{1}{2}}, \quad S_{j \mu} := q^{(j-\mu)/2} \frac{[j + \mu] \frac{1}{2}}{[2j] \frac{1}{2}}.
\]

\(\Rightarrow\) expressions for \(\pi'\) in basis \(\{|j \mu n \uparrow\rangle, |j \mu n \downarrow\rangle\}\) contain off-diagonal terms.
Invariant Dirac operator

**Proposition.** Any self-adjoint operator on $\mathcal{H} = (V \otimes \mathbb{C}^2)^{\text{cl}}$, that commutes with both actions $\rho'$, $\lambda'$ of $\mathcal{U}_q(su(2))$ is of the form

\[
D |j\mu n\uparrow\rangle = d_{j}^\uparrow |j\mu n\uparrow\rangle, \quad D |j\mu n\downarrow\rangle = d_{j}^\downarrow |j\mu n\downarrow\rangle.
\]

Restrict form of eigenvalues by imposing bounded commutator condition:

\[
[D, \pi'(x)] \in \mathcal{B}(\mathcal{H}), \quad (x \in \mathcal{A}).
\]

- $D$ with as eigenvalues $q$-analogues of the classical Dirac operator (like $[j]$) gives unbounded commutators (cf. [Bib-Kul]).
• ‘Classical’ Dirac operator, with $D$ eigenvalues linear in $j$ with opposite signs on the $\uparrow$ and $\downarrow$-eigenspaces, respectively.

**Proposition.** If $D$ has eigenvalues linear in $j$, the commutators $[D, \pi'(x)]$ ($x \in A$) are bounded operators.
Relation with [Gos], with *unbounded commutators*?
Difference in definition of spinor space: $\mathbb{C}^2 \otimes V$ (instead of $V \otimes \mathbb{C}^2$).
Define on $\mathbb{C}^2 \otimes V$:

\[
\begin{align*}
\pi'(x) &= \text{id} \otimes \pi(x); \\
\rho'(h) &= \text{id} \otimes \rho(h); \\
\lambda'(h) &= \sigma_{\frac{1}{2}}(h(1)) \otimes \lambda(h(2)).
\end{align*}
\]

Let us (naïvely) define the Dirac operator to be diagonal in the $\uparrow - \downarrow$ basis obtained from the Clebsch-Gordan decomposition, with $j$-linear eigenvalues. This is exactly [Gos]. A computation shows that $[D, \pi'(x)]$ is unbounded.

But,..., the abovely defined representation $\pi'$ is not $(\lambda', \rho')$-equivariant. In other words, the principle of $\mathcal{U}$-equivariance does not allow $\mathbb{C}^2 \otimes V$ as a spinor space.
Modular conjugation operator

Definition. The modular conjugation operator $J$ is the antilinear operator on $\mathcal{H}$ which is defined on the orthonormal spinor basis by

\[ J |j \mu n \uparrow\rangle := i^{2(2j+\mu+n)} |j, -\mu, -n, \uparrow\rangle; \]
\[ J |j \mu n \downarrow\rangle := i^{2(2j-\mu-n)} |j, -\mu, -n, \downarrow\rangle. \]

Proposition. With $D$ as before and $J$ as above, the commutant property and the first-order condition are satisfied up to compact operators:

\[ [\pi'(x), J\pi'(y)J^{-1}] \in \mathcal{K}(\mathcal{H}); \]
\[ [\pi'(x), [D, J\pi'(y)J^{-1}]] \in \mathcal{K}(\mathcal{H}); \quad (\forall x, y \in \mathcal{A}(SU_q(2))) \]
References


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