(Ramanujan-style mathematics for) Mahler measures

Wadim Zudilin
(based on joint work with Mat Rogers)

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Logarithmic Mahler measure

For a Laurent polynomial $P(x_1,\dots,x_n)\in\mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}]$ in n variables, the *logarithmic Mahler measure* is defined as

$$m(P)=\int_0^1\cdots\int_0^1\log|P(e^{2\pi it_1},\dots,e^{2\pi it_n})|\,\mathrm{d}t_1\cdots\mathrm{d}t_n$$

and its *Mahler measure* as $M(P)=e^{m(P)}$, the geometric mean of $|P|$ on the torus $\{(x_1,\dots,x_n)\in\mathbb{C}^n:|x_1|=\dots=|x_n|=1\}$.
The one-variate Mahler measure has important applications in transcendental number theory which I will not discuss in the talk.
In case $n=1$ where one has a simpler alternative expression

$$m(P)=\log|a_0|+\sum_{j=1}^{d}\max\{0,\log|\alpha_j|\}$$

for a polynomial $P(x)=a_0\prod_{j=1}^{d}(x-\alpha_j)$, thanks to Jensen's formula.

Lehmer's problem

For polynomials $P(x)$ with integer coefficients, clearly $m(P) \geq 0$ with $m(P) = 0$ only if P is monic ($a_0 = 1$) and has all its zeros inside the unit circle (hence is a product of a monomial x^a and a cyclotomic polynomial, by Kronecker's theorem).

D. Lehmer asked (already in 1933) whether $m(P)$ can be arbitrary small but positive for $P(x) \in \mathbb{Z}[x]$; the smallest value he was able to find was

$$\begin{aligned} m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) &= \log(1.17628081\dots) \\ &= 0.16235761\dots . \end{aligned}$$

This still stands as the smallest positive value of $m(P)$, in spite of extensive computation by D. Boyd, M. Mossinghoff and others. Although Lehmer's question is a completely different story in the study of Mahler measures, it inspired the above definition of $m(P)$ to the multi-variable case because of the following limit formula proven by Boyd in 1981:

$$m(P(x, x^N)) \to m(P(x, y)) \quad \text{as } N \to \infty.$$

Smyth's evaluation

It was not realised until 1981 that the multivariable Mahler measure could have some "geometric" roots. Namely, C. Smyth gave an elegant formula

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

where

$$L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \cdots$$

is the L-function attached to the real odd Dirichlet character modulo 3.

Smyth's evaluation

The proof of Smyth's formula is worth noting. Since $1+x+y$ is a linear function of y, Jensen's formula applied to one of the integrals in the double integral defining the corresponding Mahler measure shows that

$$\begin{aligned} m(1+x+y) = m(1-x+y) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+|e^{it}-1|\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_0^{2\pi/3} \log|e^{it}-1|\,\mathrm{d}t \end{aligned}$$

where $\log^+ x = \max\{0, \log x\}$. Thus, $m(1+x+y)$ is given by a special value of the Clausen integral

$$\mathrm{Cl}_2(\theta) = -\int_0^{\theta} \log|e^{it}-1|\,\mathrm{d}t = \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{k^2},$$

and the result follows.

L-series evaluations of Mahler measures

A similar computation applies to many polynomials $P(x,y)=A(x)y+B(x)$ if $A(x)$ and $B(x)$ are cyclotomic and if the solutions of $|A(x)|=|B(x)|$ on $|x|=1$ are roots of unity.
For example,

$$m(1+x+y-xy)=\frac{2}{\pi}L(\chi_{-4},2)=L'(\chi_{-4},-1),$$

where $L(\chi_{-4},2)=G$ is Catalan's constant,

$$\begin{aligned} m(1+x+x^2+y)&=\frac{3}{2}L'(\chi_{-4},-1),\\ m(1+x+y+x^2y)&=\frac{3}{2}L'(\chi_{-3},-1). \end{aligned}$$

L-functions of elliptic curves

It is not completely mysterious to expect that more sophisticated polynomials $P(x, y)$ give rise to analogous Mahler measures expressed through special values of L-functions of elliptic curves.
Here the counterpart to

$$\frac{d^{3/2}L(\chi_{-d}, 2)}{4\pi} = L'(\chi_{-d}, -1)$$

is given by

$$b_E = \frac{NL(E, 2)}{4\pi^2} = L'(E, 0)$$

where N is the conductor of the elliptic curve E and where the latter equality is only valid if E is a *modular* curve (that is, a smooth cubic curve over $\mathbb{Q}$ that has a rational point; the Shimura–Taniyama conjecture — the theorem now — says that all elliptic curves over $\mathbb{Q}$ are modular). In other words, there exist polynomials $P_E(x, y)$ for which $m(P_E)/b_E$ is (presumably) rational.

Deninger's example

I am not going to explain deep K-theoretic reasons for expecting such formulae to exist, but to provide one example in this direction (due to C. Deninger) and to support it by several other instances.
Consider

$$P(x,y) = 1 + x + \frac{1}{x} + y + \frac{1}{y}.$$

Let $x = e^{it}$ and treat $P(x,y)$ as a polynomial in y to see that

$$|P(x,y)| = |1 + y(1 + 2\cos t) + y^2| = |(y - y_1(t))(y - y_2(t))|,$$

where $y_1(t) = -b - \sqrt{b^2 - 1}$ with $b = b(t) = \frac{1}{2} + \cos t$. With the help of Jensen's formula,

$$m(P) = \frac{1}{\pi}\int_0^\pi \log^+ |y_1(t)|\,\mathrm{d}t.$$

Since the product of the roots $y_1(t)$ and $y_2(t)$ is 1, we will have $|y_1(t)| > 1 > |y_2(t)|$ exactly when the roots are real and unequal, that is, when $\cos t > \frac{1}{2}$, so $|t| < \frac{\pi}{3}$.

Deninger's example

Thus

$$m(P) = \int_0^{\pi/3} \log(b + \sqrt{b^2 - 1})\,\mathrm{d}t.$$

This integral can be integrated numerically but, of course, there are various other ways to represent it, for example,

$$\begin{aligned} m(P) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} \log(1 + 2\cos t + 2\cos s)\,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{4} \cdot {}_3F_2\left(\left.\begin{matrix} \frac12, \frac12, \frac12 \\ 1, \frac32 \end{matrix}\right| \frac{1}{16}\right) = 4 \sum_{k=0}^{\infty} \binom{2k}{k}^2 \frac{(1/16)^{2k+1}}{2k+1}. \end{aligned}$$

Here I use a standard notation for the hypergeometric series,

$${}_mF_{m-1}\left(\left.\begin{matrix} a_1, a_2, \dots, a_m \\ b_2, \dots, b_m \end{matrix}\right| z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ denotes the Pochhammer symbol.

K-theoretical interpretation

On the other hand, using a cohomological interpretation of $m(P(x,y))$, Deninger was able to evaluate $m(P)$ as an Eisenstein–Kronecker series of the elliptic curve E of conductor 15 given by

$$1 + x + \frac{1}{x} + y + \frac{1}{y} = 0,$$

and then assuming a conjecture of Beilinson, to conjecture that one should have

$$m(P) = r\frac{15}{(2\pi)^2}L(E,2) = rL'(E,0),$$

where r is a rational number (unspecified by Beilinson's conjecture). Finally, Boyd checked numerically that $r = 1.00000000\ldots$ (up to 200 places), so that presumably

$$m\left(1 + x + \frac{1}{x} + y + \frac{1}{y}\right) = \frac{15}{(2\pi)^2}L(E,2) = L'(E,0).$$

Modularity theorem

The Shimura–Taniyama conjecture implies that, for the L-function $L(E,s)=\sum_{k=1}^\infty a_k k^{-s}$ attached to an elliptic curve E of conductor N, the function $f(\tau)=\sum_{k=1}^\infty a_k q^k$, where $q=e^{2\pi i\tau}$, is a cusp form for the modular group $\Gamma_0(N)$.
In Deninger's case $N=15$, so that

$$f(\tau)=\sum_{k=1}^\infty a_k q^k = q\prod_{m=1}^\infty (1-q^m)(1-q^{3m})(1-q^{5m})(1-q^{15m}).$$

In view of Euler's pentagonal number formula

$$\eta(\tau)=q^{1/24}\prod_{m=1}^\infty (1-q^m)=\sum_{n\in\mathbb{Z}}(-1)^n q^{(6n+1)^2/24}$$

and the hypergeometric evaluation above, the conjecture can be stated as

Equivalent form of Deninger's example

$$\frac{540}{\pi^2}\sum_{\substack{n_j=-\infty\\ j=1,2,3,4}}^{\infty}\frac{(-1)^{n_1+n_2+n_3+n_4}}{\big((6n_1+1)^2+3(6n_2+1)^2+5(6n_3+1)^2+15(6n_4+1)^2\big)^2}$$
$$=4\sum_{k=0}^{\infty}\binom{2k}{k}^2\frac{(1/16)^{2k+1}}{2k+1}.$$

In spite of the origin of the formula, we do not have the Mahler measure any more but a (hypergeometric) single sum evaluation of a quadruple lattice sum.

Quadruple lattice sums

Define

$$F(a,b,c,d) = (a+b+c+d)^2 \times \sum_{\substack{n_j=-\infty \\ j=1,2,3,4}}^{\infty} \frac{(-1)^{n_1+n_2+n_3+n_4}}{\big(a(6n_1+1)^2+b(6n_2+1)^2+c(6n_3+1)^2+d(6n_4+1)^2\big)^2}$$

where the method of summation is $\lim_{M\to\infty}\sum_{n_1=-M}^{M}\cdots\sum_{n_4=-M}^{M}$, and also set

$$F(b,c) = F(1,b,c,bc).$$

Many cases are known when $F(a,b,c,d)$ can be (sometimes conjecturally) reduced to a single sum, like

$$F(3,5) = \frac{\pi^2}{15}\cdot {}_3F_2\left(\left.\begin{matrix}\frac12,\ \frac12,\ \frac12\\ 1,\ \frac32\end{matrix}\right|\frac{1}{16}\right)$$

in the above example.

L-functions and Boyd's conjectures

The finite amount of elliptic curves, whose L-functions are related to the quadruple sums, is given by the following list. Suppose that E_N is an elliptic curve of conductor N, then

$$L(E_N, 2) = F(b, c)$$

for the following values of N and (b, c):

N	(b, c)
11	$(1, 11)$
14	$(2, 7)$
15	$(3, 5)$
20	$(1, 5)$
24	$(2, 3)$
27	$(1, 3)$
32	$(1, 2)$
36	$(1, 1)$

Latest results

All Boyd's conjectural evaluations corresponding to the entries in the table are now rigorously established. (However, Boyd's complete list contains many more.)

The "lacunary" sums $F(1,1)$, $F(1,2)$ and $F(1,3)$ are settled by F. Rodríguez-Villegas (1999). These correspond to CM elliptic curves of conductors 36, 32 and 27, respectively.

F. Brunault (2006) and A. Mellit (2011) gave K-theoretic proofs of the formulae for $F(1,11)$ and $F(2,7)$ (conductors 11 and 14), respectively.

Finally, the expected relations for $F(1,5)$, $F(2,3)$ and $F(3,5)$ are proved in my recent joint papers with M. Rogers.

Our strategy to prove Boyd's evaluations

(i) reducing $L(E,2)$ to a (suitable) linear combination of integrals of the form
$$I = -\int_0^1 \frac{\log q}{q} f(q)\,\mathrm{d}q,$$
where $f(e^{2\pi i\tau})$ is a modular cusp form of weight 2 on a congruence subgroup of $SL_2(\mathbb{Z})$;

(ii) finding modular functions $x(q)$, $y(q)$, and $z(q)$ (which, of course, depend on $f(q)$) such that
$$-\int_0^1 \frac{\log q}{q} f(q)\,\mathrm{d}q = 2\pi \int_0^1 x(q)\log y(q)\,\mathrm{d}z(q);$$

(iii) expressing x and y as algebraic functions of z: if we write $x(q) = X(z(q))$ and $y(q) = Y(z(q))$, then the substitution reduces I to a (usually complicated) integral of elementary functions;

(iv) relating the resulting integral to hypergeometric functions which represent the desired Mahler measures.

Conductor 20 example

Here is an outline for a conductor 20 elliptic curve.
(a) Reduction of the L-series for a conductor 20 elliptic curve to an elementary integral results in

$$L(E_{20},2) = -\frac{\pi}{20}\int_0^1 \frac{(1-6t)\log(1+4t)}{\sqrt{t(1-t)(1+4t^2)}}\,\mathrm{d}t.$$

(b) Using the hypergeometric functions we show the following evaluation of Mahler measure for $k \in [2,8]$:

$$\frac{1}{2\pi}\int_0^1 \frac{(2-k+3kt)\log(1+kt)}{\sqrt{t(1-t)(4+(4-k)kt+k^2t^2)}}\,\mathrm{d}t = m\big((1+X)(1+Y)(X+Y)-kXY\big).$$

(c) Combining the results from (a) and (b) (with $k=4$) the conjecture of Boyd follows:

$$m\big((1+X)(1+Y)(X+Y)-4XY\big) = \frac{10}{\pi^2}L(E_{20},2).$$

Principal novelty

The main ingredient of our proofs is the reduction

$$I = -\int_0^1 \frac{\log q}{q} f(q)\,\mathrm{d}q = 2\pi \int_0^1 x(q)\log y(q)\,\mathrm{d}z(q)$$

for a weight 2 cusp form f, where x, y, and z are certain modular forms.
The only method known before was due to Beilinson. It requires introducing a certain double integral instead, computing it in two different ways as products of two single integrals and then canceling out one of the integrals in the result.
Our method is purely analytic in nature: a simple change of variable in a single integral.

Outline of the method

We first write a cusp form f as a product of two Eisenstein series of weight 1 (sometimes as a $\mathbb{Q}$-linear combination of several such products), say $E(q)$ and $E'(q)$, where the first one vanishes at the origin (this corresponds to the cusp at $\tau = 0$) and the second one has zero at $q = 1$ (this corresponds to the cusp at $\tau = i\infty$).
Here and it what follows $q = \exp(2\pi i\tau)$ and I also use the notation $\tau = it$, where t ranges from 0 to ∞, to keep integration "real":

$$I = -\int_0^1 \frac{\log q}{q} f(q)\,\mathrm{d}q = 4\pi^2 \int_0^\infty E(e^{-2\pi t})E'(e^{-2\pi t})t\,\mathrm{d}t.$$

Outline of the method

The modular involution $\tau \mapsto -1/\tau$ (equivalently, $t \mapsto 1/t$) translates the second Eisenstein series $E'(e^{-2\pi t})$ into $\widehat{E}(e^{-2\pi/t})/t$, another weight 1 Eisenstein series:

$$I = 4\pi^2 \int_0^\infty E(e^{-2\pi t})\widehat{E}(e^{-2\pi/t})\,\mathrm{d}t,$$

and the general form of these two Eisenstein series (vanishing at the origin) is

$$E(q) = \sum_{m,n=1}^{\infty} \chi_1(m)\chi_2(n)q^{mn},$$

$$\widehat{E}(q) = \sum_{r,s=1}^{\infty} \chi_3(r)\chi_4(s)q^{rs}$$

for some quadratic characters $\chi_1, \dots, \chi_4$.

Outline of the method

The substitution of the q-expansions and interchange of integration and summation (which is eligible because of the vanishing properties) lead us to

$$I = 4\pi^2 \sum_{m,n,r,s\ge1} \int_0^\infty \chi_1(m)\chi_2(n)\chi_3(r)\chi_4(s)\exp\biggl(-2\pi\Bigl(mnt+\frac{rs}{t}\Bigr)\biggr)\,\mathrm{d}t.$$

The change of variable $t \mapsto rt/n$ remains the form of the integrand but affects the differential:

$$\begin{aligned} I &= 4\pi^2 \sum_{m,n,r,s\ge1} \int_0^\infty \chi_1(m)\chi_2(n)\chi_3(r)\chi_4(s)\exp\biggl(-2\pi\Bigl(mrt+\frac{ns}{t}\Bigr)\biggr)\frac{r\,\mathrm{d}t}{n} \\ &= 4\pi^2 \int_0^\infty E_2(e^{-2\pi t})\widehat{E}_0(e^{-2\pi/t})\,\mathrm{d}t \end{aligned}$$

where

$$E_2(q) = \sum_{m,r\ge1} r\chi_1(m)\chi_3(r)q^{mr}, \quad \widehat{E}_0(q) = \sum_{n,s\ge1} \frac{\chi_2(n)\chi_4(s)}{n}\,q^{ns}.$$

Outline of the method

The function

$$E_2(q) = \sum_{m,r\ge1} r\chi_1(m)\chi_3(r)q^{mr}$$

is an Eisenstein series of weight 2, hence the logarithmic derivative of a modular function z:

$$E_2(q)\,\frac{\mathrm{d}q}{q} = \frac{\mathrm{d}z(q)}{z(q)}.$$

The function

$$\widehat{E}_0(q) = \sum_{n,s\ge1} \frac{\chi_2(n)\chi_4(s)}{n}\,q^{ns}$$

is a weak modular form — an Eisenstein series of weight 0, that is, a logarithm of a modular function. The involution $\tau \mapsto -1/\tau$ (or $t \mapsto 1/t$) translates $\widehat{E}_0(e^{-2\pi/t})$ into another weak modular form $E_0(e^{-2\pi t}) = \log y(q)$ for some modular function y.

Outline of the method

Thus,

$$\begin{aligned} I &= 4\pi^2 \int_0^\infty E_2(e^{-2\pi t})\widehat{E}_0(e^{-2\pi/t})\,\mathrm{d}t \\ &= 2\pi \int_0^1 \log y(q)\,\frac{\mathrm{d}z(q)}{z(q)}, \end{aligned}$$

and the required form follows by taking $x(q) = 1/z(q)$.

My final remark is about decomposition of a given cusp form f into a linear combination of products of Eisenstein series of weight 1. The principal source of this in our proofs of Boyd's conjectures was *Ramanujan's Notebooks*.