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On the Power of a Singular Set in Binary Additive Problems with Prime Numbers

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Binary additive problems with primes are problems connected with the representation of the positive integers n as the sum $n = a(p) + b(q)$. Here p and q are prime numbers, and $a(x)$ and $b(x)$ are defined integer-valued functions of a natural argument. Morever it is usually assumed that the number n is subjected to certain natural additional conditions of an arithmetical nature. The problem of the infinity of twin primes in a natural series and the binary Goldbach problem on the representation of any even number as the sum of two prime numbers, which are not yet solved, are classical examples of problems of this kind.

One of the important aspects in the investigation of a binary additive problem is the estimation from above of the density of its singular (exceptional) set, i.e., the set of numbers which do not admit the defined representation.

The solution of Goldbach's ternary problem, which was found in 1937 by Vinogradov [1, 2], allowed some authors to obtain, at the same time, the first results concerning the power of a singlar set for Goldbach's binary problem. Later on many papers were devoted to the improvement of these estemates. In particular, in 1975 Montgomery and Vaughan [4] obtained, for the first time, a lowering of the power in these estimates although the numerical value of this lowering was not computed. At present the best result in this direction belongs to Cheng Jing-run and Liu Jian Min [5]. They proved in 1988 that for the amout $T(x)$ of even positive integers m, which do not exceed x and are not representable as the sum of two prime numbers, as $x \to +\infty$, the estimate

$$T(x) \ll x^{1-1/20}$$

was valid.

In this article we estimate the powers $T_1(x)$ and $T_2(x)$ of singular sets for two binary additive problems with primes [6–8]. The results that we obtained can be formulated as the following two theorems.

Theorem 1. *Let $\beta > 0$ be an irrational number and let $T_1(x)$ be the amount of numbers $n \le x$ which cannot be represented as*

$$n = p + [\beta q],$$

where p and q are prime numbers.

Then, as $x \to \infty$, the following estimates are valid:

(a) *if the partial denominators of continued fractions of the numbers β are bounded in their totality*, *then*

$$T_1(x) \ll x^{1-2/9}(\ln x)^8,$$

(b) *if β is an algebraic number*, *then the estimate*

$$T_1(x) \ll_\varepsilon x^{1-2/9+\varepsilon},$$

where $\varepsilon > 0$ is arbitrarily small, holds true. The constant under the sign $\ll_\varepsilon$ is ineffective.

Theorem 2. *Let $c > 1$ be a noninteger, $c\gamma = 1$. We denote by $T_2(x)$ the amout of positive integers $n \le x$, which cannot be represented as*

$$n = p + [q^c],$$

where p and q are prime numbers.

Then, for the quantity $T_2(x)$ we have the estimate

$$T_2(x) \ll_\varepsilon x^{1-2\delta+\varepsilon},$$

where ε is arbitrarily small and the quantity $\delta > 0$ is defined by the condition

$$\sum_{q \le y^\gamma} e^{2\pi i\alpha[q^c]} \ll y^{\gamma-\delta},$$

the parameter α satisfying the inequality $y^{-0.8} \le \alpha \le 1 - y^{-0.8}$.

In order to prove this theorem, we shall first consider Theorem 1.

Proof of Theorem 1. We denote by Ω the singular set of the given problem, i.e., the collection of positive integers y which cannot be represented as

$$y = p + [\beta q].$$

Next we define the intervals $P = P_x$, $Q = Q_x$, $Y = Y_x$ by the conditions

$$P_x = \left[\frac{x}{4}, \frac{3x}{4}\right], \qquad Q_x = \left[\frac{x}{4}, \frac{x}{2}\right], \qquad Y_x = \left[\frac{3x}{4}, x\right].$$

The amount of numbers $y \in \Omega$, which satisfy the condition $y \in Y_x$, will be denoted by $m = m(x)$.

Lemma 1. *Let a, b be numbers subjected to the conditions $0 < a < 1$, $b > 0$. Then the estimate*

$$m(t) \ll t^{1-a} \ln^b t,$$

which is valid for all t under the conditions $1 < t \le x$, yields the estimate

$$T_1(x) \ll x^{1-a} \ln^b x.$$

Proof. According to the definition, $T_1(x)$ is the number of $y \in \Omega$, satisfying the inequality $y \le x$. Next, it is abvious that for every $z \in [1, x]$ there exists a positive integer k defined by the condition

$$z \in \left(x\left(\frac{3}{4}\right)^k, x\left(\frac{3}{4}\right)^{k-1}\right) = Y_{x(3/4)^{k-1}}.$$

Therefore we have the estimate

$$\begin{aligned} T_1(x) &= m(x) + m\left(x \times \frac{3}{4}\right) + m\left(x \times \left(\frac{3}{4}\right)^2\right) + \cdots \\ &\ll x^{1-a} \ln^b x\left(1 + \left(\frac{3}{4}\right)^{1-a} + \left(\frac{3}{4}\right)^{2(1-a)} + \cdots\right) \ll x^{1-a} \ln^b x. \end{aligned}$$

We have proved the lemma.

It follows from Lemma 1 that in order to prove Theorem 1, it suffices to obtain or the quantity $m(x)$ essentially the same estimate which is required for $T_1(x)$. In this connection, we consider the Diophantine equation

$$y = p + [\beta q]$$

under the condition that, as before, $y \in \Omega$ and p and q are prime numbers.

In this case where $y \in \Omega$, the number of solutions of this equation is zero. On the other hand, we can formally express this quantity in terms of trigonometric sums. We assume that $y \in Y_x$, $p \in P_x$, $[\beta q] \in Q_x$.

Then, obviously, we have the relation

$$J = \int_0^1 SVW\, d\alpha,$$

where

$$S = S(\alpha) = \sum_{p \le P_x} e^{2\pi i \alpha p} \ln p, \quad V = V(\alpha) = \sum_{[\beta q] \in P_x} e^{2\pi i \alpha [\beta q]} \ln q, \quad W = W(\alpha) = \sum_{y \in \Omega,\ y \in Y_x} e^{2\pi i \alpha y}.$$

Next we set $\tau = x^{7/9}$, $\varkappa = \tau^{-1}$, $E_1 = [-\varkappa, \varkappa]$, $E_2 = [\varkappa, 1]$. Then, by virtue of the periodicity of the function SVW we have

$$0 = \int_0^1 SVW\, d\alpha = \int_{-\varkappa}^{\varkappa} \cdots + \int_{\varkappa}^{1} \cdots = J_1 + J_2,$$

where the quantities J_1 and J_2 are obviously defined by the last relation. It is clear that $|J_1| = |J_2|$.

The plan of our further resoning consists in an appropriate estimation from below the quantity $|J_1|$ of the form

$$|J_1| \gg mx,$$

and the estimation from above of the quantuty $|J_2|$ of the form

$$|J_2| \ll m^{1/2} x^{1/2} \Delta,$$

where $m = m(x)$, $\Delta = x^{8/9} (\ln x)^4$.

It is easy to see that, indeed, these estimates yield the required estimate for the quantity m.

We shall first consider the estimate $|J_2|$. We denote by $F(\alpha)$ the quantity

$$F(\alpha) = \min\bigl(|S(\alpha)|, |V(\alpha)|\bigr).$$

Then, obviously, for all α we have

$$|S(\alpha)V(\alpha)| \le F(\alpha)\bigl(|S(\alpha)| + |V(\alpha)|\bigr).$$

It follows that

$$|J_2| = \left| \int_{\varkappa}^{1-\varkappa} SVW\, d\alpha \right| \le \max_{\alpha \in E_2} F(\alpha) \biggl(\int_0^1 |S(\alpha)W(\alpha)|\, d\alpha + \int_0^1 |V(\alpha)W(\alpha)|\, d\alpha \biggr).$$

Next, using the Cauchy inequality, we get

$$\int_0^1 |SW|\,d\alpha \le \left(\int_0^1 |S|^2\,d\alpha \int_0^1 |W|^2\,d\alpha\right)^{1/2} = \left(\sum_{p\in P_x} \ln^2 p \times \sum_{\substack{y\in\Omega\\ y\in Y_x}} 1\right)^{1/2} \ll (mx\ln x)^{1/2}.$$

Using similar arguments, we derive an estimate of the form

$$\int_0^1 |VW|\,d\alpha \ll (mx\ln x)^{1/2}.$$

Now we must estimate the quantity $F(\alpha)$, for which purpose we use Vinogradov's estimate for a trigonometric sum with prime numbers in the following form.

Lemma 2. *Let*

$$S(\alpha, x) = \sum_{n\le x} \Lambda(n) e^{2\pi i\alpha n},$$

where

$$\alpha = \frac{a}{q} + \frac{\theta}{q^2}, \qquad (a,q)=1, \quad |\theta|\le 1.$$

Then the inequality

$$S(\alpha,x) \ll x\ln^{3.5} x(q^{1/2}x^{-1/2} + q^{-1/2} + x^{-0.2})$$

holds true.

Proof (see [9, theorem 5.12, p. 74]).

In turn, the application of the last lemma is based on Dirichlet's lemma on the approximation of real numbers by rational numbers which is given below.

Lemma 3. *Let $\tau \ge 1$. Then every real number α can be represented as*

$$\alpha = \frac{a}{q} + \frac{\theta}{q\tau}, \qquad 0<q\le\tau, \quad (a,q)=1, \quad |\theta|<1.$$

Proof. With an accuracy to within the notation, this lemma coincides with Question 4 on p. 21 of Vinogradov's book [3].

Using Lemma 3, we represent every real number $\alpha \in E_2$ in the form

$$\alpha = \frac{a}{q} + \frac{\theta}{q\tau},$$

where $0<q\le\tau$, $(a,q)=1$, $|\theta|<1$ and $\tau = x^{7/9}$.

We divide the further consideration into two cases:

(1) $x^{2/9} \le q \le x^{7/9}$,

(2) $1 \le q < x^{2/9}$.

In the first case we estimate the quantity $F(\alpha)$ in terms of $S(\alpha)$ with the use of the obvious inequality $|F(\alpha)| \le |S(\alpha)|$ and reduce the estimation of the sum $S(\alpha)$ to Lemma 2. We have

$$S(\alpha) = \sum_{p\in P_x} e^{2\pi i\alpha p}\ln p = \sum_{n\in P_x} \Lambda(n)e^{2\pi i\alpha n} + O(x^{1/2}) = S\Big(\alpha, \frac{3x}{4}\Big) - S\Big(\alpha, \frac{x}{4}\Big) + O(x^{1/2}),$$

where $S(\alpha, y)$ is the sum considered in Lemma 2.

We arrive at the estimate

$$S(\alpha) \ll x(\ln x)^{3.5}(q^{1/2}x^{-1/2} + q^{-1/2} + x^{-0.2}) + x^{1/2} \ll x^{1-1/9}(\ln x)^{3.5}$$

since $x^{2/9} \le q \le x^{7/9}$.

Consequently, the same estimate is valid for $F(\alpha)$, i.e.,

$$F(\alpha) \ll x^{1-1/9}(\ln x)^{3.5}.$$

We have thus obtained the estimate of the quantity $F(\alpha)$ required in the first case.

Suppose now that the quantity q is in the limits $1 \le q \le x^{2/9}$. In this case, we reduce the estimation of the quantity $F(\alpha)$ to the estimatation of the sum $V(\alpha)$ with the aid of the inequality

$$|F(\alpha)| \le |V(\alpha)|.$$

Then we apply the following lemma to the quantity $V(\alpha)$.

Lemma 4. *Suppose that α is a real number, $0 < |\alpha| \le 1/2$, P is a positive integer, and $P_1 \le 2P$, and $f(x)$ is any function of a natural argument,*

$$V_0(\alpha) = \sum_{P<n\le P_1} \Lambda(n)e^{2\pi i\alpha[f(n)]}.$$

Then, for any positive integer $N \ge 2$, $M = [(2N+1)\pi/2] + 1$, $M_0 = [M\log M]$, the estimate

$$|V_0(\alpha)| \le \sum_{|m|\le N}\left|\frac{\sin\pi\alpha}{\alpha+m}T(\alpha+m)\right| + 36\sum_{0<|m|\le M_0}|\sin\pi\alpha T(m)|\frac{\ln M}{M} + 136|\sin\pi\alpha|\frac{P\ln M}{M},$$

where

$$T(\lambda) = \sum_{P<n\le P_1}\Lambda(n)e^{2\pi i\lambda f(n)},$$

holds true.

Proof. This lemma is a slight modification of Lemma 3 from [10, p. 28].

In this lemma we set $f(n) = \beta n$ and choose the numbers P and P_1 such that the condition $[\beta n] \in Q_x$ is fulfilled. Then we have

$$V_0(\alpha) = \sum_{n\in Q_x}\Lambda(n)e^{2\pi i\alpha[\beta n]} = \sum_{q\in Q_x} e^{2\pi i\alpha[\beta q]}\ln q + O(x^{1/2}) = V(\alpha) + O(x^{1/2}).$$

Suppose that $N = x^{1/9}$. Then Lemma 4 implies the validity of an inequality of the form

$$V(\alpha) \ll \sum_{0<k\leq N} \frac{|\alpha|}{\alpha+k}\bigl|T\bigl((\alpha+k)\beta\bigr)\bigr| + x^{-1/9}\ln x \sum_{k=1}^{M_0} |T(k\beta)| + x^{1-1/9}\ln x.$$

Here the number M_0 satisfies the condition $M_0 \ll x^{1/9}\ln x$, and, for, $\lambda = \alpha + k$ or $\lambda = k$ the sum $T(\lambda)$ has the form

$$T(\lambda) = \sum_{[\beta n]\in Q_x} \Lambda(n) e^{2\pi i \lambda \beta n}.$$

When estimating the quantity $T(\lambda)$, we must make an essential use of the arithmetic nature of the number β. For this purpose, we shall need one more lemma on Diophantine approximations.

Lemma 5. *Let q_n and q_{n+1} be the denominators of successive convergents of continued fractions of the arbitrary irrational number μ. Then the inequality*

$$\|q_n\mu\| > \frac{1}{2q_{n+1}}$$

holds true. As usual, we denote here by the symbol $\|t\|$ the distance from t to the nearest integer.

Proof. See inequality (16) on p. 16 of the book [11].

Note that if the partial denominators of continued fractions of the number μ are bounded, then the inequalities $q_n \ll q_{n+1} \ll q_n$ hold true, and if μ is an algebraic number, then it follows from Roth's theorem (see [11, p. 128]) that $q_n \ll q_{n+1} \ll q_n^{1+\varepsilon}$. In addition, by virtue of the property of convergents of continued fractions, for any natural k under the conditions $q_n \leq k < q_{n+1}$, the inequalities $\|k\mu\| \geq \|q_n\mu\| > \|q_{n+1}\mu\|$ hold true.

Taking this into account, we divide the further estimation of the quantity $T(\lambda)$ into three cases:

(1) $\lambda = k\beta$, $k \ll x^{1/9}\ln x$,

(2) $\lambda = \alpha\beta$,

(3) $\lambda = (\alpha+k)\beta$, $k \ll x^{1/9}$.

We begin with the case $\lambda = k\beta$. Using the Dirichlet lemma, we represent the number $k\beta$, for the fixed k, in the form

$$k\beta = \frac{a_1}{q_1} + \frac{\theta_1}{q_1\tau_1},$$

where $(a_1, q_1) = 1$, $|\theta_1| < 1$, $a_1 \geq 1$, $q_1 \leq \tau_1 = x^{5/9}$.

Hence we have

$$q_1 k\beta = a_1 + \theta_1\tau_1^{-1}, \qquad q_1 k\beta - a_1 = \theta_1\tau_1^{-1}.$$

However, according to Lemma 5 and the remark to it, the inequalities

$$\tau_1^{-1} \geq \|q_1 k\beta\| \geq \|r_n\beta\| > \frac{1}{2r_{n+1}}$$

hold true, whence it follows that $r_{n+1} > \tau_1/2$. Here the numbers r_n and r_{n+1} are the denomintors of the convergents of continued fractions to the number β under the condition $r_n \leq q_1 k < r_{n+1}$.

If the partial denominators of continued fractions of the number β are bounded, then we have

$$r_{n+1} \ll r_n \ll q_1 k,$$

whence it follows that

$$q_1 k \gg \tau_1 = x^{5/9}, \qquad q_1 \gg x^{5/9} k^{-1}.$$

However, since $q_1 \le \tau_1$ and $k_1 \le x^{1/9} \ln x$, we have

$$\frac{x^{4/9}}{\ln x} \ll q_1 \ll x^{5/9}.$$

Therefore, reducing the estimatation of the sum $T(k\beta)$ to the use of Lemma 2 just as we did when deriving the estimate of the sum $S(\alpha)$ above, we arrive at the inequality

$$T(k\beta) \ll x^{0.8} (\ln x)^{3.5}.$$

In the case where β is an algebraic number, we must replace the preceding estimates of the product $q_1 k$ by the estimates

$$r_n \le q_1 k < r_{n+1} \ll_\varepsilon r_n^{1+\varepsilon} \le (q_1 k)^{1+\varepsilon}.$$

It follows, as before, that

$$(q_1 k)^{1+\varepsilon} \gg_\varepsilon r_{n+1} \ge \frac{\tau_1}{2} \gg x^{5/9}, \qquad q_1 k \gg_\varepsilon x^{5/(9(1+\varepsilon))}, \qquad q_1 \gg_\varepsilon k^{-1} x^{5/(9(1+\varepsilon))-1/9} \gg_{\varepsilon_1} x^{4/9-\varepsilon_1},$$

where $\varepsilon_1 > 0$ is also arbitrarily small. Without any stipulations, we shall assume in what follows that the numbers $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ also possess this property.

Thus the number q_1 is contained in an interval of the form

$$x^{4/9-\varepsilon_1} \ll_{\varepsilon_1} q_1 \ll x^{5/9}.$$

Consequently, in this case as well, an estimate of the form $T(k\beta)$ follows from Lemma 2 for the sum

$$T(k\beta) \ll x^{0.8} (\ln x)^{3.5}.$$

Consider now the sum $T(\lambda)$ for $\lambda = \alpha\beta$ under the condition that

$$\alpha = \frac{a}{q} + \frac{\theta}{q\tau}, \qquad (a, q) = 1, \quad \tau = x^{7/9} \quad \text{and} \quad q < x^{2/9}.$$

Again proceeding from Dirichlet's lemma, we represent the number λ in the form

$$\lambda = \alpha\beta = \frac{a_2}{q_2} + \frac{\theta_2}{q_2 \tau_2},$$

where $\tau_2 = \tau = x^{7/9}$, $q_2 \le \tau_2$, $1 \le a_2 < q_2$, $(a_2, q_2) = 1$.

Performing obvious transformations, we obtain

$$\alpha\beta = \left(\frac{a}{q} + \frac{\theta}{q\tau}\right)\beta = \frac{a_2}{q_2} + \frac{\theta_2}{q_2\tau}, \qquad \frac{a\beta}{q} - \frac{a_2}{q_2} \ll \frac{1}{q\tau} + \frac{1}{q_2\tau}, \qquad \|a q_2 \beta\| \ll a q_2 \beta - a_2 q \ll \frac{q_2 + q}{\tau}.$$

The arguments similar to those we used above when considering the sum $T(\lambda) = T(k\beta)$ give

$$r_{n+1} \gg \frac{\tau}{q_2 + q}, \qquad r_n \le a q_2 < r_{n+1},$$

where r_n and r_{n+1} are the denominators of the successive convergents of continued fractions to the number β.

If β is a number with bounded partial denominators of continued fractions, then

$$aq_2 \gg \frac{\tau}{q_2+q}, \qquad q_2(q_2+q) \gg \frac{\tau}{a} \gg x^{5/9}.$$

If $q_2 \gg q$, then

$$q_2 \gg q_2+q, \qquad q_2^2 \gg (q_2+q)q_2 \gg x^{5/9}, \qquad q_2 \gg x^{5/18} > x^{2/9}.$$

Now if $q_2 \ll q$, then, since $q \ll x^{2/9}$, we also have here

$$q_2 q \gg (q_2+q)q_2 \gg x^{5/9}, \qquad q_2 \gg x^{5/9}q^{-1} \gg x^{2/9}.$$

Thus always $\tau = x^{7/9} \geq q_2 \gg x^{2/9}$, and therefore, proceeding from Lemma 2, we arrive at the estimate

$$T(\alpha\beta) \ll x^{1-1/9}(\ln x)^{3.5}.$$

Now if β is an algebraic number, then we have

$$r_{n+1} \gg \frac{\tau}{q_2+q}, \qquad r_n \leq aq_2 < r_{n+1} \ll_\varepsilon r_n^{1+\varepsilon} \leq (aq_2)^{1+\varepsilon},$$
$$(aq_2)^{1+\varepsilon}(q_2+q) \gg \tau = x^{7/9}, \qquad a \leq q < x^{1/9}.$$

Hence, for $q_2 \gg q$ we obtain

$$q_2^{2+\varepsilon}a^{1+\varepsilon} \gg_\varepsilon x^{7/9}, \quad q_2^{2+\varepsilon} \gg_\varepsilon x^{7/9}a^{-(1+\varepsilon)} \gg x^{5/9-\varepsilon}, \quad q_2^2 \gg_\varepsilon x^{5/9-2\varepsilon}, \quad q_2 \gg_\varepsilon x^{5/18-\varepsilon} \gg x^{2/9}.$$

If $q_2 \ll q$, then $q \gg q_2+q$,

$$(aq_2)^{1+\varepsilon}q \gg aq_2(q_2+q) \gg_\varepsilon x^{7/9}, \qquad q_2^{1+\varepsilon} \gg_\varepsilon x^{7/9}a^{-(1+\varepsilon)}q^{-1} \gg x^{7/9-4/9-\varepsilon}, \qquad q_2 \gg x^{2/9}.$$

Thus, in the case of the algebraic number β we also have

$$x^{2/9} \ll q_2 \leq x^{7/9}.$$

Therefore the use of Lemma 2 here gives the same estimate, i.e.,

$$T(\alpha\beta) \ll x^{1-1/9}(\ln x)^{3.5}.$$

It remains to consider the case $\gamma = (\alpha+k)\beta$ for $k \ll x^{1/9}$ and $q < x^{2/9}$.

Using the Dirichlet lemma, we represent the number γ as

$$\gamma = (\alpha+k)\beta = \frac{a_3}{q_3} + \frac{\theta_3}{q_3\tau_3}, \qquad (a_3,q_3)=1, \quad |\theta_3|<1, \quad q_3 \leq \tau_3 = x^{1-5/18}k^{1/2}.$$

In addition,

$$\alpha = \frac{a}{q} + \frac{\theta}{q\tau}, \qquad \gamma = \frac{a+kq}{q}\beta + \frac{\theta\beta}{q\tau} = \frac{a_3}{q_3} + \frac{\theta_3}{q_3\tau_3},$$

whence we have

$$q_3(a+kq)\beta - a_3q \ll \frac{q_3}{\tau} + \frac{q}{\tau_3}.$$

If the number β have bounded convergents of continued fractions, then, repeating the same arguments presented above for $\gamma = \alpha\beta$, we arrive at the inequality

$$q_3(a+kq) \gg \left(\frac{q_3}{\tau} + \frac{q}{\tau_3}\right)^{-1}.$$

Next, if $q_3/\tau \geq q/\tau_3$, then

$$\left(\frac{q_3}{\tau} + \frac{q}{\tau_3}\right)^{-1} \gg \frac{\tau}{q_3},$$

whence, taking into account that $a < q$, we obtain

$$q_3(a+kq) \gg \frac{\tau}{q_3}, \qquad q_3^2 \gg \frac{\tau}{kq} \geq \frac{x^{5/9}}{k}, \qquad q_3 \gg x^{5/18}k^{-1/2}.$$

Now if $q_3/\tau \leq q/\tau_3$, then

$$\left(\frac{q_3}{\tau} + \frac{q}{\tau_3}\right)^{-1} \gg \frac{\tau_3}{q}, \qquad q_3(a+kq) \gg \frac{\tau_3}{q}, \qquad q_3 \gg \frac{\tau_3}{q^2k} \geq \frac{x^{1-5/18}k^{1/2}}{x^{4/9}k} = x^{5/18}k^{-1/2}.$$

Taking into account that $q_3 \leq \tau_3$, in both cases we have

$$x^{5/18}k^{-1/2} \ll q_3 \ll x^{1-5/18}k^{1/2}.$$

Therefore, estimating the sum $T((\alpha+k)\beta)$ with the use of Lemma 2, we get the inequality

$$T\big((\alpha+k)\beta\big) \ll x^{1-5/36}k^{1/4}(\ln x)^{3.5}.$$

In the case where β is an algebraic number our reasoning obviously changes so that we arrive at the inequality

$$q_3(a+kq) \gg_\varepsilon \left(\frac{q_3}{\tau} + \frac{q}{\tau_3}\right)^{-1/(1+\varepsilon)}.$$

We again consider two cases: $q_3/\tau \geq q/\tau_3$ and $q_3/\tau < q/\tau_3$.

In the first case we have

$$q_3^{2+\varepsilon} \gg_\varepsilon \frac{\tau^{1/(1+\varepsilon)}}{kq} \gg \frac{x^{7/(9(1+\varepsilon))-2/9}}{k}.$$

However, since $k \ll x^{1/9}$, it follows that

$$q_3 \gg_{\varepsilon_1} x^{2/9-\varepsilon_1},$$

where ε_1 is arbitrarily small together with $\varepsilon > 0$.

Now if $q_3/\tau < q/\tau_3$, then, since $\tau_3 = x^{1-5/18}k^{1/2}$, we obtain

$$q_3 \gg_\varepsilon \frac{1}{kq}\left(\frac{\tau_3}{q}\right)^{1/(1+\varepsilon)} \gg_{\varepsilon_1} x^{2/9-\varepsilon_1}.$$

Using Lemma 2, we arrive at the estimate

$$T\big((\alpha+k)\beta\big) \ll_{\varepsilon_2} x^{1/9-\varepsilon_2}.$$

We have thus obtained the desired estimate of the quantity $T(\lambda)$ and can return to the estimate of $V(\alpha)$. Somewhat earlier we proved the inequality

$$V(\alpha) \ll T(\alpha\beta) + \sum_{1\le k\le N} \frac{|\alpha|}{\alpha+k}\big|T\big((\alpha+k)\beta\big)\big| + x^{-1/9}\ln x \sum_{k=1}^{M_0} |T(k\beta)| + x^{1-1/9}\ln x.$$

Here $N = x^{1/9}$, $M_0 \ll x^{1/9}\ln x$,

$$\alpha = \frac{a}{q} + \frac{\theta}{q\tau}, \qquad (a,q)=1, \quad q < x^{2/9}, \quad \tau = x^{7/9}, \quad |\theta| < 1.$$

We use the estimates of the sums $T(\lambda)$ obtained earlier with due account of the specified properties of the number β. Let us first assume that the number β is a number with bounded partial denominators of continued fractions. Then we have the estimates

$$\begin{aligned} V(\alpha) &\ll x^{1-1/9}(\ln x)^{3.5} + \sum_{1\le k\le N} x^{1-5/36}k^{-3/4}(\ln x)^{3.5} + x^{-1/9}\ln x \times x^{0.8}(\ln x)^{3.5} \times M \\ &\ll x^{1-1/9}(\ln x)^{3.5} + x^{1-5/36}N^{1/4}(\ln x)^{3.5} + x^{0.8}(\ln x)^{4.5} \ll x^{1-1/9}(\ln x)^{3.5}. \end{aligned}$$

Thus, in the first case we obtained the estimate of the sum $V(\alpha)$. Suppose now that β is an algebraic number. In this case, the estimatation of the quantity $T(\lambda)$ gives the following result:

$$V(\alpha) \ll_{\varepsilon_2} x^{1-1/9}(\ln x)^{3.5} + \sum_{1\le k\le N} \frac{x^{1-1/9+\varepsilon_2}}{k} + x^{-1/9}\ln x \times x^{0.8}(\ln x)^{3.5} \times M \ll_{\varepsilon_3} x^{1-1/9+\varepsilon_3}.$$

The inequalities obtained for $V(\alpha)$ allow us to complete the estimation of the quantities $F(\alpha)$ and J_2.

Indeed, we have proved that

$$|F(\alpha)| \le \begin{cases} |S(\alpha)| \ll x^{1-1/9}(\ln x)^{3.5} \ \text{ for } q \ge x^{2/9} \\ |V(\alpha)| \ll x^{1-1/9}(\ln x)^{3.5} \ \text{ for } q < x^{2/9} \\ \qquad (\beta \text{ is a number with bounded denominators of continued fractions}) \\ |V(\alpha)| \ll_{\varepsilon_3} x^{1-1/9+\varepsilon_3} \qquad \text{for } q < x^{2/9} \\ \qquad (\beta \text{ is an algebraic number}). \end{cases}$$

Consequently, in the case (a) of Theorem 1 we have an estimate of the form

$$\begin{aligned} |J_2| &\le \max_{\alpha\in E_2} |F(\alpha)| \bigg(\int_0^1 |SW|\,d\alpha + \int_0^1 |VW|\,d\alpha \bigg) \\ &\ll x^{1-1/9}(\ln x)^{3.5} x^{1/2}(\ln x)^{1/2} m^{1/2} \ll x^{3/2-1/9}(\ln x)^4 m^{1/2}. \end{aligned}$$

In the case (b) of this theorem the same estimate has the form

$$|J_2| \ll_{\varepsilon_4} x^{3/2-1/9+\varepsilon_4} m^{1/2}.$$

This completes the investigation of the quantity J_2, and we pass to the estimation of $|J_1|$ from below.

We denote by $S_0(\alpha)$ the sum

$$S_0(\alpha) = S_0 = \sum_{n \in P_x} \Lambda(n) e^{2\pi i \alpha n}.$$

The elementary estimate $\psi(x) \ll x$ of Chebyshev's function $\psi(x) = \sum_{n \le x} \Lambda(n)$ gives the obvious equality

$$S(\alpha) = S_0(\alpha) + O(x^{1/2}).$$

Therefore we have

$$J_1 = \int_{-\varkappa}^{\varkappa} SWV\,d\alpha = \int_{-\varkappa}^{\varkappa} S_0 WV\,d\alpha + R_0 = J_0 + R_0,$$

where J_0 and R_0 are defined by the last relation, with

$$R_0 \ll x^{1/2} \int_{-\varkappa}^{\varkappa} |WV|\,d\alpha \ll x^{1/2} \varkappa x m \ll m x^{1/2+2/9} = mx \times x^{-5/16}.$$

For $\alpha \in E_2$ we transform the sum $S_0(\alpha)$ with the use of Abel's integral formula of summation and the representation of Chebyshev's function $\psi(x)$ as the sum over zeros of Riemann's zeta function. We give these formulas in the form of the following lemmas.

Lemma 6. *Suppose that $\psi(x) = \sum_{n \le x} \Lambda(n)$ is Chebyshev's function and let $\rho = \sigma + it$ be one of the zeros of Riemann's zeta function lying in the critical strip $0 < \sigma < 1$.*

Then, for $T \ge 2$ and $x \ge 2$ we have the relation

$$\psi(x) = x - \sum_{|\gamma| \le T} \frac{x^\rho}{\rho} + R(x,T),$$

where $R(x,T) \ll \dfrac{x}{T} \ln^2(xT) + \ln x$, as well as

$$R(x,T) \ll \begin{cases} \dfrac{x}{T}\left(\ln^2(xT) + \dfrac{\ln x}{\|x\|}\right) & \text{for noninteger } x \\ \dfrac{x}{T} \ln^2(xT) & \text{for integer } x. \end{cases}$$

Proof. This lemma is a special case of Theorem 4.5 [12, p. 264].

Lemma 7 (Abel's summation formula in integral form). *Let $g(x)$ be a function which has a continous derivative on the interval $[x_0, x_1]$. We shall consider the nondecreasing sequence of real numbers $b_1 \le b_2 \le \cdots$ and the arbitrary number sequence $\{a_n\}$. Let us define the function $A(t)$ by the relation*

$$A(t) = \sum_{\lambda_n \le t} a_n.$$

Then the relation

$$\sum_{\lambda_n \le x} a_n g(\lambda_n) - A(x)g(x) = -\int_{\lambda_1}^{x} A(t)g'(t)\,dt$$

holds true.

Proof. The left-hand side of the relation regarded as a function x is continuous and piecewise smooth by virtue of the conditions imposed on $g(x)$, and therefore, according to the generalized Newton–Leibniz formula, it can be represented as a definite integral of its formal derivative, and this is precisely the right-hand side of the relation.

We have completed the proof of the lemma.

The application of Lemma 7 to the sum $S_0(\alpha)$ leads to the relation

$$S_0(\alpha) = \sum_{x/4 \le n \le 3x/4} \Lambda(n) e^{2\pi i \alpha n} = \Big(\psi\Big(\frac{3x}{4}\Big) - \psi\Big(\frac{x}{4}\Big)\Big) e^{2\pi i \alpha n} - \int_{x/4}^{3x/4} \Big(\psi(z) - \psi\Big(\frac{x}{4}\Big)\Big) e^{2\pi i \alpha t} 2\pi i \alpha\, dt.$$

We apply Lemma 6 to Chebyshev's function in the last relation setting $A(t) = \psi(t) - \psi(x/4)$ and $T = x^a$. The exact value of the parameter a, $0 < a < 1$, will be given later. We obtain

$$\begin{aligned} S_0(\alpha) = {} & \Big(\frac{3x}{4} - \frac{x}{4}\Big) e^{2\pi i \alpha 3x/4} - \sum_{|\gamma| \le T} \frac{1}{\rho}\Big(\Big(\frac{3x}{4}\Big)^{\rho} - \Big(\frac{x}{4}\Big)^{\rho}\Big) e^{2\pi i \alpha 3x/4} \\ & + \Big(R\Big(\frac{3x}{4}, T\Big) - R\Big(\frac{x}{4}, T\Big)\Big) e^{2\pi i \alpha 3x/4} - \int_{x/4}^{3x/4} \Big(t - \frac{x}{4}\Big)\, de^{2\pi i \alpha t} \\ & + \int_{x/4}^{3x/4} \sum_{|\gamma| \le T} \frac{1}{\rho}\Big(t^{\rho} - \Big(\frac{x}{4}\Big)^{\rho}\Big)\, de^{2\pi i \alpha t} + \int_{x/4}^{3x/4} \Big(R\Big(\frac{x}{4}, T\Big) - R(t, T)\Big) 2\pi i \alpha e^{2\pi i \alpha t}\, dt. \end{aligned}$$

Integrating by parts and estimating the remainder $R(t, T)$ with the use of Lemma 6, we obtain

$$S_0(\alpha) = \int_{x/4}^{3x/4} e^{2\pi i \alpha t}\, dt - \int_{x/4}^{3x/4} e^{2\pi i \alpha t} \sum_{|\gamma| \le T} t^{\rho - 1}\, dt + R_1 = S_1(\alpha) + R_1.$$

Here the quantities $S_1(\alpha)$ and R_1 are defined by the last relation. In this case we have

$$R_1 = \Big(R\Big(\frac{3x}{4}\Big) - R\Big(\frac{x}{4}\Big)\Big)e^{2\pi i\alpha 3x/4} + \int\limits_{x/4}^{3x/4}\Big(R\Big(\frac{x}{4},T\Big) - R(t,T)\Big)2\pi i\alpha e^{2\pi i\alpha t}\,dt$$

$$\ll \frac{x}{T}\ln^2 x + |\alpha|\frac{x}{T}\ln^2 x\int\limits_{x/4}^{3x/4} dt \ll \frac{|\alpha|x^2}{T}\ln^2 x.$$

Next we have to estimate the quantity R_2, where

$$R_2 = \int\limits_{-\varkappa}^{\varkappa} R_1 V W\,d\alpha.$$

To this end, we first estimate the quantity $V(\alpha)$ for $\alpha \in E_1$. If $|\alpha| \le x^{-1}$, then we restrict ourselves to the trivial estimate

$$V(\alpha) \ll x.$$

Suppose now that $\alpha > x^{-1}$. Then we have

$$\begin{aligned} V(\alpha) &= \sum_{[\beta q]\in Q_x} \ln q e^{2\pi i\alpha[\beta q]} = \sum_{\beta n\in Q_x} \Lambda(n)e^{2\pi i\alpha[\beta n]} + O(x^{1/2}) \\ &= \sum_{\beta n\in Q_x} \Lambda(n)e^{2\pi i\alpha\beta n}e^{-2\pi i\alpha\{\beta n\}} + O(x^{1/2}) \\ &= \sum_{\beta n\in Q_x} \Lambda(n)e^{2\pi i\alpha\beta n}\big(1 + O(|\alpha|)\big) + O(x^{1/2}) \\ &= \sum_{\beta n\in Q_x} \Lambda(n)e^{2\pi i\alpha\beta n} + R_3 = V_0(\alpha) + R_3. \end{aligned}$$

Here $V_0(\alpha)$ and R_3 are defined by the last relations, and an estimate of the form

$$R_3 \ll x^{1/2} + |\alpha|x \ll x^{1/2} + x\tau^{-1} \ll x^{1/2}.$$

is valid for the remainder R_3. As before, we reduce the estimation of the sum $V_0(\alpha)$ to Lemma 2. In this case we must take into account that $x^{-1} < \alpha < x^{-7/9} = \tau^{-1}$. The last inequalities show that $x^{-1} \ll \alpha\beta \ll x^{-7/9}$, and therefore in the representation

$$\alpha\beta = \frac{a}{q} + \frac{\theta}{q^2}$$

for $(a,q) = 1$ and $|\theta| < 1$ the number q satisfies the inequalities

$$q \ll \frac{1}{\alpha} \ll q$$

since we can easily see that in this case either $a/q = 1/[\alpha\beta]$ or $a/q = 1/([\alpha\beta]+1)$. Therefore the use of Lemma 2 gives the following estimate for the sum $V_0(\alpha)$:

$$V_0(\alpha) \ll x\ln^{3.5} x q^{1/2} x^{-1/2} \ll x^{1/2} q^{1/2} \ln^{3.5} x \ll \left(\frac{x}{\alpha}\right)^{1/2} \ln^{3.5} x.$$

Combining all estimates obtained above for the quantities R_3 and $V_0(\alpha)$ on the whole set E_1, we have the inequality

$$V(\alpha) \ll \left(\frac{x}{|\alpha| + x^{-1}}\right)^{1/2} \ln^{3.5} x = x(|\alpha|x+1)^{-1/2} \ln^{3.5} x.$$

It follows from this inequality and from the estimate for R_1 obtained earlier that

$$R_2 = \int_{-\varkappa}^{\varkappa} R_1 V W \, d\alpha \ll \frac{x^2 \ln^2 x}{T} \int_{-\varkappa}^{\varkappa} |\alpha| x (|\alpha| x + 1)^{-1/2} \ln^{3.5} x m \, d\alpha$$

$$\ll \frac{mx^3 \ln^{5.5} x}{T} \int_0^{x^{-1}} \alpha \, d\alpha + \frac{mx^{2.5} \ln^{5.5} x}{T} \int_{x^{-1}}^{\tau^{-1}} \alpha^{1/2} \, d\alpha$$

$$\ll \frac{mx \ln^{5.5} x}{T} + \frac{mx^{2.5} \ln^{5.5} x}{T} \left(x^{-7/9}\right)^{3/2} \ll mx \frac{x^{1/3} \ln^{5.5} x}{T}.$$

We choose now the value of the parameter $a = 0.4$. Then $T = x^a = x^{0.4}$ and

$$R_2 \ll mx \ln^{5.5} x \times x^{1/3 - 0.4} \ll mx^{1-1/20}.$$

Using the notation introduced above, we can write the relation

$$J_1 = \int_{-\varkappa}^{\varkappa} S_1 V W \, d\alpha + R_0 + R_2 = \int_{-\varkappa}^{\varkappa} S_1 V W \, d\alpha + O(mx^{1-1/20}).$$

We have used here the abridged notation by setting

$$S_1 = S_1(\alpha) = \int_{x/4}^{3x/4} e^{2\pi i \alpha t} \, dt - \sum_{|\gamma|<T} \int_{x/4}^{3x/4} e^{2\pi i \alpha t} t^{\rho-1} \, dt.$$

Next we must replace each integral appearing in the sum $S_1(\alpha)$ by a trigonometric sum. For this purpose we shall need the Euler–Maclaurin summation formula, to be more precise, its integral form.

Lemma 8. *Let the function $f(t)$ have the derivative $f'(t)$ continuous on the interval $[a, x]$, and let $\rho(x) = \frac{1}{2} - \{x\}$. Then we have the relation*

$$\sum_{a<n\le x} f(n) - \rho(x) f(x) = \int_a^x f(t) \, dt - \int_a^x f'(t) \rho(t) \, dt - \rho(a) f(a).$$

Proof. We can prove this lemma in the same way as we proved Lemma 7.

Applying Lemma 8 to each of the integrals appearing in the sum $S_1(\alpha)$, we obtain

$$\sum_{x/4<n\le 3x/4} n^{\rho-1}e^{2\pi i\alpha n}-\rho\Big(\frac{3x}{4}\Big)\Big(\frac{3x}{4}\Big)^{\rho-1}=\int\limits_{x/4}^{3x/4} e^{2\pi i\alpha t}t^{\rho-1}\,dt-\int\limits_{x/4}^{3x/4}\rho(t)\,d\big(e^{2\pi i\alpha t}t^{\rho-1}\big)-\rho\Big(\frac{x}{4}\Big)\Big(\frac{x}{4}\Big)^{\rho-1}.$$

Note that

$$\big(e^{2\pi i\alpha t}t^{\rho-1}\big)'_t=\big(2\pi i\alpha t^{\rho-1}+(\rho-1)t^{\rho-2}\big)e^{2\pi i\alpha t}\ll|\alpha|x^{\sigma-1}+(1+|\gamma|)x^{\sigma-2},$$

and, consequently,

$$\Bigg|\int\limits_{x/4}^{3x/4}\rho(t)\,d\big(e^{2\pi i\alpha t}t^{\rho-1}\big)\Bigg|\ll|\alpha|x^{\sigma}+(1+|\gamma|)x^{\sigma-1}\ll\varkappa x^{\sigma}+Tx^{\sigma-1}\ll x^{\sigma-7/9}+x^{\sigma-0.6}\ll x^{\sigma-0.6}.$$

Therefore we have the relation

$$\int\limits_{x/4}^{3x/4} e^{2\pi i\alpha t}t^{\rho-1}\,dt=\sum_{n\in P_x}e^{2\pi i\alpha n}n^{\rho-1}+O(x^{\sigma-0.6}).$$

It follows that

$$S_1(\alpha)=\sum_{n\in P_x}e^{2\pi i\alpha n}-\sum_{|\gamma|<T}\sum_{n\in P_x}e^{2\pi i\alpha n}n^{\rho}+O(x^{-0.6})\sum_{|\gamma|<T}x^{\sigma}.$$

In other words, the relation

$$S_1(\alpha)=S_2+R_4$$

is valid, where the quantity $S_2=S_2(\alpha)$ includes the first two terms of the right-hand side of the preceding formula and R_4 is its remainder, with

$$R_4\ll x^{-0.6}\sum_{|\gamma|<T}x^{\sigma}.$$

Note that for the quantity $\sum_{|\gamma|<T}x^{\sigma}$ we have the well-known estimate which we present as the following lemma.

Lemma 9. *There exists a constant $c>0$ for which the estimate*

$$\sum_{|\gamma|<y^{0.4}}y^{\sigma}\ll ye^{-c(\ln y)^{0.2}}$$

is valid.

Proof. This lemma is a special case of the statement of Lemma 11.20, p. 210 in [9].

The use of Lemma 9 for the estimation of the remainder R_4 leads to the inequality

$$R_4\ll x^{0.4}\exp\big(-c(\ln^{0.2}x)\big).$$

Therefore, replacing the quantity S_1 in the expression for J_1 by the sum $S_1 = S_2 + R_4$, we have

$$J_1 = \int_{-\varkappa}^{\varkappa} S_2 VW\, d\alpha + \int_{-\varkappa}^{\varkappa} R_4 VW\, d\alpha + O(mx^{1-1/20})$$

$$= \int_{-\varkappa}^{\varkappa} S_2 VW\, d\alpha + O(x^{0.4} x m \varkappa) + O(mx^{1-1/20}) = \int_{-\varkappa}^{\varkappa} S_2 VW\, d\alpha + O(mx^{1-1/20}) = J_3 + R_5.$$

Here we denote by J_3 the next in turn form of the principal term for J_1 and by R_5 the next in turn remainder, with

$$J_3 = \int_{-\varkappa}^{\varkappa} S_2 VW\, d\alpha, \qquad R_5 \ll mx^{1-1/20}.$$

Next we extend the integration in J_3 from the interval $E_1 = [-\varkappa, \varkappa]$ to the interval $E = E_1 \cup E_2 = [-1/2, 1/2]$. To do this, we shall need the lemma on the estimation of trigonometric sum from the derivatuves of the function appearing in the exponent.

Lemma 10. *Suppose that in the interval $q < x \le r$ the function $f''(x)$ is continuous and satisfies the condition*

$$\frac{1}{A} \ll |f''(x)| \ll \frac{1}{A}.$$

In addition, let $\varphi(x)$ be monotonic and $\varphi(x) \ll H$.

Then the inequality

$$\sum_{q<x\le r} \varphi(x) e^{2\pi i f(x)} \ll H\left(\frac{r-q}{\sqrt{A}} + \sqrt{A} + \ln U\right),$$

where $U = \max(r-q, A)$, holds true.

If, moreover, the inequality

$$\delta \le |f'(x)| < 0.5$$

holds, then the estimate

$$\sum_{q<x\le r} \varphi(x) e^{2\pi i f(x)} \ll H\left(\frac{1}{\delta} + \ln(r-q)\right)$$

is valid.

Proof. This lemma is a direct corollary of Lemmas 1 and 1a [2, pp. 14–16].

Applying the estimate of Lemma 10 to the sum

$$\sigma_\rho = \sum_{n \in P_x} e^{2\pi i \alpha n} n^{\rho - 1}$$

for $\alpha \in E_2$, we have

$$\sigma_\rho = \sum_{n \in P_x} e^{2\pi(i\alpha n + \gamma \ln n)} n^{\sigma-1}, \qquad \varphi(n) = n^{\sigma-1},$$

$$f(n) = \alpha n + \gamma \ln n, \quad f'(n) = \alpha + \frac{\gamma}{n}, \quad f''(n) = -\frac{\gamma}{n^2}, \qquad A = \frac{n^2}{|\gamma|} U \ll \ln x.$$

Next, since $n \in P_x$, it follows that $x/4 \le n \le 3x/4$. In addition, by virtue of the condition $\alpha \in E_2$, we can assume that

$$x^{-7/9} < |\alpha| < \frac{1}{2}.$$

We divide the set E_2 into two parts, $E_2 = E_{21} \cup E_{22}$, referring to E_{21} the points α subjected to the condition $|\alpha| > 10|\gamma|/x$.

For the points $\alpha \in E_{21}$ we apply the first estimate of Lemma 10 to the sum σ_ρ. We must take into account that then

$$\frac{|\gamma|}{x} \gg |\alpha| > \varkappa = x^{-7/9},$$

i.e., $|\gamma| \gg x^{2/9}$, and therefore

$$A = \frac{n^2}{|\gamma|} \ll x^{2-2/9},$$

and, in addition, $A \gg x^{2-0.4}$. Consequently,

$$\sigma_\rho \ll x^{\sigma-1}\left(\frac{x}{x^{1-0.2}} + x^{1-1/9} + \ln x\right) \ll x^{\sigma-1/9}.$$

Now if $\alpha \in E_{22}$, i.e., if $|\alpha| > 10|\gamma|/x$, then

$$|f'(\alpha)| = \left|\alpha + \frac{\gamma}{n}\right| \ge |\alpha| - \frac{|\gamma|}{n} > 0.9|\alpha| \gg |\alpha|.$$

Therefore the second estimate of Lemma 8 leads to the inequality

$$\sigma_\rho \ll x^{\sigma-1}\left(\frac{1}{\|\alpha\| + \ln x}\right) \ll x^{\sigma-1}x^{7/9} = x^{\sigma-2/9}.$$

Thus, for all $\alpha \in E_2$ we have the estimate

$$\sigma_\rho \ll x^{\sigma-1/9}.$$

Obviously, for $\sigma = 1$ the same estimate is valid for the sum σ_1, where

$$\sigma_1 = \sum_{n \in P_x} e^{2\pi i \alpha n},$$

i.e.,

$$\sigma_1 \ll \frac{1}{\|\alpha\|} \ll x^{1-2/9}.$$

Then however an estiamte of the form

$$R_6 = \int\limits_{E_2} S_2 VW\, d\alpha \ll \max_{\alpha \in E_2} |S(\alpha)| \int\limits_0^1 |VW|\, d\alpha \ll \left(x^{1-2/9} + x^{-1/9} \sum_{|\gamma|<T} x^\sigma\right) \int\limits_0^1 |VW|\, d\alpha$$

holds true.

Applying the estimates obtained above for the quantities $\sum_{|\gamma|<T} x^\sigma$ and $\int_0^1 |VW|\,d\alpha$, we have the inequality

$$R_6 \ll x^{1-1/9} x^{1/2} m^{1/2}.$$

We have thus established the relations

$$J_1 = \int\limits_{-1/2}^{1/2} S_2 VW\,d\alpha + R_7 = J_4 + R_7,$$

where $R_7 \ll mx^{1-1/20} + m^{1/2} x^{3/2-1/9}$, for the quantity J_1.

Note now that the integral in the last expression for J_1 can be computed explicitly. Since the relation

$$\int\limits_0^1 e^{2\pi i\alpha z}\,d\alpha = \begin{cases} 1 & \text{for } z = 0 \\ 0 & \text{for } z \neq 0 \end{cases}$$

holds for the integer z, we have

$$J_4 = \int\limits_0^1 S_2 VW\,d\alpha = \sum_{\substack{n\in P_x,\ q\in Q_x \\ y\in Y_x,\ y\in\Omega \\ n+[\beta q]=y}} \ln q\biggl(1 + \sum_{|\gamma|<T} n^{\rho-1}\biggr).$$

By virtue of the conditions imposed on the intervals P_x, Q_x, and Y_x, every pair of values (q, y) is in one-to-one correspondence with its own value $n \in P_x$. We must also keep in mind that $n = y - [\beta q] \geq x/4 \gg x$. We have

$$\begin{aligned} J_4 &= \sum_{[\beta q]\in Q_x} m \ln q\biggl(1 + \sum_{|\gamma|<T} (y - [\beta q])^{\rho-1}\biggr) \\ &= m\biggl(\psi\Bigl(\frac{x}{2\beta}\Bigr) - \psi\Bigl(\frac{x}{4\beta}\Bigr)\biggr) + O(mx^{1/2}) + O\biggl(x^{-1} \sum_{|\gamma|<T} x^\sigma m\psi(x)\biggr). \end{aligned}$$

Here, as usual, we denote by $\psi(x)$ Chebyshev's function.

Next, by virtue of the asymptotic law of distribution of prime numbers and Lemma 9, we have the estimates

$$\psi\Bigl(\frac{x}{2\beta}\Bigr) - \psi\Bigl(\frac{x}{4\beta}\Bigr) \gg x, \qquad \sum_{|\gamma|<T} x^\sigma \ll x\exp(-c\ln^{0.2} x).$$

Consequently,

$$J_4 = J_5 + R_8,$$

where J_5 is a certain positive number, with

$$J_5 \gg mx,$$

and the estimate

$$R_8 \ll mx\exp(-c\ln^{0.2} x)$$

is valid for the remainder R_8.

The relations between the quantities J_1, J_4, J_5, R_7 and R_8 obtained above allow us, finally, to obtain the required estimate for the quantity $|J_1|$. We have

$$\begin{aligned}|J_1| &\geq |J_4| - |R_7| \geq |J_5| - |R_7| - |R_8| \\ &\gg mx + O(mx^{1-1/20}) + O(m^{1/2}) + O\big(mx\exp(-c\ln^{0.2} x)\big) \gg mx + O(m^{1/2}x^{3/2-1/9}).\end{aligned}$$

Consequently, using the obtained upper estimate for the quantity $|J_2| = |J_1|$, we have, in the case (a) of Theorem 1,

$$mx + O(m^{1/2}x^{3/2-1/9}) \ll |J_1| = |J_2| \ll m^{1/2}x^{3/2-1/9}\ln^4 x.$$

Hence

$$mx \ll m^{1/2}x^{3/2-1/9}\ln^4 x, \qquad m^{1/2} \ll x^{1/2-1/9}\ln^4 x, \qquad m \ll x^{1-2/9}\ln^8 x.$$

As was pointed out earlier, it follows from this estimate and from Lemma 1 that

$$T_1(x) \ll x^{1-2/9}\ln^8 x.$$

We have thus proved the case (a) of Theorem 1. In the same way, in the case (b) we arrive at the desired estimate of the form

$$T_1 \ll_\varepsilon x^{1-\varepsilon-2/9}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. Passing to the proof of Theorem 2, we must say that its derivation is simillar, in many respects, to the proof of Theorem 1, and therefore, in order to avoid repetitions, we shall make our further reasoning more schematic.

We shall again consider the singular set Ω_1 consisting of all natural n, which cannot be represented as

$$n = p + [q^c],$$

where p and q are prime numbers and c is a noninteger, and also the amount $m = m(x)$ of the numbers $y \in \Omega_1$, which belong to the interval $Y = Y_x = [3x/4, x]$ introduced above. According to Lemma 1, we shall have the estimate

$$T_2(x) \ll m.$$

Next, the Diophantine equation

$$y = p + [q^c]$$

has no solution for $y \in \Omega_1$. Nevertheless, we can formally express the number of its salutions, which is equal to zero, in terms of the integral of the trigonometric sum, i.e.,

$$0 = \int_0^1 SKW\,d\alpha = \int_{-\varkappa}^{1-\varkappa} SKW\,d\alpha.$$

The number $\varkappa > 0$ appearing in this realtion will be defined somewhat later. The sums S and W have the same sense as in Theorem 1 and the sum $K = K(\alpha)$ is defined as

$$K(\alpha) = \sum_{[q^c]\in Q_x} e^{2\pi i\alpha[q^c]}.$$

We shall change slightly the division of the integration interval into the points of the first class (E_1) and the points of the second class (E_2) setting

$$\varkappa = x^{-0.8}, \qquad E_1 = [-\varkappa, \varkappa), \qquad E_2 = [\varkappa, 1-\varkappa], \qquad E = E_1 \cup E_2 = [-\varkappa, 1-\varkappa].$$

We again denote by J_1 and J_2 the integrals

$$J_1 = \int\limits_{E_1} SKW\,d\alpha = \int\limits_{-\varkappa}^{\varkappa} SKW\,d\alpha, \qquad J_2 = \int\limits_{E_2} SKW\,d\alpha = \int\limits_{\varkappa}^{1-\varkappa} SKW\,d\alpha.$$

Then, obviously, the relations

$$0 = J_1 + J_2, \qquad |J_1| = |J_2|$$

hold true.

The general scheme of reasoning remains the same, i.e., we estimate the quantity $|J_2|$ from above and te quantity $|J_1|$ from below and derive, from these estimates, the required inequality for $m = m(x)$.

We begin with the quantity $|J_2|$. Applying the Cauchy inequality and using the estimate of the sum $K(\alpha)$, stipulated by the hypothesis of Theorem 2, we obtain

$$|J_2| = \left| \int\limits_{\varkappa}^{1-\varkappa} SKW\,d\alpha \right| \le \max_{\alpha\in E_2} |K(\alpha)| \int\limits_0^1 |SW|\,d\alpha$$

$$\ll x^{\gamma-\delta} \left(\int\limits_0^1 |S|^2\,d\alpha \int\limits_0^1 |W|^2\,d\alpha \right)^{1/2} = x^{\gamma-\delta+1/2}(m\ln x)^{1/2}.$$

At the same time, for the quantity $|J_1|$ we shall try to obtain the lower estimate of the form

$$|J_1| \gg x^{\gamma} m.$$

For this purpose, we transform the sum $S = S(\alpha)$ for $\alpha \in E_1$ so that we can express it in terms of the sum over zeros of Riemann's zeta function like we did when proving Theorem 1. The only difference is in the choice of the value of parameter $T = x^a$. Here we set $a = 0.41$. It is important to note that in this case we have the inequalities

$$a = 0.41 < \frac{5}{12} = 0.41(6).$$

As a result, we obtain a relation of the form

$$S(\alpha) = \int\limits_{x/4}^{3x/4} e^{2\pi i\alpha t}\,dt - \sum_{|\gamma|<T} \int\limits_{x/4}^{3x/4} t^{\rho-1} e^{2\pi i\alpha t}\,dt + R_1 = S_0(\alpha) + R_1,$$

and the estimate

$$R_1 \ll \frac{|\alpha| x^2 \ln^2 x}{T}$$

is valid for the remainder R_1.

Then, for the quantity J_1 we have the relation

$$J_1 = \int_{-\varkappa}^{\varkappa} SKW\,d\alpha = \int_{-\varkappa}^{\varkappa} S_0 KW\,d\alpha + R_2,$$

where

$$R_2 = \int_{-\varkappa}^{\varkappa} R_1 KW\,d\alpha \ll x^{\gamma} m \frac{x^2 \ln^2 x}{T} \int_0^{\varkappa} \alpha\,d\alpha$$
$$\ll x^{\gamma+2} \ln^2 x m x^{-a} \varkappa^2 = x^{\gamma+2-0.41-1.6} \ln^2 x m \ll x^{\gamma} m x^{-\varepsilon},$$

Here $\varepsilon > 0$ is a positive constant.

Next we have to replace each of the integrals appearing in $S_0(\alpha)$ by a trigonometric sum and estimate the error introduced by this substitution into the integral J_1. Reasoning as before, we have

$$\int_{x/4}^{3x/4} e^{2\pi i\alpha t}\,dt = \sum_{n\in P_x} e^{2\pi i\alpha n} + O(1), \qquad \int_{x/4}^{3x/4} e^{2\pi i\alpha t} t^{\rho-1}\,dt = \sum_{n\in P_x} e^{2\pi i\alpha n} n^{\rho-1} + O(x^{\sigma-1}).$$

Consequently,

$$S_0(\alpha) = S_1(\alpha) + R_3,$$

where

$$S_1(\alpha) = \sum_{n\in P_x} e^{2\pi i\alpha n} - \sum_{n\in P_x}\sum_{|\gamma|<T} e^{2\pi i\alpha n} n^{\rho-1} = L_1 - \sum_{|\gamma|<T} L_\rho, \qquad R_3 \ll 1 + \sum_{|\gamma|<T} x^{\sigma-1}.$$

Note now that the estimate of the sum $\sum_{|\gamma|<T} x^{\sigma-1}$ of the form

$$\sum_{|\gamma|<T} x^{\sigma-1} \ll \exp(-c \ln^{0.2} x)$$

that we used in proving Theorem 1 holds for any $a < 5/12$. This fact is stated in Lemma 11.20, p. 210 in [9].

Therefore we have $R_3 \ll 1$,

$$\int_{-\varkappa}^{\varkappa} S_0 KW\,d\alpha = \int_{-\varkappa}^{\varkappa} S_1 KW\,d\alpha + O\Big(\int_{-\varkappa}^{\varkappa} |KW|\,d\alpha\Big) = \int_{-\varkappa}^{\varkappa} S_1 KW\,d\alpha + O\big(x^{-0.8+\gamma} m\big),$$

whence it follows that

$$J_1 = \int_{-\varkappa}^{\varkappa} S_1 KW\,d\alpha + O(x^{\gamma} m x^{-\varepsilon}),$$

where $\varepsilon > 0$ is a constant.

We shall now estimate the error which arises when we extend the integration domains in the last integral to the whole set E_2. Here we have an estimate of the form

$$\biggl|\int\limits_{\varkappa}^{1-\varkappa} S_1KW\,d\alpha\biggr| \le \max_{\alpha\in E_2}|K(\alpha)|\int\limits_0^1 |S_1W|\,d\alpha \le x^{\gamma-\delta}\Bigl(\int\limits_0^1 |L_1W|\,d\alpha + \sum_{|\gamma|<T}\int\limits_0^1 |L_\rho W|\,d\alpha\Bigr),$$

$$\int\limits_0^1 |L_\rho W|\,d\alpha \ll \Bigl(\int\limits_0^1 |L_\rho|^2\,d\alpha\int\limits_0^1 |W|^2\,d\alpha\Bigr)^{1/2} \ll \bigl(x^{2\sigma-1}m\bigr)^{1/2} = x^{\sigma-1}x^{1/2}m^{1/2},$$

$$\int\limits_0^1 |L_1W|\,d\alpha \ll \Bigl(\int\limits_0^1 |L_1|^2\,d\alpha\int\limits_0^1 |W|^2\,d\alpha\Bigr)^{1/2} \ll x^{1/2}m^{1/2}.$$

However, since

$$\sum_{|\gamma|<T} x^{\sigma-1} \ll \exp(-c\ln^{0.2}x) \ll 1,$$

it follows that

$$\biggl|\int\limits_{\varkappa}^{1-\varkappa} S_1KW\,d\alpha\biggr| \ll x^{1/2}m^{1/2}x^{\gamma-\delta}.$$

Thus we arrive at a relation of the form

$$J_1 = \int\limits_{-\varkappa}^{1-\varkappa} S_1KW\,d\alpha + O\bigl(x^\gamma mx + x^{1/2}m^{1/2}x^{\gamma-\delta}\bigr) = J_4 + R_4,$$

where J_4 denotes the integral and R_4 denotes the remainder in the last formula.

Carrying out direct termwise integration in J_4, we get the relation

$$J_4 = \sum_{[q^c]\in Q_x}\ \sum_{y\in\Omega_1,\ y\in Y_x}\Bigl(1 - \sum_{|\gamma|<T}(y-[q^c])^{\rho-1}\Bigr).$$

Next, since the inequalities

$$x \ll y-[q^c] \ll x$$

are valid for all $y\in Y_x$ and $[q^c]\in Q_x$, the inner sum is estimated as

$$\sum_{|\gamma|<T}(y-[q^c])^{\rho-1} \ll \sum_{|\gamma|<T}x^{\sigma-1} \ll \exp(-c\ln^{0.2}x).$$

Consequently, we have

$$|J_4| \ge \sum_{[q^c]\in Q_x}\ \sum_{y\in\Omega_1,\ y\in Y_x}\bigl(1+O\bigl(\exp(-c\ln^{0.2}x)\bigr)\bigr) = x^\gamma m\bigl(1+O\bigl(\exp(-c\ln^{0.2}x)\bigr)\bigr).$$

Therefore

$$|J_1| \ge |J_4| - |R_4| \gg x^\gamma m + O\bigl(x^\gamma m\exp(-c\ln^{0.2}x) + x^{1/2}m^{1/2}x^{\gamma-\delta}\bigr).$$

Using the upper estimate for $|J_2| = |J_1|$ obtained earlier, we have the following chain of inequalities:

$$x^\gamma m + O\big(x^\gamma m \exp(-c\ln^{0.2} x) + x^{1/2}m^{1/2}x^{\gamma-\delta}\big) \ll |J_1| = |J_2| \ll x^{\gamma-\delta}m^{1/2}x^{1/2}\ln^{1/2} x,$$
$$m^{1/2} \ll x^{1/2-\delta}\ln^{1/2} x, \qquad m \ll x^{1-2\delta}\ln x, \qquad T_2(x) \ll m \ll x^{1-2\delta}\ln x \ll x^{1-2\delta+\varepsilon},$$

where $\varepsilon > 0$ is arbitrarily small.

We have thus completed the proof of Theorem 2.

Two remarks are due in conclusion.

Remark 1. Actually, when proving Theorem 2 we established a slightly more exact result, namely,

$$T_2(x) \ll m \ll x^{1-2\delta}\ln x.$$

Remark 2. Theorem 2 reduced the problem of power lowering in the estimate of the power of the singular set for the binary additive problem of the representation of the numbers n as

$$n = p + [q^c]$$

to the power lowerings in the estimates of trigonometric sums with prime numbers $V(\alpha)$ on a set of points of the second class. These estimates can actually be obtained by means of the application of Lemma 4, Vinogradov's method of estimating trigonometric sums with prime numbers, and different methods of estimating sums over the entire summation interval such, for instance, as the estimates of the sums with respect to the kth derivative. It should be pointed out that the problem of finding the best estimates for the sums $V(\alpha)$ on the whole range of values of the noninteger parameter $c > 1$ is both very difficult and very cumbersome. Nevertheless, the realization of the scheme presented above makes it possible to obtain concrete values of the exponent δ of power lowering in the estimates of these sums.

Now we shall consider the principal moments of obtaining these estimates.

Recall that we denoted by $V(\alpha)$ the sum

$$V = V(\alpha) = \sum_{x/4 \le [q^c] \le x/2} e^{2\pi i\alpha[q^c]}\ln q.$$

Here q runs through a set of prime numbers and α satisfies the condition $x^{-0.8} \le \alpha \le 1 - x^{-0.8}$. It is obvious that in order to estimate the modulus $|V|$ it suffices to restrict ourselves to the values of α from the interval $x^{-0.8} \le \alpha \le 1/2$. The replacement of the coefficient $\ln q$ by $\Lambda(n)$ in the sum $V(\alpha)$ gives an error of order $O(x^{\gamma/2})$, i.e.,

$$V_0 = V_0(\alpha) = \sum_{[n^c]\in Q_x} \Lambda(n)e^{2\pi i\alpha[n^c]} = V(\alpha) + O(x^{\gamma/2}).$$

The application of Lemma 4 to the sum $V_0(\alpha)$ leads to the estimate

$$V_0(\alpha) \ll \sum_{k=0}^{N} \frac{\alpha}{\alpha+k}|T(\alpha+k)| + \frac{\ln N}{N}\sum_{k=1}^{M_0}|T(k)| + \alpha\frac{x^\gamma \ln N}{N},$$

where N is an arbitrarily chosen positive integer, and $M_0 \ll N\ln N \ll M_0$. It is expedient to choose the quantity N such that the estimate

$$\frac{\ln N}{N} \ll x^{-\delta}, \qquad N \ll x^\delta \ln x,$$

should be satisfied, where $\delta > 0$ is the power exponent of the expected lowering in the estimate of the sum $V(\alpha)$ of the form

$$V(\alpha) \ll x^{\gamma-\delta}.$$

The sum $T(\lambda)$ for $\lambda = \alpha + k$ or $\lambda = k$ is defined as

$$T(\lambda) = \sum_{n \in Q_x} \Lambda(n) e^{2\pi i \lambda n^c}.$$

The first step in the estimation of the sum $T(\lambda)$ is to use Vinogradov's sieve method. We use its version based on Vaughan's combinatorial identity which constitutes the content of the next lemma.

Lemma 11. *Let $1 \le u < N$, and then the identity*

$$\sum_{u<n\le N} \Lambda(n) f(n) = \sum_{d\le u} \mu(d) \sum_{l\le Nd^{-1}} \ln l \times f(ld) - \sum_{d\le u} \mu(d) \sum_{n\le u} \Lambda(n) \sum_{r\le N(dn)^{-1}} f(ndr)$$
$$- \sum_{n<m\le Nu^{-1}} \Biggl(\sum_{\substack{d|m \\ d\le u}} \mu(d) \Biggr) \sum_{u<n\le Nm^{-1}} \Lambda(n) f(mn)$$

is valid for any complex-valued function $f(n)$.

Proof. See Problem 9 to Ch. III, p. 60, of the book [13].

Here are two more auxiliary statements concerning the estimation of trigonometric sums from the derivatives of functions appearing in the exponent.

Lemma 12. *Let $k \ge 2$, $K = 2^{k-1}$, and let the function $f(x)$ have the kth continuous derivative on the interval $[a, b]$, with*

$$0 < \lambda_k \le f^{(k)}(\xi) \le h\lambda_k \qquad \text{for all} \quad \xi \in [a, b].$$

Then the estimate

$$\Biggl| \sum_{a<n\le b} e^{2\pi i f(n)} \Biggr| \ll (b-a)\lambda_k^{1/(2K-2)} h^{2/K} + (b-a)^{1-2/K} \lambda_k^{-1/(2K-2)}$$

holds true, where the constant under the sign $\ll$ is absolute.

Proof (see [13, p. 27, Theorem 5]).

Lemma 13. *Suppose that in the hypothesis of Lemma 12 we have estimates $h \ll 1 \ll h$, $(b-a)^{-2} \ll \lambda_k \ll (b-a)^{-1}$, and let $k \ge 9$. Then, for the sum S of Lemma 12 we have the esimate*

$$|S| \ll (b-a)\lambda_k^{\omega(k)},$$

where $\omega(k) = 1/(4k^2 \ln k)$.

Proof. This lemma is a simple corollary of Theorem 9.1, p. 109 from [14].

The following statement can be immediatly derived from Lemmas 12 and 13.

Corollary. *Suppose that for a certain natural $k \geq 3$ the inequality*

$$t^{-2} \ll f^{(k)}(t) \ll t^{-1}, \qquad t \ll b-a \ll t$$

holds for the function $f(t)$ satisfying the conditions of Lemma 12. Then the estimate

$$|S| \ll t^{1-\Delta},$$

where

$$\Delta = \begin{cases} \dfrac{1}{4k^2 \ln k} & \text{for } k \geq 9 \\ \dfrac{1}{2^{k-1}-2} & \text{for all } k \geq 3, \end{cases}$$

is valid for the trigonometric sum S of this lemma.

Let us now consider the estimate of the sum $T(\lambda)$.

Running somewhat ahead, we can state that our method does not make it possible to obtain values of δ exceeding 0.2, and therefore we shall assume at once that $\delta \leq 0.2$.

In Lemma 11 we choose the value $u = t^{0.25}$ of the parameter and the number $t = N$ satisfying the condition $N = t \ll x^\gamma \ll N$. We set $f(n) = e^{2\pi i\lambda n^c}$ and denote the appearing trigonometric sum by $T_1(\lambda)$. It is easy to see that the sum $T(\lambda)$ can be expressed as the difference of two sums $T_1(\lambda)$, and therefore the estimate

$$|T(\lambda)| \ll |T_1(\lambda)|$$

holds true.

Applying Lemma 11 to $T_1(\lambda)$, we obtain

$$\begin{aligned} T_1(\lambda) &= \sum_{d\leq u}\mu(d)\sum_{l\leq Nd^{-1}} \ln l \times e^{2\pi i\lambda l^c d^c} - \sum_{d\leq u}\mu(d)\sum_{n\leq u}\Lambda(n)\sum_{r\leq N(dn)^{-1}} e^{2\pi i\lambda n^c d^c r^c} \\ &\quad - \sum_{n<m\leq Nu^{-1}}\Bigg(\sum_{\substack{d|m\\ d\leq u}}\mu(d)\Bigg)\sum_{u<n\leq Nm^{-1}}\Lambda(n)e^{2\pi i\lambda m^c n^c} \\ &= \Sigma_1 - \Sigma_2 - \Sigma_3, \end{aligned}$$

where the sums $\Sigma_1, \Sigma_2, \Sigma_3$ are defined, in a natural way, by the last relation. The required estimate of the sum $T_1(\lambda)$ follows from the estimates of these three sums.

We shall begin with considering the sum Σ_2. We arbitrarily fix in it the summation variables d and n and divide the enitre summation interval with respect to r into $\ll \ln t$ intervals of the form $t_1 < r \leq t_2 \ll t_1$.

If it turns out that $t_1 \leq tx^{-\delta}u^{-2}$, then the sum over r trivially, by the number of term. Obviously, the contribution σ_{21} of these sums into the estimate of the sum Σ_2 satisfies the estimate

$$\sigma_{21} \ll \sum_{d\leq u}\sum_{n\leq u}\Lambda(n)\sum_{r\leq tx^{-\delta}u^{-2}} \ll u^2tx^{-\delta}u^{-2} \ll x^{\gamma-\delta}.$$

In order to estimate the contribution σ_{22} of the remaining part of the sum Σ_2, we apply, to the summation over r on each interval, the corollary of Lemmas 12 and 13. We must bear in mind that in this interval the function $f(ndr) = \lambda n^c d^c r^c$ satisfies the inequalities

$$\lambda(ndt_1)^c \ll f(ndr) = \lambda(ndr)^c \ll \lambda(ndt_1)^c \ll x^{1+\gamma}\ln^2 x.$$

In addition,

$$x^{-\delta+\gamma/2} \ll t_1 \ll r \ll t_1 \ll x^{\gamma}(nd)^{-1}, \qquad \gamma = \frac{1}{c} < 1,$$

and, for any natural k, we have

$$\lambda(nd)^c t_1^{c-k} \ll \lambda_k \ll \lambda(nd)^c t_1^{c-k} \ll x^{1+\delta} t_1^{-k} \ln^2 x.$$

Consequently, there exists $k = k_0 \geq 1$ such that the inequalities

$$t_1^{-2} \ll \lambda_k \ll t_1^{-1}$$

hold true, and then, according to the corollary of Lemmas 12 and 13, for every sum over r we get an estimate of the form

$$H = H(t_1) = \sum_{t_1 < r \leq t_2} e^{2\pi i \lambda (ndr)^c} \ll t_1^{1-\Delta} = t_1 \times t_1^{-\Delta} \ll t_1 x^{-\Delta(-\delta+\gamma/2)},$$

where

$$\Delta = \Delta(k_0) = \begin{cases} \dfrac{1}{4k_0^2 \ln k_0} & \text{for } k_0 \geq 9 \\ \dfrac{1}{2^{k_0} - 2} & \text{for } k_0 \geq 3. \end{cases}$$

Note that the inequalities indicated above make it possible to obtain the lower estimate for $k = k_0$. Indeed, we have

$$x^{1+\delta} t_1^{-k} \ln^2 x \gg t_1^{-2}, \qquad x^{1+\delta} \ln^2 x \gg t_1^{k-2} \gg x^{(-\delta+\gamma/2)(k-2)}.$$

By means of the inequality $\delta < \gamma/6$ we impose one more constraint on the quantity δ. Then we get

$$x^{1+\delta} \ln^2 x \gg x^{\gamma(k-2)/3}, \qquad 1 + \delta \geq \frac{\gamma(k-2)}{3},$$

$$k - 2 \leq 3c(1+\delta) < 3c\frac{7}{6} = 3.5c, \qquad k_0 = k < 3.5c + 2.$$

By virtue of the monotone decrease of the quantity $\Delta(k)$, it follows that for $c > 2$ the inequalities

$$3.5c + 2 > 9, \qquad \Delta = \Delta(k) > \frac{1}{(7c+4)^2 \ln(3.5c+2)},$$

$$\Delta\Big(-\delta + \frac{\gamma}{2}\Big) > \Delta\frac{\gamma}{3} > \frac{1}{3c(7c+4)^2 \ln(3.5c+2)} = \delta_1$$

hold true.

Now if we only have the inequality $c > 1$, then

$$\Delta = \Delta(k) > \frac{1}{2^{3.5c+2} - 2}, \qquad \Delta\Big(-\delta + \frac{\gamma}{2}\Big) > \Delta\frac{\gamma}{3} > \frac{1}{6c(2^{3.5c+1} - 1)} = \delta_2.$$

Note that for $1 < c < 2$ we always have $\delta_2 < \delta_1$, and therefore, setting $\delta_3 = \max(\delta_1, \delta_2)$, we obtain the estimate

$$H = H(t_1) \ll t_1 x^{-\delta_3}.$$

Summing up this estimate over all intervals of variation of the summation variable r, we get

$$\sigma_{22} \ll \sum_{d\le u}\sum_{n\le u} \Lambda(n) x^{\gamma-\delta_3}(nd)^{-1} \ll x^{\gamma-\delta_3}\ln^2 x.$$

It follows that

$$\Sigma_2 \ll \sigma_{21}+\sigma_{22} \ll x^{\gamma-\delta_3}\ln^2 x.$$

We have thus obtained the estimate for the sum Σ_2.

As to the estimate of the sum Σ_1, by virtue of the fact that the summation variable l runs in it through the entire interval of the lenght not less than $t^{3/4}$, the repetition of the arguments just given in the connection with this sum leads to a better estimate than that obtained for Σ_2 since there the entire summation interval over r has the length of the order exceeding $t^{1/2}$. The presence of the factor $\ln l$ in that sum is really not an essiential obstacle since the standard application of a partial summation makes it possible to get rid of this factor. Thus it remains to rstimate the sum Σ_3.

This sum does not include any variables over the entire summation intervals at all, and therefore we use Vinogradov's method to "smoothe" the multiple sums. However, we shall first divide Σ_3 into two parts as fallows:

$$\Sigma_3 = \sum_{\substack{m>u\\ mn\le t}}\sum_{n>u}\left(\sum_{\substack{d|m\\ d\le u}}\mu(d)\right)\Lambda(n)e^{2\pi i\lambda m^c n^c} = \sum_{u<m\le n}\cdots + \sum_{u<n\le m}\cdots = \sigma_{31}+\sigma_{32}.$$

Both sums σ_{31} and σ_{32} can be estimates in the same way, and therefore we restrict our consideration to the sum σ_{31}.

We can write the sum σ_{31} in the form

$$\sigma_{31} = \sum_{u<n<tu^{-1}} \Lambda(n) \sum_{u<m\le v} a(m) f(mn),$$

where

$$v=v(n)=\min(n, Nn^{-1}), \qquad f(mn)=e^{2\pi i\lambda n^c m^c}, \qquad a(m)=\sum_{\substack{d|m\\ d\le u}}\mu(d),$$

with $|a(m)|\le \tau(m)=\sum_{d|m}1$.

We can divide the summation intervals over n into intervals of the form $t_1<n\le t_2\ll t_1$ which are $\ll \ln x$ in number. The part of the sum σ_{31} corresponding to one of these intervals will be denoted by σ. To make the notation more conveinient, we shall suppose that the point $t_0=t^{0.5}$ is an endpoint of two adjacent intervals. Then, for the sum σ, we can write one of the relations

$$\sigma = \sum_{t_1<n\le t_2} \Lambda(n) \sum_{u<m\le n} a(m) f(mn)$$

if $t_2\le t_0$, or

$$\sigma = \sum_{t_1<n\le t_2} \Lambda(n) \sum_{u<m\le tn^{-1}} a(m) f(mn)$$

if $t_1>t_0=t^{0.5}$.

Both cases can be analyzed quite similarly, and therefore we shall restrict the consideration to only the second case since it is more cumbersome.

Applying the Cauchy inequality to the sum σ, we obtain

$$|\sigma|^2 \leq \sum_n \Lambda^2(n) \sum_n \left| \sum_m a(m) f(mn) \right|^2 \ll t_1 \ln x \sum_{t_1 < n \leq t_2} \sum_{u < m_1, m_2 \leq tn^{-1}} a(m_1) a(m_2) e^{2\pi i \lambda n^c (m_1^c - m_2^c)}$$

$$\ll t_1 \ln x \left(t_1 \sum_{u \leq m \leq tt_1^{-1}} a^2(m) + \sum_{\substack{u \leq m_1, m_2 \leq tt_1^{-1} \\ m_1 \neq m_2}} |a(m_1)| \times |a(m_2)| \sum_{t_1 < n \leq v_1} e^{2\pi i \lambda n^c (m_1^c - m_2^c)} \right),$$

where $v_1 = \min(t_2, tm_1^{-1}, tm_2^{-1})$.

The first sum can be estimated with the aid of the well-known inequality of the form

$$\sum_{m \leq y} \tau^2(m) \ll y \ln^3 y$$

by the quantity $tt_1 \ln^4 x$.

We reduce the estimation of the second sum to the estimation of its inner sum over n with the aid of the corollary of Lemmas 12 and 13. We have here the relations

$$n > t_1 > t_0 = t^{0.5}, \qquad \lambda n^c (m_1^c - m_2^c) \ll x^{1+\delta} \ln x.$$

For fixed m_1 and $m_2 \neq m_1$ we set

$$f(n) = \lambda n^c (m_1^c - m_2^c).$$

Then, for the kth derivative of the function $f(n)$, we have the inequalities

$$\lambda n^{c-k} |m_1^c - m_2^c| \ll f^{(k)}(n) \ll \lambda n^{c-k} |m_1^c - m_2^c| \ll x^{1+\delta} t_1^{-k} \ln x.$$

Consequently, we can state that there exists a positive integer $k = k_0$ for which the inequalities

$$t_1^{-2} \ll f^{(k)}(n) \ll t_1^{-1}$$

are satisfied. For such a k we have the estimates

$$\sum_{t_1 < n \leq v_1} e^{2\pi i f(n)} \ll t_1^{1-\Delta} \ll t_1 x^{-\Delta\gamma/2},$$

where the quantity Δ is defined in the same way as in the estimation of the sum Σ_2.

Next we have to estimate the quantity $k = k_0$ from below. Indeed, for such a k the condition

$$x^{1+\delta} t_1^{-k} \ln x \gg f^{(k)}(n) \gg t_1^{-2}$$

is fulfilled, and, consequently,

$$x^{1+\delta} \ln x \gg t_1^{k-2} \gg x^{\gamma(k-2)/2},$$

whence it follows that

$$1 + \delta \geq \frac{\gamma(k-2)}{2}, \qquad k - 2 \leq 2c(1+\delta) < \frac{7c}{3}, \qquad k < \frac{7c}{3} + 2.$$

In accordance with the corollary of Lemmas 12 and 13, for such a k the lowering $\gamma\Delta/2 = \delta_4$ satisfies the condition

$$\delta_4 = \frac{\Delta}{2c} > \delta_5 \qquad \text{for} \quad c > 3,$$

where

$$\delta_5 = \frac{1}{2c \times 4k^2 \ln k} > \frac{1}{8c(7c/3+2)^2 \ln(7c/3+2)},$$

or

$$\delta_4 = \frac{\Delta}{2c} = \delta_6 = \frac{1}{2c(2^{k_0}-2)} > \frac{1}{2c(2^{7c/3+2}-2)}$$

for all nonintegers $c > 1$.

Combining the estimates obtained for the contribution to the quantity $|\sigma|^2$ for its first and second terms, we get an inequality of the form

$$|\sigma|^2 \ll tt_1 \ln^4 x + t_1^2 x^{-\delta_4} \ln x \sum_{\substack{m_1 \neq m_2 \\ m_1, m_2 \leq tt_1^{-1}}} \tau(m_1)\tau(m_2) \ll tt_1 \ln^4 x + t^2 x^{-\delta_4} \ln^3 x$$

$$\ll t^2 \ln^3 x (u^{-1} \ln x + x^{-\delta_4}) \ll t^2 x^{-\delta_4} \ln^3 x \ll x^{2\gamma - \delta_4} \ln^3 x.$$

Consequently,

$$|\sigma| \ll x^{\gamma - \delta_4/2} \ln^{3.5} x.$$

Summing up over all intervals of variation of the variable n, we have, for Σ_3, an estiamte of the form

$$\Sigma_3 \ll x^{\gamma - \delta_4/2} \ln^{2.5} x \ll x^{\gamma - \delta_7/2},$$

where the number $\delta_7 > 0$ satisfies the same inequalities as δ_4.

Comparing the estimates obtained for the sums $\Sigma_1, \Sigma_2, \Sigma_3$, we arrive at the inequality

$$|T(\lambda)| \ll |T_1(\lambda)| \ll x^{\gamma - \delta_7/2}.$$

Let us substitute this estimate into the expression for $V_0(\alpha)$. As a result we get

$$V_0(\alpha) \ll x^{\gamma - \delta_7/2} \ln x + x^{\gamma - \delta_7/2} \ln^2 x + x^{\gamma - \delta}.$$

Since the quantity δ_7, just as the quantity δ_4, satisfies srict inequalities, the inequality

$$V_0(\alpha) \ll x^{\gamma - \delta_8/2} + x^{\gamma - \delta}$$

is satisfied for a certain δ_8. If we choose the quantity $\delta = \delta_8/2$, as the value of δ, then the sums $V_0(\alpha)$ and $V(\alpha)$ will be estimated as

$$V_0(\alpha) \ll x^{\gamma - \delta} \qquad \text{and} \qquad V(\alpha) \ll x^{\gamma - \delta}.$$

We have thus established the following result.

Theorem 3. *There exists a constant $\delta > 0$ such that*

(1) *for the sum $V(\alpha)$ of the form*

$$V(\alpha) = \sum_{[q^c]\in Q_x} e^{2\pi i\alpha[q^c]} \ln q,$$

where q runs through successive prime numbers, the estimate

$$V(\alpha) \ll x^{\gamma-\delta}$$

holds true. Here, for the noninteger $c > 3$, the number $\delta > 0$ satisfies the inequality

$$\delta > \frac{1}{16c(7c/3+2)^2 \ln(7c/3+2)}$$

and one more inequality, namely,

$$\delta > \frac{1}{4c(2^{7c/3+2}-2)},$$

is valid for all nonintegers $c > 1$,

(2) *we have the estimate of the power $T_2(x)$ of the set of positive integers $n \le x$ which cannot be represented as $n = p + [q^c]$, where p and q are prime numbers, $c > 1$ is noninteger:*

$$T_2(x) \ll x^{1-2\delta}.$$

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