Fields Institute Communications
Volume **54**, 2008

Apéry Limits of Differential Equations of Order 4 and 5

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Abstract. The concept of *Apéry limit* for second and third order differential equations is extended to fourth and fifth order equations, mainly of Calabi–Yau type. For those equations obtained from Hadamard products of second and third order equations we can prove that the limits are determined in terms of the factors by a certain formula. Otherwise the limits are found by using PSLQ in Maple and are only conjectural. All identified limits are rational linear combinations of the following numbers: π^2, Catalan's constant G, $\sum_{n=1}^{\infty} \frac{\left(\frac{n}{3}\right)}{n^2}$, π^3, $\zeta(3)$, $\pi^3\sqrt{3}$, π^4.

1 Introduction

In 1978 Apéry proved that $\zeta(3)$ is irrational by considering the recursion

$$(n+2)^3 A_{n+2} - (2n+3)(17n^2+51n+39)A_{n+1} + (n+1)^3 A_n = 0$$

2000 *Mathematics Subject Classification.* Primary 14J32, 11Y60, 14D05; Secondary.

The work of the third-named author was supported by a fellowship of the Max Planck Institute for Mathematics (Bonn).

with $A_{-1}=0$, $A_0=1$. Let B_n satisfy the same recursion with $B_0=0$, $B_1=1$. Then

$$\frac{B_n}{A_n}\to\frac{\zeta(3)}{6}\qquad\text{as } n\to\infty.$$

He also considered the recursion

$$(n+2)^2A_{n+2}-(11n^2+33n+25)A_{n+1}-(n+1)^2A_n=0$$

with $A_{-1}=0$, $A_0=1$. Let B_n satisfy the same recursion with $B_0=0$, $B_0=1$. Then

$$\frac{B_n}{A_n}\to\frac{\pi^2}{30}\qquad\text{as } n\to\infty.$$

Note that $y_0=\sum_{n=0}^{\infty}A_nx^n$ is annihilated by the differential operator

$$\theta^2-x(11\theta^2+11\theta+3)-x^2(\theta+1)^2,\qquad\text{where}\quad\theta=x\frac{d}{dx}.$$

This is denoted by (b) in [1] and we note that

$$A_n=\sum_k\binom{n}{k}^2\binom{n+k}{n}.$$

Later Zudilin [8] considered the recursion

$$\begin{aligned}&(n+2)^5A_{n+2}-3(2n+3)(3n^2+9n+7)(15n^2+45n+34)A_{n+1}\\&\quad-3(n+1)^3(3n+2)(3n+4)A_n=0\end{aligned}$$

with $A_{-1}=0$, $A_0=1$. Let $B_0=0$, $B_1=1$. Then

$$\frac{B_n}{A_n}\to\frac{\pi^4}{1170}=\frac{\zeta(4)}{13}.$$

It turns out that

$$w_0=\sum_{n=0}^{\infty}A_nx^n$$

is the Wronskian

$$w_0=x\begin{vmatrix}y_0 & y_1\\ y_0' & y_1'\end{vmatrix},$$

where y_0 and y_1 are solutions of a fourth order differential equation (this was the start of [1]). It is equation #32 in the "big" table [2] of Calabi–Yau differential equations (for a definition see Section 2). Further cases of connections between Diophantine approximations of di- and trilogarithms and Catalan's constant G led to Calabi–Yau equations #195, #209 and #257. These will be treated in section 2.1 (see Zudilin [9, 10]).

In an unpublished paper [7], Zagier considered the recursion

$$(n+2)^2A_{n+2}-(a(n+1)^2+a(n+1)+b)A_{n+1}+c(n+1)^2A_n=0$$

trying to find integer solutions A_n. He finds the cases denoted by (a), (b), (c), (d), (e), (f), (g) and (h) in [1]. In [1] we find two more cases (i) and (j); e.g., in case (j) we have

$$A_n=432^n\sum_k(-1)^k\binom{-5/6}{k}\binom{-1/6}{n-k}^2$$

which satisfies Zagier's recursion with $a=864$, $b=372$, $c=186624$. The sequence (A_n^2), the Hadamard square of (A_n), satisfies a recursion of order 3 and $y_0=\sum_{n=0}^{\infty}A_n^2x^n$ satisfies a fourth order Calabi–Yau equation. If (A_n') and (A_n'') are two sequences from the list (a)–(j), then $(A'_n A''_n)$, the Hadamard product of (A'_n) and (A''_n), also results in a Calabi–Yau equation. In Section 3 we find formulas for how the Apéry limits are transformed under various Hadamard products.

There are also ten cases of third order differential equations (α), (β), (γ), (δ), (ϵ), (ζ), (η), (ϑ), (ι), (κ) with recursions similar to those of the second order:

$$(n+2)^3 A_{n+2} - (2n+3)(a(n+1)^2 + a(n+1) + b)A_{n+1} + c(n+1)^3 A_n = 0.$$

For instance, in case (κ)

$$A_n = 432^n \sum_k \binom{-1/6}{k}^2 \binom{-5/6}{n-k}^2$$

satisfies the above recursion with $a = 432$, $b = 312$, $c = 186624$. Let B_n satisfy the same recursion with $B_0 = 0$, $B_1 = 1$. Then

$$\frac{B_n}{A_n} \to \frac{1}{432} \sum_{j=1}^{\infty} \frac{1}{(j-\frac{1}{6})^3} = \frac{91}{432}\zeta(3) - \frac{1}{216}\pi^3\sqrt{3}.$$

This will be proved in Section 2.3.

In Section 4 we list all the conjectured Apéry limits identified by PSLQ. We have also some limits which we cannot identify (usually listed with 50 digits).

2 Known results

2.1 Calabi–Yau differential equations with known Apéry limits. Consider a fourth order differential equation

$$y^{(4)} + a_3 y''' + a_2 y'' + a_1 y' + a_{0y} = 0,$$

where the a_j are polynomials. We assume it is MUM (*maximal unipotent monodromy*) at the origin, so we have the Frobenius solutions

$$y_0 = 1 + A_1 x + \cdots, \qquad y_1 = y_0 \log x + B_1 x + \cdots,$$
$$y_2 = \frac{1}{2} y_0 \log^2 x + \cdots, \qquad y_3 = \frac{1}{6} y_0 \log^3 x + \cdots.$$

We also assume that the coefficients satisfy the relation

$$a_1 = \frac{1}{2}a_2 a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3 a_3' - \frac{1}{2}a_3''.$$

Let

$$q = \exp\left(\frac{y_1}{y_0}\right) = x + c_2 x^2 + \cdots.$$

Solving for x we get the *mirror map*

$$x = q - c_2 q^2 + \cdots.$$

Then we have the Yukawa coupling

$$K(q) = \left(q\frac{d}{dq}\right)^2 \left(\frac{y_2}{y_0}\right) = 1 + \sum_{n=1}^{\infty} \frac{n^3 N_n q^n}{1-q^n}.$$

The N_n are called the instanton numbers. If there exists a fixed integer c such that cN_n are integers for all n we call the differential equation Calabi–Yau.

The maximal degree of the a_j is called the degree of the equation. Observe that the degree of the equation becomes the order of the corresponding recursion for the coefficients A_n. When computing the Apéry limit we usually try to use the recursion of lowest order.

The following two equations are in Zudilin [9].

Example (#209) Let

$$A_n = \binom{2n}{n} \sum_k \binom{n}{k}^2 \binom{n+k}{n} \binom{n+2k}{n}.$$

It satisfies the recursion

$$\begin{aligned}
&(n+3)^4(946n^2+3053n+2475)A_{n+3} \\
&\quad - (208120n^6+2752860n^5+1506697n^4+43651558n^3 \\
&\qquad + 70564960n^2+60321212n+21297414)A_{n+2} \\
&\quad + 4(2n+3)(3784n^5+36808n^4+141179n^3 \\
&\qquad + 267255n^2+250336n+93060)A_{n+1} \\
&\quad - 4(n+1)^2(2n+1)(2n+3)(946n^2+4945n+6474)A_n = 0
\end{aligned}$$

with $A_{-2}=0$, $A_{-1}=0$, $A_0=1$. Let B_n satisfy the same recursion with $B_{-1}=0$, $B_0=0$, $B_1=1$; then

$$\frac{B_n}{A_n} \to \frac{\pi^2}{138}.$$

But if we start with $B_0=0$, $B_1=17$, $B_2=\frac{9405}{8}$ then

$$\frac{B_n}{A_n} \to \zeta(3),$$

so we have simultaneous approximation of $\zeta(2)$ and $\zeta(3)$.

Example (#195) Let $H_n = \sum_{j=1}^n \frac{1}{j}$ if $n>0$ and 0 otherwise be the harmonic numbers. Let

$$\begin{aligned}
A_n &= \binom{2n}{n}^2 \binom{3n}{n}^2 \sum_k \binom{n}{k}^5 \binom{3n}{n+k}^{-2} \\
&\qquad \times \{1+k(-5H_k+5H_{n-k}+2H_{n+k}-2H_{2n-k})\} \\
&= \sum_{i,j} \binom{n}{i}^2 \binom{n}{j}^2 \binom{i+j}{j} \binom{n+i}{n}
\end{aligned}$$

which satisfies

$$\begin{aligned}
&(n+3)^4(1457n^2+4465n+3450)A_{n+3} \\
&\quad + (148614n^6+1941570n^5+10489565n^4+29970066n^3 \\
&\qquad + 47717965n^2+40114368n+13906092)A_{n+2} \\
&\quad + (97619n^6+1080107n^5+4934487n^4+11925999n^3 \\
&\qquad + 16102866n^2+115316n+3425652)A_{n+1} \\
&\quad + 3(n+1)^2(3n+2)(3n+4)(1457n^2+7379n+9372)A_n = 0
\end{aligned}$$

with $A_{-2}=0$, $A_{-1}=0$, $A_0=1$. Let B_n satisfy the same recursion with $B_{-1}=0$, $B_0=0$, $B_1=1$; then

$$\frac{B_n}{A_n} \to -\frac{\pi^2}{78}.$$

But if we take $B_0 = 0$, $B_1 = \frac{29}{13}$, $B_2 = -\frac{7617}{104}$, then

$$\frac{B_n}{A_n} \to -\frac{3}{13}\zeta(3).$$

Example (#257) Let

$$A_n = (-1)^n \binom{2n}{n}^2 \sum_k \binom{n}{k}\binom{n+k}{n-k}\binom{2n-k}{k}\binom{2n+2n}{n+k}\binom{4n-2k}{2n-k}$$
$$\times \{1 + k(-2H_k + 2H_{n-k} - H_{n+k} + H_{2n-k} - 2H_{2k} + 2H_{2n-2k} + 2H_{2n+2k} - 2H_{4n-2k})\}.$$

Then

$$\begin{aligned}(n+2)^4&(20n^2+32n+13)A_{n+2}\\ &-(56320n^6+428032n^5+1328384n^4+2153472n^3\\ &\quad+1923136n^2+897408n+171184)A_{n+1}\\ &-2^{12}(2n+1)^4(20n^2+72n+65)A_n = 0.\end{aligned}$$

Then with $A_{-1} = 0$, $A_0 = 1$ and $B_0 = 0$, $B_1 = 1$ we have

$$\frac{B_n}{A_n} \to \frac{G}{108}$$

where G is Catalan's constant,

$$G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)^2}.$$

This is proved in Zudilin [10].

2.2 Differential equations of order 2. In [1] we considered the following differential equations of order 2 as building blocks.

Hypergeometric:

	Coefficients	Differential operator
(A)	$A_n = \binom{2n}{n}^2$	$\theta^2 - 4x(2\theta+1)^2$
(B)	$A_n = \frac{(3n)!}{n!^3}$	$\theta^2 - 3x(3\theta+1)(3\theta+2)$
(C)	$A_n = \frac{(4n)!}{(2n)!n!^2}$	$\theta^2 - 4x(4\theta+1)(4\theta+3)$
(D)	$A_n = \frac{(6n)!}{(3n)!(2n)!n!}$	$\theta^2 - 12x(6\theta+1)(6\theta+5)$

Sporadic:

	Coefficients	Differential operator
(a)	$A_n=\sum_k \binom{n}{k}^3$	$\theta^2-x(7\theta^2+7\theta+2)-8x^2(\theta+1)^2$
(b)	$A_n=\sum_k \binom{n}{k}^2\binom{n+k}{n}$	$\theta^2-x(11\theta^2+11\theta+3)-x^2(\theta+1)^2$
(c)	$A_n=\sum_k \binom{n}{k}^2\binom{2k}{k}$	$\theta^2-x(10\theta^2+10\theta+3)+9x^2(\theta+1)^2$
(d)	$A_n=\sum_k \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}$	$\theta^2-4x(3\theta^2+3\theta+1)+32x^2(\theta+1)^2$
(e)	$A_n=\sum_k 4^{n-k}\binom{2k}{k}^2\binom{2n-2k}{n-k}$	$\theta^2-4x(8\theta^2+8\theta+3)+256x^2(\theta+1)^2$
(f)	$A_n=\sum_k(-1)^k 3^{n-3k}\binom{n}{3k}\frac{(3k)!}{k!^3}$	$\theta^2-3x(3\theta^2+3\theta+1)+27x^2(\theta+1)^2$
(g)	$A_n=\sum_{i,j} 8^{n-i}(-1)^i\binom{n}{i}\binom{i}{j}^3$	$\theta^2-x(17\theta^2+17\theta+6)+72x^2(\theta+1)^2$
(h)	$A_n=27^n\sum_k(-1)^k\binom{-2/3}{k}\binom{-1/3}{n-k}^2$	$\theta^2-3x(18\theta^2+18\theta+7)+3^6x^2(\theta+1)^2$
(i)	$A_n=64^n\sum_k(-1)^k\binom{-3/4}{k}\binom{-1/4}{n-k}^2$	$\theta^2-4x(32\theta^2+32\theta+13)$ $+2^{12}x^2(\theta+1)^2$
(j)	$A_n=432^n\sum_k(-1)^k\binom{-5/6}{k}\binom{-1/6}{n-k}^2$	$\theta^2-12x(72\theta^2+72\theta+31)$ $+2^8 3^6x^2(\theta+1)^2$

Then we have the following limits:

	Limit
(a)	$\pi^2/24$
(b)	$\pi^2/30$
(c)	$L(\chi_3,2)/2$
(d)	$G/2$
(e)	slow convergence
(f)	not convergent
(g)	$5L(\chi_3,2)/8$
(h)	slow convergence
(i)	slow convergence
(j)	slow convergence

The cases with four singular points (a), (b), (c), (d), (f) and (g) are treated in Zagier [7]. In the cases with three singular points convergence is slow and the limit hard to guess. In case (e) we found some evidence for the limit being $G/2$.

For case (h) Arne Meurman conjectures that the limit is $\frac{1}{2}L(\chi_3,2)-\frac{2}{81}\pi^2$. His computations suggest the following Ramanujan-like conjecture

$$\sum_{n=1}^{\infty}\frac{\left(\frac{n}{3}\right)(-\exp(-\pi/\sqrt{3}))^n}{n^2(1-(-\exp(-\pi/\sqrt{3}))^n)}=\frac{2}{81}\pi^2-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\left(\frac{n}{3}\right)}{n^2}.$$

This was proved subsequently by Yifan Yang [6], who also settled case (e). He uses, as Beukers [4] and Zagier [7], the fact that these equations admit modular parametrisations. In [1] there are three more second order differential equations which give rise to some fourth order Calabi–Yau equations by Hadamard product:

	Coefficients	Differential operator
(k)	$A_n = 9^n \sum_k (-1)^k \binom{n}{k}\binom{-1/3}{n-k}\binom{-2/3}{k}$	$\theta^2 - 3x(2\theta+1) - 81x^2(\theta+1)^2$
(l)	$A_n = 8^n \sum_k (-1)^k \binom{n}{k}\binom{-1/4}{n-k}\binom{-3/4}{k}$	$\theta^2 - 4x(2\theta+1) - 64x^2(\theta+1)^2$
(m)	$A_n = 36^n \sum_k (-1)^k \binom{n}{k}\binom{-1/6}{n-k}\binom{-5/6}{k}$	$\theta^2 - 24x(2\theta+1) - 1296x^2(\theta+1)^2$

Then we have the following limits:

	Limit
(k)	$\frac{\pi^2}{27} - \frac{1}{4}L(\chi_3, 2)$
(l)	$\frac{\pi^2}{16} - \frac{G}{2}$
(m)	$\frac{\pi^2}{36} - \frac{5}{16}L(\chi_3, 2)$

Proof of case (l) Let

$$R_n(t) = (-1)^n \frac{n! \prod_{j=0}^{n-1}(t+\frac{1}{4}+j)}{\prod_{j=0}^{n}(t+j)^2} = \sum_{k=0}^{n}\left\{\frac{U_k}{(t+k)^2} + \frac{\widetilde{U}_k}{t+k}\right\},$$

where

$$\begin{aligned} U_k = (R_n(t)(t+k)^2)_{t=-k} &= (-1)^{n-k} n! \frac{(1/4)_{n-k}(3/4)_k}{(k!(n-k)!)^2} \\ &= (-1)^k \binom{n}{k}\binom{-1/4}{n-k}\binom{-3/4}{k}. \end{aligned}$$

Let

$$\begin{aligned} r_n &= \sum_{j=1}^{\infty}(R_n(t))_{t=j-1/4} = \sum_{j=1}^{\infty}\sum_{k=0}^{n}\left\{\frac{U_k}{(j+k-1/4)^2} + \frac{\widetilde{U}_k}{j+k-1/4}\right\} \\ &= a_n \sum_{j=1}^{\infty}\frac{1}{(j-\frac{1}{4})^2} - b_n, \end{aligned}$$

where

$$a_n = \sum_{k=0}^{n} U_k = \sum_{k=0}^{n}(-1)^k \binom{n}{k}\binom{-1/4}{n-k}\binom{-3/4}{k} = \frac{1}{8^n}A_n$$

and

$$b_n = \sum_{k=0}^{n}\sum_{j=1}^{k}\frac{U_k}{(j-1/4)^2} + \sum_{k=0}^{n}\sum_{j=1}^{k}\frac{\widetilde{U}_k}{j-1/4}$$

as in the proof of Theorem 1 in [10].

Here we used

$$\sum_{k=0}^{n}\widetilde{U}_k = \sum_{k=0}^{n}\mathrm{Res}_{t=-k} R_n(t) = -\mathrm{Res}_{t=\infty} R_n(t) = 0,$$

since

$$R_n(t) = O(t^{-(n+2)}) \qquad \text{as } t \to \infty.$$

Using for instance Maple's `Zeilberger` we find that a_n and $\sum_t R_n(t)$ both satisfy the recursion

$$2(n+2)^2N^2 - (2n+3)N - 2(n+1)^2.$$

(Here N is the shift operator, that is: $Nf(n) := f(n+1)$.) It follows that r_n and hence b_n also satisfy this recursion and we have the start values

$$a_0 = 1, \quad a_1 = \frac{1}{2}, \qquad b_0 = 0, \quad b_1 = -2.$$

Now we show that

$$\frac{r_n}{a_n} \to 0 \qquad \text{as} n \to \infty.$$

It follows from general considerations that the coefficients a_n behave asymptotically as

$$a_n \sim Cn^{-d}\left\{1 + \frac{e_1}{n} + \frac{e_2}{n^2} + \cdots\right\}$$

as $n \to \infty$. Substituting this in the recursion formula above we obtain

$$d = \frac{1}{2}, \qquad e_1 = -\frac{1}{4}, \quad e_2 = \frac{1}{16}.$$

Numerical evidence suggests $C = 2/3$, but we will not need this.

Consider

$$F_n(t) = R_n(t)\frac{\pi}{\sin(\pi t)} = (-1)^n\Gamma(t)^2\Gamma\Big(\frac{3}{4} - t\Big)n^{-3/4-t}(1 + O(n^{-1})).$$

Then

$$\mathrm{Res}_{t=k-1/4}\, F_n(t) = R_n(t)_{t=k-1/4}$$

and

$$r_n = \frac{1}{2\pi i}\int_{\mathscr{L}} F_n(t)\, dt,$$

where $\mathscr{L}$ is a rectangle with vertices in $\varepsilon + iM$, $\varepsilon - iM$, $M + \frac{1}{4} + iM$, $M + \frac{1}{4} - iM$, and M is a positive integer $> n$. It follows that

$$\frac{r_n}{a_n} \le Cn^{-1/4} \to 0$$

as $n \to \infty$. □

2.3 Third order differential equations. Here we also have ten differential equations suitable for Hadamard products:

	Coefficients	Differential operator
(α)	$A_n = \sum_k \binom{n}{k}^2 \binom{2k}{k}\binom{2n-2k}{n-k}$	$\theta^3 - 2x(2\theta+1)(5\theta^2+5\theta+2)$ $+ 64x^2(\theta+1)^3$
(β)	$A_n = \sum_k \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2$	$\theta^3 - 8x(2\theta+1)(2\theta^2+2\theta+1)$ $+ 256x^2(\theta+1)^3$
(γ)	$A_n = \sum_k \binom{n}{k}^2 \binom{n+k}{n}^2$	$\theta^3 - x(2\theta+1)(17\theta^2+17\theta+5)$ $+ x^2(\theta+1)^3$
(δ)	$A_n = \sum_k (-1)^k 3^{n-3k} \binom{n}{3k}\binom{n+k}{n}\frac{(3k)!}{k!^3}$	$\theta^3 - x(2\theta+1)(7\theta^2+7\theta+3)$ $+ 81x^2(\theta+1)^3$
(ϵ)	$A_n = \sum_k \binom{n}{k}^2 \binom{2k}{n}^2$	$\theta^3 - 4x(2\theta+1)(3\theta^2+3\theta+1)$ $+ 16x^2(\theta+1)^3$
(ζ)	$A_n = \sum_{i,j} \binom{n}{i}^2 \binom{n}{j}\binom{i}{j}\binom{i+j}{j}$	$\theta^3 - 3x(2\theta+1)(3\theta^2+3\theta+1)$ $- 27x^2(\theta+1)^3$
(η)	A_n is not known	$\theta^3 - x(2\theta+1)(11\theta^2+11\theta+5)$ $+ 125x^2(\theta+1)^3$
(θ)	$A_n = 64^n \sum_k \binom{-1/4}{k}^2 \binom{-3/4}{n-k}^2$	$\theta^3 - 8x(2\theta+1)(8\theta^2+8\theta+5)$ $+ 4096x^2(\theta+1)^3$
(ι)	$A_n = 27^n \sum_k \binom{-1/3}{k}^2 \binom{-2/3}{n-k}^2$	$\theta^3 - 3x(2\theta+1)(9\theta^2+9\theta+5)$ $+ 729x^2(\theta+1)^3$
(κ)	$A_n = 432^n \sum_k \binom{-1/6}{k}^2 \binom{-5/6}{n-k}$	$\theta^3 - 24x(2\theta+1)(18\theta^2+18\theta+13)$ $+ 186624x^2(\theta+1)^3$

Then we find the following limits:

	Limit
(α)	$\frac{7}{24}\zeta(3)$
(β)	slow convergence
(γ)	$\frac{1}{6}\zeta(3)$
(δ)	not convergent
(ϵ)	$\frac{7}{32}\zeta(3)$
(ζ)	$\frac{1}{3}L(\chi_3, 3)$
(η)	not convergent
(θ)	$\frac{7}{16}\zeta(3) - \frac{1}{64}\pi^3$
(ι)	$\frac{13}{27}\zeta(3) - \frac{2}{243}\pi^3\sqrt{3}$
(κ)	$\frac{91}{432}\zeta(3) - \frac{1}{216}\pi^3\sqrt{3}$

where

$$\frac{1}{3}L(\chi_3,3) = \sum_{j=0}^{\infty}\left\{\frac{1}{(3j+1)^3} - \frac{1}{(3j+2)^3}\right\} = \frac{4}{729}\pi^3\sqrt{3}.$$

Proof of case (θ) Consider

$$R_n(t) = \frac{\prod_{j=0}^{n-1}(t+\frac{1}{4}+j)^2}{\prod_{j=0}^{n}(t+j)^2} = \sum_{k=0}^{n}\left\{\frac{U_k}{(t+k)^2} + \frac{\widetilde{U}_k}{t+k}\right\},$$

where

$$U_k = (R_n(t)(t+k)^2)_{t=-k} = \left\{\frac{(3/4)_k(1/4)_{n-k}}{k!(n-k)!}\right\}^2$$

and

$$\widetilde{U}_k = \frac{d}{dt}(R_n(t)(t+k)^2)_{t=-k} \in \mathbb{Q}.$$

Let

$$\begin{aligned}
r_n &= -\frac{1}{2}\sum_{j=1}^{\infty}\left(\frac{d}{dt}R_n(t)\right)_{t=j-1/4} \\
&= \sum_{j=1}^{\infty}\sum_{k=0}^{n}\left\{\frac{U_k}{(j+k-\frac{1}{4})^3} + \frac{1}{2}\frac{\widetilde{U}_k}{(j+k-\frac{1}{4})^2}\right\} \\
&= \sum_{k=0}^{n}U_k\left\{\sum_{j=1}^{\infty}\frac{1}{(j-\frac{1}{4})^3} - \sum_{j=1}^{k}\frac{1}{(j-\frac{1}{4})^3}\right\} \\
&\qquad + \frac{1}{2}\sum_{k=0}^{n}\widetilde{U}_k\left\{\sum_{j=1}^{\infty}\frac{1}{(j-\frac{1}{4})^2} - \sum_{j=1}^{n}\frac{1}{(j-\frac{1}{4})^2}\right\} \\
&= a_n\sum_{j=1}^{\infty}\frac{1}{(j-\frac{1}{4})^3} - b_n,
\end{aligned}$$

where

$$a_n = \sum_{k=0}^{n}U_k = \sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2 = \frac{1}{64^n}A_n$$

and

$$b_n = \sum_{k=0}^{n}\sum_{j=1}^{k}\frac{U_k}{(j-\frac{1}{4})^3} + \frac{1}{2}\sum_{k=0}^{n}\sum_{j=1}^{k}\frac{\widetilde{U}_k}{(j-\frac{1}{4})^2}.$$

It follows that

$$\frac{1}{2}\sum_{k=0}^{n}\widetilde{U}_k = \frac{1}{2}\sum_{k=0}^{n}\operatorname{Res}_{t=-k}R_n(t) = -\frac{1}{2}\operatorname{Res}_{t=\infty}R_n(t) = 0,$$

since

$$R_n(t) = O(t^{-2}) \qquad \text{as } t\to\infty.$$

Using Maple's `Zeilberger` we find that a_n and $\sum_t R_n(t)$ are annihilated by the difference operator

$$8(n+2)^3N^2 - (16n^3+72n^2+114n+63)N + 8(n+1)^3.$$

It follows that r_n, and hence also b_n, satisfy the same difference equation. We also have the start values

$$a_0 = 1, \quad a_1 = \frac{5}{8}, \quad \dots, \qquad b_0 = 0, \quad b_0 = 1, \quad \dots .$$

We want to show that

$$\frac{r_n}{a_n} \to 0 \qquad \text{as } n \to \infty.$$

First we note that

$$a_n > \binom{-3/4}{n}^2 = \Gamma(3/4)^{-2} n^{-1/2} + O(n^{-3/2}).$$

Consider the function

$$\begin{aligned} F_n(t) = R_n(t) \left\{ \frac{\pi}{\sin(\pi(t+1/4)} \right\}^2 &= \left\{ \frac{\Gamma(t)\Gamma(3/4-t)\Gamma(t+n+1/4)}{\Gamma(t+n+1)} \right\}^2 \\ = \Gamma(t)^2 \Gamma(3/4-t)^2 n^{-3/2} + O(n^{-5/2}) & \qquad \text{as } n \to \infty. \end{aligned}$$

We note that

$$\operatorname{Res}_{t=k-1/4} F_n(t) = \left(\frac{d}{dt} R_n(t) \right)_{t=k-1/4}$$

and hence

$$-2r_n = \frac{1}{2\pi i} \int_{\mathscr{L}} F_n(t)\, dt,$$

where the path of integration $\mathscr{L}$ is taken as in the proof of case (l) above.

It follows that

$$|r_n| < Cn^{-3/2} \qquad \text{as } n \to \infty$$

and

$$\left| \frac{r_n}{a_n} \right| = O(n^{-1}).$$

Finally we note that

$$\sum_{j=0}^{\infty} \frac{1}{(4j+3)^3} = \frac{1}{2} \left\{ \sum_{j=0}^{\infty} \frac{1}{(2j+1)^3} - \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^3} \right\} = \frac{1}{2} \left(\frac{7}{8}\zeta(3) - \frac{\pi^3}{32} \right)$$

and, since $A_n = 64^n a_n$, we have to divide the result by 64. □

Remark: As was pointed out by the referee, some of the cases in the table can be proven using modular parametrisations as in Beukers [4]. For instance, the function $A(x) = \sum_{n=1}^{\infty} a_n x^n$ for case (ϵ) has the parametrisation

$$x(\tau) = \left(\frac{\eta(\tau)\eta(8\tau)}{\eta(2\tau)\eta(4\tau)} \right)^8, \quad A(\tau) = \frac{(\eta(2\tau)\eta(4\tau))^6}{(\eta(\tau)\eta(8\tau))^4},$$

modular forms for $\Gamma_0(8) + w_8$. Case (β) is proven along these lines in [6].

3 Hadamard products

It is well known that the Hadamard product of D-finite sequences is D-finite (see [5, p. 194] for a proof), i.e., given two sequences A'_n and A''_n that satisfy difference equations with polynomial coefficients then the same is the case for the sequence $A'_n A''_n$. Here we mostly consider the case when

$$\begin{aligned} A'_{n+2} &= P_1(n) A'_{n+1} + Q_1(n) A'_n, \\ A''_{n+2} &= P_2(n) A''_{n+1} + Q_2(n) A''_n, \end{aligned}$$

where the P, Q are rational functions of degree 2.

Proposition 1 *The coefficients*

$$C_n = A'_n A''_n$$

satisfy the recursion

$$T_4 C_{n+4} + T_3 C_{n+3} + T_2 C_{n+2} - W_1 C_{n+1} - W_0 C_n = 0,$$

where for $j = 1, 2$

$$\begin{aligned} R_j(n) &= P_j(n+1)P_j(n) + Q_j(n+1), \\ S_j(n) &= P_j(n+1)Q_j(n), \\ U_j(n) &= P_j(n+2)R_j(n) + Q_j(n+2)P_j(n), \\ V_j(n) &= P_j(n+2)S_j(n) + Q_j(n+2)Q_j(n), \end{aligned}$$

and (here $P_j = P_j(n)$ *etc.)*

$$\begin{aligned} T_2 &= R_2 S_1 U_1 V_2 - R_1 S_2 U_2 V_1, \\ T_3 &= P_1 Q_2 U_2 V_1 - P_2 Q_1 U_1 V_2, \\ T_4 &= R_1 S_2 P_2 Q_1 - R_2 S_1 P_1 Q_2, \\ W_0 &= Q_1 Q_2 T_2 + S_1 S_2 T_3 + V_1 V_2 T_4, \\ W_1 &= P_1 P_2 T_2 + R_1 R_2 T_3 + U_1 U_2 T_4. \end{aligned}$$

Proof We have

$$A'_{n+3} = R_1 A'_{n+1} + S_1 A'_n, \qquad A'_{n+4} = U_1 A'_{n+1} + V_1 A'_n,$$

and similarly formulas for A''_{n+3} and A''_{n+4}.

Multiplying the recursion formulas for A'_{n+2} and A''_{n+2} (respectively, for A'_{n+3} and A''_{n+3}, and for A'_{n+4} and A''_{n+4}) we have

$$\begin{aligned} C_{n+2} &= P_1 P_2 C_{n+1} + Q_1 Q_2 C_n + P_1 Q_2 A'_{n+1} A''_n + P_2 Q_1 A'_n A''_{n+1}, \\ C_{n+3} &= R_1 R_2 C_{n+1} + S_1 S_2 C_n + R_1 S_2 A'_{n+1} A''_n + R_2 S_1 A'_n A''_{n+1}, \\ C_{n+4} &= U_1 U_2 C_{n+1} + V_1 V_2 C_n + U_1 V_2 A'_{n+1} A''_n + U_2 V_1 A'_n A''_{n+1}. \end{aligned}$$

Considering $A'_{n+1} A''_n$ and $A'_n A''_{n+1}$ as unknowns we find

$$\begin{vmatrix} P_1 Q_2 & P_2 Q_1 & -C_{n+2} + P_1 P_2 C_{n+1} + Q_1 Q_2 C_n \\ R_1 S_2 & R_2 S_1 & -C_{n+3} + R_1 R_2 C_{n+1} + S_1 S_2 C_n \\ U_1 V_2 & U_2 V_1 & -C_{n+4} + U_1 U_2 C_{n+1} + V_1 V_2 C_n \end{vmatrix} = 0.$$

Expanding the determinant we get the wanted recursion formula. □

Let A'_n be the solution to

$$A'_{n+2} = P_1(n) A'_{n+1} + Q_1(n) A'_n$$

with

$$A_{-1} = 0 \quad \text{and} \quad A_0 = 1$$

(and similarly for A''_n). Let further B'_n be the solution to the same recursion but with

$$B_0 = 0 \quad \text{and} \quad B_1 = 1$$

(and similarly for B''_n). Let $C_n = A'_n A''_n$ as above and let D_n be the solution to the recursion in Proposition 1 with

$$D_n = 0 \qquad \text{for} \quad n \leq 0 \quad \text{and} \quad D_1 = 1.$$

Assume that

$$P_j(n) = \frac{a_j(n+1)^2 + a_j(n+1) + b_j}{(n+2)^2}, \qquad Q_j(n) = \frac{c_j(n+1)^2}{(n+2)^2}$$

for $j = 1, 2$. Define

$$f_1 = \frac{b_2 c_1}{b_2^2 c_1 - b_1^2 c_2}, \qquad f_2 = \frac{b_1 c_2}{b_1^2 c_2 - b_2^2 c_1}.$$

Theorem *We have for $n \geq 1$*

$$\frac{D_n}{C_n} = f_1 \frac{B'_n}{A'_n} + f_2 \frac{B''_n}{A''_n}.$$

Proof We use induction on n. First we let Maple check the identity for $n = 0, 1, 2, 3$. Assume that

$$\frac{D_k}{C_k} = f_1 \frac{B'_k}{A'_k} + f_2 \frac{B''_k}{A''_k},$$

i.e., $D_k = f_1 A''_k B'_k + f_2 A'_k B''_k$ for $k = n, n+1, n+2, n+3$. We have

$$\begin{aligned}
\frac{D_{n+4}}{C_{n+4}} &= -\frac{1}{T_4} \frac{T_3 D_{n+3} + T_2 D_{n+2} - W_1 D_{n+1} - W_0 D_n}{A'_{n+4} A''_{n+4}}, \\
&= -f_1 \frac{1}{T_4 A'_{n+4} A''_{n+4}} \left\{ \begin{array}{c} T_3 A''_{n+3} B'_{n+3} + T_2 A''_{n+2} B'_{n+2} \\ -W_1 A''_{n+1} B'_{n+1} - W_0 A''_n B'_n \end{array} \right\} \\
&\quad - f_2 \{\text{similar formula with } ' \text{ replaced by } ''\} \\
&= -f_1 \frac{1}{T_4 A'_{n+4} A''_{n+4}} \left\{ \begin{array}{l} T_3 (R_2 A''_{n+1} + S_2 A''_n)(R_1 B'_{n+1} + S_1 B'_n) \\ \quad + T_2 (P_2 A''_{n+1} + Q_2 A''_n)(P_1 B'_{n+1} + Q_1 B'_n) \\ \quad - W_1 A''_{n+1} B'_{n+1} - W_0 A''_n B'_n \end{array} \right\} \\
&\quad - f_2 \{\text{similar formula}\} \\
&= -f_1 \frac{1}{T_4 A'_{n+4} A''_{n+4}} \left\{ \begin{array}{l} (T_3 R_2 R_1 + T_2 P_1 P_2 - W_1) A''_{n+1} B'_{n+1} \\ \quad + (T_3 S_1 S_2 + T_2 Q_1 Q_2 - W_0) A''_n B'_n \\ \quad + (T_3 S_1 R_2 + T_2 P_2 Q_1) A''_{n+1} B'_n \\ \quad + (T_3 R_1 S_2 + T_2 P_1 Q_2) A''_n B'_{n+1} \end{array} \right\} \\
&\quad - f_2 \{\text{similar formula}\} \\
&= -f_1 \frac{1}{T_4 A'_{n+4} A''_{n+4}} \left\{ -T_4 (U_2 A''_{n+1} + V_2 A''_n)(U_1 B'_{n+1} + V_1 B'_n) \right\} \\
&\quad - f_2 \{\text{similar formula}\} \\
&= f_1 \frac{B'_{n+4}}{A'_{n+4}} + f_2 \frac{B''_{n+4}}{A''_{n+4}}.
\end{aligned}$$

□

In [1] we consider the second order differential operators

$$\theta^2 - z(a\theta^2 + a\theta + b) - cz^2(\theta+1)^2, \qquad \text{where} \quad \theta = z\frac{d}{dz}.$$

This corresponds to the recursion formula

$$(n+2)^2 A_{n+2} = (a(n+1)^2 + a(n+1) + b) A_{n+1} + c(n+1)^2 A_n$$

We consider the following cases:

#	a	b	c
(a)	7	2	8
(b)	11	3	1
(c)	10	3	−9
(d)	12	4	−32
(e)	32	12	−256
(f)	9	3	−27
(g)	17	6	−72
(h)	54	21	−729
(i)	128	52	−4096
(j)	864	372	−186624

We get the following table for (f_1, f_2):

	(b)	(c)	(d)	(e)	(f)
(a)	$(\frac{6}{17}, -\frac{1}{34})$	$(\frac{2}{9}, \frac{1}{6})$	$(\frac{1}{8}, \frac{1}{4})$	$(\frac{3}{68}, \frac{4}{17})$	$(\frac{2}{15}, \frac{3}{10})$
(b)		$(\frac{1}{30}, \frac{3}{10})$	$(\frac{1}{76}, \frac{6}{19})$	$(\frac{1}{204}, \frac{16}{51})$	$(\frac{1}{84}, \frac{9}{28})$
(c)			$(-\frac{1}{4}, \frac{2}{3})$	$(-\frac{3}{28}, \frac{16}{21})$	$(-\frac{1}{6}, \frac{1}{2})$
(d)				$(\frac{3}{4}, -2)$	$(-\frac{2}{3}, \frac{3}{4})$
(e)					$(-\frac{16}{33}, \frac{9}{44})$

	(g)	(h)	(i)	(j)
(a)	$(\frac{1}{12}, \frac{1}{4})$	$(\frac{14}{537}, \frac{81}{358})$	$(\frac{13}{1188}, \frac{64}{297})$	$(\frac{31}{19308}, \frac{324}{1609})$
(b)	$(\frac{1}{114}, \frac{6}{19})$	$(\frac{7}{2334}, \frac{243}{778})$	$(\frac{13}{9892}, \frac{768}{2473})$	$(\frac{31}{151500}, \frac{3888}{12625})$
(c)	$(-\frac{1}{6}, \frac{2}{3})$	$(-\frac{7}{96}, \frac{27}{32})$	$(-\frac{13}{348}, \frac{256}{261})$	$(-\frac{31}{4020}, \frac{432}{335})$
(d)	∞	$(\frac{14}{51}, -\frac{81}{68})$	$(\frac{13}{164}, -\frac{32}{41})$	$(\frac{31}{3756}, -\frac{162}{313})$
(e)	$(-\frac{4}{3}, \frac{3}{4})$	$(\frac{112}{165}, -\frac{243}{220})$	$(\frac{13}{100}, -\frac{12}{25})$	$(\frac{31}{2784}, -\frac{243}{928})$
(f)	$(\frac{1}{2}, -\frac{2}{3})$	$(\frac{7}{66}, -\frac{9}{22})$	$(\frac{39}{1004}, -\frac{256}{753})$	$(\frac{31}{6348}, -\frac{144}{529})$
(g)		$(\frac{14}{51}, -\frac{27}{34})$	$(\frac{13}{164}, -\frac{64}{123})$	$(\frac{31}{3756}, -\frac{108}{313})$
(h)			$(\frac{1053}{4580}, -\frac{1792}{3445})$	$(\frac{31}{2124}, -\frac{112}{531})$
(i)				$(\frac{124}{5061}, -\frac{1053}{6748})$

We can also consider Hadamard products of third order differential equations, i.e., we have recursions for $j = 1, 2$

$$A'_{n+2} = \frac{(2(n+1)+1)(a_1(n+1)^2 + a_1(n+1) + b_1)}{(n+2)^3} A'_{n+1} + c_1 \frac{(n+1)^3}{(n+2)^3} A'_n$$

and similarly for A''_n. Then we can imitate everything we did for second order equations and the results are the same except that the degree of the difference equation is higher. In particular, the weights f_1 and f_2 are the same as before.

For a Hadamard square we must modify the construction of the recurrence. Let

$$A_{n+2} = P(n)A_{n+1} + Q(n)A_n$$

with $A_{-1} = 0$, $A_0 = 1$. Let B_n satisfy the same recursion with $B_0 = 0$, $B_1 = 1$. Then

$$A_{n+3} = R(n)A_{n+1} + S(n)A_n,$$

where

$$R(n) = P(n+1)P(n) + Q(n+1), \qquad S(n) = P(n+1)Q(n+1).$$

Put

$$T(n) = R(n)Q(n) - S(n)P(n);$$

then $C_n = A_n^2$ satisfies

$$PQC_{n+3} - RSC_{n+2} - PRTC_{n+1} + QSTC_n = 0.$$

Proposition 2 *Let $C_{-2} = 0$, $C_{-1} = 0$, $C_0 = 1$ and let D_n satisfy the same recursion with $D_{-1} = 0$, $D_0 = 0$, $D_1 = 1$. Then if*

$$P(n) = \frac{a(n+1)^2 + a(n+1) + b}{(n+2)^2}, \qquad Q(n) = c\frac{(n+1)^2}{(n+2)^2},$$

we have

$$\frac{D_n}{C_n} = \frac{1}{b}\frac{B_n}{A_n} \qquad \textit{for} \quad n > 0.$$

Proof Assume that

$$\frac{D_k}{C_k} = \frac{1}{b}\frac{B_k}{A_k} \qquad \text{for} \quad k = n, n+1, n+2.$$

Then

$$\begin{aligned} \frac{D_{n+3}}{C_{n+3}} &= \frac{RSA_{n+2}B_{n+2} + PRTA_{n+1}B_{n+1} - QSTA_nB_n}{bPQA_{n+3}^2} \\ &= \frac{1}{bPQA_{n+3}^2}\left\{ \begin{array}{c} RS(PA_{n+1} + QA_n)(PB_{n+1} + QB_n) \\ +PRTA_{n+1}B_{n+1} - QSTA_nB_n \end{array} \right\} \\ &= \frac{PQ(RA_{n+1} + SA_n)(RB_{n+1} + SB_n)}{bPQA_{n+3}^2} = \frac{1}{b}\frac{B_{n+3}}{A_{n+3}} \end{aligned}$$

Maple checks that the formula is true for $n = 0, 1, 2$. □

We still have another case of Hadamard products, namely, the product of the second order differential equations (a), (b), (c), (d), (e), (f), (g), (h), (i), (j) (respectively, third order (α), (β), (γ), (δ), (ε), (ζ), (η), (θ), (ι), (κ)) with the second order differential equations (A), (B), (C), (D), i.e., multiplying the coefficients with

$$\binom{2n}{n}, \quad \frac{(3n)!}{n!^3}, \quad \frac{(4n)!}{(2n)!n!^2}, \quad \frac{(6n)!}{(3n)!(2n)!n!}.$$

The corresponding differential operators are $\theta^2 - zQ(\theta)$, where $Q(\theta)$ is given by

(A)	$(2\theta+1)^2$
(B)	$3(3\theta+1)(3\theta+2)$
(C)	$4(4\theta+1)(4\theta+3)$
(D)	$12(6\theta+1)(6\theta+5)$

We have the formulas

$$\begin{aligned} &\left\{\theta^2 - zP(\theta) - cz^2(\theta+1)^2\right\} * \left\{\theta^2 - zQ(\theta)\right\} \\ &\quad = \theta^4 - zP(\theta)Q(\theta) - cz^2Q(\theta)Q(\theta+1) \end{aligned}$$

$$\begin{aligned} &\left\{\theta^3 - z(2\theta+1)P(\theta) - cz^2(\theta+1)^3\right\} * \left\{\theta^2 - zQ(\theta)\right\} \\ &\quad = \theta^5 - z(2\theta+1)P(\theta)Q(\theta) - cz^2(\theta+1)Q(\theta)Q(\theta+1). \end{aligned}$$

Let now A_n solve the equation

$$\theta^2 - zP(\theta) - cz^2(\theta+1)^2$$

with $A_{-1}=0, A_0=1$. Then A_n satisfies the recursion

$$(n+2)^2 A_{n+2} = P(n+1)A_{n+1} + c(n+1)^2 A_n.$$

Let B_n satisfy the same recurrence but with $B_0=0, B_1=1$. Let further C_n satisfy the recursion

$$(n+2)^4 C_{n+2} = P(n+1)Q(n+1)C_{n+1} + cQ(n)Q(n+1)C_n$$

with $C_{-1}=0, C_0=1$. Similarly, let D_n satisfy the same recursion but with $D_0=0, D_1=1$.

Proposition 3 *We have*

$$\frac{D_n}{C_n} = \frac{1}{Q(0)}\frac{B_n}{A_n}.$$

Proof We first show that

$$\frac{C_{n+1}}{C_n} = \frac{Q(n)}{(n+1)^2}\frac{A_{n+1}}{A_n}.$$

Assume this is true for n. Then

$$\begin{aligned}(n+2)^4\frac{C_{n+2}}{C_{n+1}} &= P(n+1)Q(n+1) + cQ(n)Q(n+1)\frac{C_n}{C_{n+1}}\\ &= P(n+1)Q(n+1) + cQ(n)Q(n+1)\frac{(n+1)^2}{Q(n)}\frac{A_n}{A_{n+1}}\\ &= \frac{Q(n+1)}{A_{n+1}}\{P(n+1)A_{n+1} + Q(n)A_n\}\\ &= \frac{Q(n+1)}{A_{n+1}}(n+2)^2A_{n+2}.\end{aligned}$$

We get, assuming the formula true for n and $n+1$,

$$\begin{aligned}\frac{D_{n+2}}{C_{n+2}} &= \frac{P(n+1)Q(n+1)D_{n+1} + cQ(n)Q(n+1)D_n}{P(n+1)Q(n+1)C_{n+1} + Q(n)Q(n+1)C_n}\\ &= \frac{1}{Q(0)}\frac{P(n+1)B_{n+1}C_{n+1}/A_{n+1} + cQ(n)B_nC_n/A_n}{P(n+1)A_{n+1}C_{n+1}/A_{n+1} + cQ(n)A_nC_n/A_n}\\ &= \frac{1}{Q(0)}\frac{P(n+1)B_{n+1} + c(n+1)^2B_n}{P(n+1)A_{n+1} + c(n+1)^2A_n} = \frac{1}{Q(0)}\frac{B_{n+2}}{A_{n+2}}.\end{aligned}$$

□

4 Conjectured limits

Using Maple's `PSLQ` we have found many limits, where we have no proof.

4.1 Limits of form $c\pi^2$.

#	15	18	19	20	21	22	23	24	25	26	27	33	45
c	$\frac{1}{144}$	$\frac{1}{60}$	$\frac{1}{56}$	$\frac{5}{216}$	$\frac{1}{48}$	$\frac{1}{36}$	$\frac{1}{32}$	$\frac{1}{180}$	$\frac{1}{120}$	$\frac{1}{84}$	$-\frac{1}{56}$	$\frac{1}{384}$	$\frac{1}{96}$

#	51	55	62	63	68	99	109	117	118	119
c	$\frac{1}{360}$	$-\frac{1}{128}$	$\frac{1}{1440}$	$-\frac{1}{1800}$	$\frac{1}{288}$	$-\frac{1}{198}$	$-\frac{1}{474}$	$\frac{1}{64}$	$\frac{1}{160}$	$\frac{3}{224}$

#	193	195	198	208	209	210	211	212	222	224
c	$\frac{1}{108}$	$\frac{1}{78}$	$-\frac{1}{72}$	$\frac{7}{1119}$	$\frac{1}{138}$	$-\frac{1}{276}$	$-\frac{1}{2048}$	$-\frac{1}{48}$	$\frac{1}{128}$	$\frac{1}{64}$

#	232	234	235	238	241	243	248	250	256	260
c	$\frac{1}{256}$	$\frac{2}{81}$	$\frac{1}{162}$	$\frac{1}{162}$	$-\frac{1}{512}$	$-\frac{1}{168}$	$\frac{1}{40}$	$\frac{1}{80}$	$-\frac{1}{320}$	$-\frac{25}{3096}$

#	261	262	279	282	297	338	341
c	$-\frac{1}{81}$	$-\frac{1}{54}$	$\frac{1}{36}$	$-\frac{4}{729}$	$\frac{67}{3870}$	$-\frac{1}{160}$	$\frac{29}{1842}$

4.2 Limits of the form cG.

#	36	38	48	65	231	233	237	239	257	258
c	$\frac{2}{2}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{120}$	$\frac{3}{76}$	$\frac{1}{8}$	$\frac{1}{40}$	$\frac{1}{152}$	$\frac{1}{104}$	$\frac{1}{56}$

#	264	277	288	289	300	329	331
c	$\frac{1}{1144}$	$\frac{1}{248}$	$\frac{1}{200}$	$\frac{1}{472}$	$-\frac{1}{168}$	$\frac{1}{16}$	$\frac{1}{24}$

4.3 Limits of the form $L(\chi_3, 2)$.

#	58	64	69	70	137	138	139	140	183	274	278
c	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{5}{32}$	$\frac{5}{48}$	$\frac{5}{96}$	$\frac{1}{96}$	$\frac{1}{4}$	$-\frac{5}{19}$	$\frac{1}{30}$

4.4 Limits of the form $c\zeta(3)$.

#	16	28	29	37	42	44	50	52	53	60
c	$\frac{7}{48}$	$\frac{1}{7}$	$\frac{1}{12}$	$\frac{7}{288}$	$\frac{7}{64}$	$\frac{1}{24}$	$\frac{7}{144}$	$\frac{1}{72}$	$\frac{1}{76}$	$\frac{3}{23}$

#	66	67	149	182	189	205
c	$\frac{7}{1440}$	$\frac{7}{1440}$	$-\frac{1}{360}$	$\frac{3}{11}$	$\frac{1}{14}$	$\frac{21}{80}$

4.5 Limits of the form $cL(\chi_3, 3)$.

#	185
c	$\frac{1}{2}$

4.6 Limits of the form $c\pi^4$.

#	32	244
c	$\frac{1}{1170}$	$-\frac{1}{630}$

4.7 Limits of mixed form.

#	59	116	214
Limit	$-\frac{1}{208}\pi^2+\frac{7}{26}G$	$-\frac{1}{320}\pi^2+\frac{1}{10}G$	$-\frac{1}{144}\pi^2+\frac{3}{16}L(\chi_3,2)$

#	215	216 & 217	218
Limit	$-\frac{1}{736}\pi^2+\frac{3}{46}G$	$\frac{1}{54}\pi^2-\frac{1}{24}L(\chi_3,2)$	$-\frac{1}{450}\pi^2+\frac{7}{50}L(\chi_3,2)$

#	219	220	221
Limit	$\frac{1}{18}\pi^2-\frac{5}{8}L(\chi_3,2)$	$-\frac{1}{128}\pi^2+\frac{1}{8}G$	$-\frac{5}{4104}\pi^2+\frac{10}{171}G$

#	226	240	242
Limit	$-\frac{1}{216}\pi^2+\frac{5}{24}L(\chi_3,2)$	$\frac{1}{6072}\pi^2+\frac{26}{253}G$	$-\frac{1}{54}\pi^2+\frac{1}{6}L(\chi_3,2)$

#	251	252
Limit	$-\frac{1}{945}\pi^2+\frac{2}{45}G+\frac{3}{28}L(\chi_3,2)$	$\frac{1}{39888}\pi^2-\frac{20}{277}G$

#	328	340
Limit	$\frac{1}{84}\pi^2-\frac{1}{12}G$	$-\frac{1}{7712}\pi^2+\frac{9}{241}G$

Remark Equation #251 is remarkable. The recursion is of order 4 and degree 8. The corresponding differential operator (of order 8 and degree 4) factors into two of order 4 (of degree 12 and 8, respectively). It is also the only case we have found which is a linear combination of values of three L-functions.

4.8 Unidentified limits with many digits. The following limits converge very fast but we have not been able to identify them:

#	Limit
17	0.3097538790467859180612009276621093227187570662744
34	0.3328799000999687141578669041535332990987481910397
147	−0.18294864738225619912816208889600198974512074131046
206	0.034533768895242744048924842293799661180236416
207	−0.00050462505145900474057831709244307528529622730007723
$\widetilde{214}$	0.078351361063584306378644830113068845245731042453985
229	0.017891639973294587136164078700439361783026615410295

For $\#\widetilde{214}$ see [3].

Acknowledgements. We thank Vasily Golyshev and Arne Meurman for several interesting discussions about this work. We also want to thank the referee for useful comments and spotting several mistakes in the first version of this paper.

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