MULTIPLE MODULAR VALUES

WADIM ZUDILIN

Abstract. One natural (and historically approved) way to extend the values of zeta or, more generally, L-functions to a multiple setting is via their sum expressions. In a recent take of the theory of multiple modular L-values, Francis Brown defines them via iterated integrals equipped with a simple regularisation procedure. The lectures will review Brown's theory from a "practical" point of view and highlight some of its applications, in particular, to computing regulator integrals for elliptic curves.

These lecture notes source from my joint project [6] with François Brunault. Iterated integrals of modular forms appear intrinsically in the study of modular regulators. Yuri Manin pioneered the former topic in [9, 10] but a general construction and main tools were set up more recently by Francis Brown in [2, 3] whose footsteps we follow in [6].

1. (Multiple) zeta values

As we will see in the coming days, our creativity in generalising Riemann's zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},$$

in particular, it values at integers $s>1$, hardly has any boundary. One notable *multiple* generalisation

$$\zeta(\boldsymbol{s})=\zeta(s_1,s_2,\dots,s_l)=\sum_{0<n_1<n_2<\dots<n_l}\frac{1}{n_1^{s_1}n_2^{s_2}\cdots n_l^{s_l}},$$

known as multiple zeta values (MZVs), goes back to Euler's investigation of the case $l=2$ and its modern history starts from the pioneering works of Hoffman [7, 8] and Zagier [13] in the 1990s. In the definition above we assume all s_j to be integral and further $s_l\ge 2$ for convergence reasons; any such multi-index $\boldsymbol{s}$ is said to be admissible and l is called its depth (or length), while $|\boldsymbol{s}|=s_1+\dots+s_l\ge l+1$ its weight. Multiplication of two such series and rearrangement of the result as a $\mathbb{Z}$-linear combination of same kind leads to the harmonic (or stuffle) product of MZVs. There is however another representation of the MZVs as iterated integrals recorded by Zagier in [13] and attributed by him to Kontsevich. For the word $x_{\epsilon_1}x_{\epsilon_2}\cdots x_{\epsilon_k}=x_1x_0^{s_1-1}x_1x_0^{s_2-1}\cdots x_1x_0^{s_l-1}$ attached to an admissible multi-index $\boldsymbol{s}=(s_1,s_2,\dots,s_l)$, we have

$$\zeta(\boldsymbol{s})=\int\cdots\int_{0<z_1<\dots<z_k<1}\omega_{\epsilon_1}(z_1)\cdots\omega_{\epsilon_k}(z_k)$$

in a form of *Chen's iterated integrals*, where the differential 1-forms $\omega_0(z)$ and $\omega_1(z)$ are simply

$$\omega_0(z)=\frac{\mathrm{d}z}{z}\quad\text{and}\quad\omega_1(z)=\frac{\mathrm{d}z}{1-z}.$$

Multiplication of two such integral representations gives rise to another product of MZVs known as the shuffle product. I refer to other lectures for an exposition of the shuffle-stuffle

Date: 12 February 2025.
Lecture notes for the 17th MSJ symposium "Developments of Multiple Zeta Values ."

algebra of MZVs. For me an important message from this very brief review of MZVs is about two ways of generalising a series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

associated with an arithmetic object $\sum_{n=1}^{\infty} a_n q^n$, or even $\sum_{n=0}^{\infty} a_n q^n$, to multiple setting. Before proceeding with a multiple setting, let me review the single one.

2. Modular forms and their L-functions

I will review this for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ (of level 1), however definitions can be easily adapted to its congruence subgroups of level N. The group Γ is generated by $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. A *modular form* $f(\tau)$ of weight k (forced to be a non-negative positive even integer) is characterised by analyticity in the upper half-plane $H = \{\tau \in \mathbb{C} : \operatorname{Im} \tau > 0\}$, the functional equation $f((a\tau + b)/(c\tau + d)) = (c\tau + d)^k f(\tau)$ for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and a Fourier expansion $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ where $q = e^{2\pi i \tau}$ at the cusp $\tau = i\infty$. Modular forms with $a_0 = 0$ are further called *cusp forms*; forms normalised to have $a_1 = 1$ will be labeled as *normalised* modular forms. A basic example of a normalised (non-cusp!) modular form of weight $k \in 2\mathbb{Z}$, $k \ge 4$, is the normalised Eisenstein series

$$E_k(\tau) = -\frac{B_k}{2k} + \sum_{n \ge 1} \sigma_{k-1}(n) q^n = -\frac{B_k}{2k} + \sum_{d,m \ge 1} d^{k-1} q^{md},$$

where B_k denotes the kth Bernoulli number (and $\sigma_{k-1}(\,\cdot\,)$ is the sum-of-powers-of-divisors function). The series defining the L-function of a modular form,

$$L(f,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

converges in the half-plane $\operatorname{Re} s > k$ in general, and in the half-plane $\operatorname{Re} s > k/2 + 1$ in the case of cusp form f. In order to analytically continue the series one uses its Mellin transform

$$\Lambda(f,s) = \int_0^{\infty} (f(iy) - a_0)\, y^{s-1}\, \mathrm{d}y, \tag{1}$$

known as the *completed L-function*; the latter is an analytic function of s in the half-plane and satisfies $\Lambda(f,s) = (2\pi)^{-s}\Gamma(s)L(f,s)$ there. By the Hecke theory $\Lambda(f,s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies the functional equation $\Lambda(f,s) = (-1)^{k/2}\Lambda(f,k-s)$. Indeed, this follows from the following simple calculation. In the case of cusp form $f(\tau)$, the function $\Lambda(f,s)$ extends to an entire function of s.

The (single) *modular values* associated with a modular form $f(\tau) = \sum_{n=0}^{\infty} a_n q^n$ are the values of $\Lambda(f,s)$ at integers $s = m > 0$. Though they are well-defined for all such positive integers, we will pay a special attention to the critical ones, when $1 \le m \le k - 1$.

The mechanism behind the functional equation (hence also analytic continuation) is important by itself, so I will review it to motivate a regularisation procedure that follows.

Lemma 1. *Let f be modular of weight $k \in 2\mathbb{Z}$. Then*

$$\Lambda(f,s) = g(s) + i^k g(k-s) - a_0\biggl(\frac{1}{s} + \frac{i^k}{k-s}\biggr)$$

where

$$g(s) = \int_1^{\infty} (f(iy) - a_0)\, y^s \frac{\mathrm{d}y}{y} = \sum_{n \ge 1} a_n \int_1^{\infty} e^{-2\pi n y} y^s \, \frac{\mathrm{d}y}{y},$$

which converges uniformly for $\operatorname{Re} s \ge K$ *for any* K.

Proof. For all $\operatorname{Re} s \gg 0$ sufficiently large, decompose the domain of integration in the right-hand side of (1) into a path from 0 to 1 and 1 to ∞. This gives

$$\Lambda(f,s) = \int_1^\infty (f(iy) - a_0)\, y^s \frac{\mathrm{d}y}{y} + \int_0^1 \left(f(iy)y^s + a_0 i^k y^{s-k} - a_0 i^k y^{s-k} - a_0 y^s\right) \frac{\mathrm{d}y}{y}.$$

Using $f(i/y) = (iy)^k f(iy)$, apply the change of variables $y \mapsto y^{-1}$ to the first two terms in the integrand of the second integral to obtain

$$\begin{aligned}\Lambda(f,s) &= g(s) + \int_1^\infty i^k (f(iy) - a_0) y^{k-s} \frac{\mathrm{d}y}{y} - a_0 \int_0^1 (i^k y^{s-k} + y^s) \frac{\mathrm{d}y}{y} \\ &= \left(g(s) - a_0 \int_0^1 y^s \frac{\mathrm{d}y}{y}\right) + i^k \left(g(k-s) - a_0 \int_0^1 y^{s-k} \frac{\mathrm{d}y}{y}\right). \qquad (2)\end{aligned}$$

By analytic continuation, this formula holds for all values of $s \in \mathbb{C}$. □

Finally, notice that the L-function of the normalised Eisenstein series is

$$L(E_k, s) = \zeta(s)\zeta(s-k+1),$$

and if f is a Hecke normalised eigenform of weight k — for example, $f(\tau) = \Delta(\tau) = q\prod_{m=1}^\infty (1-q^m)^{24}$ with $k = 12$, then its L-function admits an Euler product expansion

$$L(f,s) = \prod_p (1 - a_p p^{-s} + p^{k/2-1-2s})^{-1}$$

which converges for $\operatorname{Re} s > k/2 + 1$.

The terms in (2) can be interpreted geometrically using tangential base-points. This brings us naturally to regularisation.

3. 'Practical' regularisation (after Brown)

One think to notice about the way our modular values were defined is that the integration is performed exclusively along the imaginary axis $]0, i\infty[= \{iy : y > 0\}$, though the function $f(\tau)$ was analytic on the upper half-plane.

A C^∞ function $f\colon\ ]0, i\infty[\ \to \mathbb{C}$ is called *admissible at* ∞ if it can be written $f(\tau) = f^\infty(\tau) + f^0(\tau)$, where $f^\infty(\tau) \in \mathbb{C}[\tau]$ is a polynomial, and $f^0(\tau)$ has exponential decay as $\operatorname{Im}\tau \to +\infty$: there exists $0 < c < 1$ such that $f^0(\tau) = O_{\tau\to\infty}(c^{\operatorname{Im}\tau})$. For such an admissible function f, its *regularised value at infinity*, denoted by $f(\infty)$, is defined as the constant term of the polynomial f^∞. Note that the decomposition $f = f^\infty + f^0$ is unique, hence $f(\infty)$ is well defined.

In parallel, a C^∞ differential form $\omega = f(\tau)\,\mathrm{d}\tau$ on $]0, i\infty[$ is called *admissible at* ∞ if f is admissible at ∞. We then write $\omega = \omega^\infty + \omega^0$ with $\omega^\infty = f^\infty(\tau)\,\mathrm{d}\tau$ and $\omega^0 = f^0(\tau)\,\mathrm{d}\tau$.

In the case of a modular form f of weight $k \geq 1$ (on Γ or some finite-index subgroup of Γ), the form $\omega = f(\tau)\tau^{m-1}\,\mathrm{d}\tau$ is admissible at ∞ for any integer $m \geq 1$. Note that if a form ω is admissible at ∞, then so are $\operatorname{Re}\omega = \frac{1}{2}(\omega + \bar\omega)$ and $\operatorname{Im}\omega = \frac{1}{2i}(\omega - \bar\omega)$.

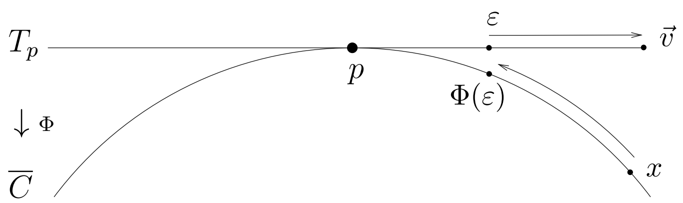
Lemma 2. *If a function f and a form ω on $]0, i\infty[$ are admissible at ∞, then so is $f\omega$.*

Now, for a differential form ω on $]0, i\infty[$ which is admissible at ∞, define

$$\int_\tau^\infty \omega = \lim_{y\to+\infty} \int_\tau^{iy} \omega + \int_{iy}^0 \omega^\infty. \qquad (3)$$

Brown's original definition deals with integration along a more general smooth complex curve $\overline{C}$ be a smooth complex curve, a point $p \in \overline{C}$ on it, and a geometric 'gluing' of the curve with its tangent space T_p. A tangential base point on the punctured curve $C = \overline{C} \backslash p$

at the point p is an element $\vec v$ of the punctured tangent space $T_p^\times = T_p\backslash\{0\}$. For an analytic isomorphism $\Phi\colon (T_p, 0) \to (\overline{C}, p)$ such that $\mathrm{d}\Phi\colon T_p \to T_p$ is the identity, we can glue the space $T_p^\times$ to C along the map Φ to obtain a space $T_p^\times \cup_\Phi C$ which is homotopy equivalent to C; then the tangential base point $\vec v$ is simply an ordinary base point on this enlarged space. A path from a point $x \in C$ to this tangential base point can be thought of as a path in $\overline{C}$ from x to a point $\Phi(\varepsilon)$ close to p, followed by a path from ε to $\vec v$ in the tangent space T_p as demonstrated on the figure:



Finally, the integral of an (admissible) meromorphic 1-form ω on C along a path from x to $\vec v$ is defined to be

$$\int_x^{\vec v} \omega = \lim_{\varepsilon\to p}\biggl(\int_x^{\Phi(\varepsilon)} \omega + \int_\varepsilon^{\vec v} P\Phi^*(\omega)\biggr),$$

where $P\Phi^*(\omega)$ denotes the projection of the pullback of the form on $T_p^\times$, itself a 1-form on the latter.

Returning to our practical interpretation (3), observe first that we can actually get rid of the limit there.

Lemma 3. *Let ω be a differential form on $]0, i\infty[$ which is admissible at ∞. The limit in (3) exists, and we have*

$$\int_\tau^\infty \omega = \int_\tau^\infty \omega^0 + \int_\tau^0 \omega^\infty. \tag{4}$$

Moreover, the error term in the convergence of (3) is $O_{y\to+\infty}(c^y)$ with $0 < c < 1$, the constant c being uniform with respect to τ on domains of the form $\{\operatorname{Im}\tau \ge y_0 > 0\}$.

Proof. Indeed,

$$\int_\tau^{iy} \omega + \int_{iy}^0 \omega^\infty = \int_\tau^{iy} \omega^0 + \int_\tau^{iy} \omega^\infty + \int_{iy}^0 \omega^\infty = \int_\tau^{iy} \omega^0 + \int_\tau^0 \omega^\infty. \qquad \square$$

I refer to the right-hand side of (4) as the *practical* regularised integral. Note that the regularised integral recovers the classical integral in the case ω is integrable on $[\tau, i\infty[$ (which happens if and only if $\omega^\infty = 0$). Lemma 3 has the following consequence.

Lemma 4. *Let ω be a differential form on $]0, i\infty[$ which is admissible at ∞. Then the function $F(\tau) = -\int_\tau^\infty \omega$ is admissible at ∞. Moreover, F is the unique primitive of ω whose regularised value at ∞ is zero.*

In particular, if a form ω is admissible at ∞, then any primitive of ω is again admissible at ∞. On the other hand, the differential of an admissible function f need not be admissible, because there is no control on the derivative of f^0.

Lemma 5. *Let $f\colon\ ]0, i\infty[\ \to \mathbb{C}$ be a function such that $\mathrm{d}f$ is admissible at ∞. Then f is admissible at ∞ and $\int_\tau^\infty \mathrm{d}f = f(\infty) - f(\tau)$, where $f(\infty) = f^\infty|_{\tau=0}$ is the regularised value at ∞.*

Proof. This follows from Lemma 4 applied to $\omega = \mathrm{d}f$. $\square$

One should be careful that in general $\int_\tau^\infty \omega$ does not converge to zero as $\tau \to \infty$: this can be seen from (4). For example, if $f(\tau) = \sum_{n\geq 0} a_n q^n$ is a modular form, then

$$\int_\tau^\infty f(\tau_1)\,\mathrm{d}\tau_1 = -\frac{1}{2\pi i}\sum_{n\geq 1}\frac{a_n}{n}q^n - a_0\tau.$$

One outcome of Lemma 5 is the following formula for integration by parts: if the forms $\mathrm{d}f$ and $\mathrm{d}g$ are admissible, then f and g are admissible as well, and

$$\int_\tau^\infty \frac{\mathrm{d}f}{\mathrm{d}\tau}(\tau_1)g(\tau_1)\,\mathrm{d}\tau_1 = f(\infty)g(\infty) - f(\tau)g(\tau) - \int_\tau^\infty f(\tau_1)\frac{\mathrm{d}g}{\mathrm{d}\tau}(\tau_1)\,\mathrm{d}\tau_1. \tag{5}$$

Once again, here $f(\infty)$ and $g(\infty)$ are the corresponding regularised values at ∞.

In a similar spirit we can introduce regularisation at zero; it makes use of involution $\sigma\colon \tau \mapsto -1/\tau$ of H. For a differential form ω on $]0, i\infty[$, we write $\omega^\sigma = \sigma^*\omega$. A C^∞ function $f\colon\ ]0, i\infty[\ \to \mathbb{C}$ is called *admissible at* 0 if the function $g(\tau) = f(-1/\tau)$ is admissible at ∞; in this case, the *regularised value of* f *at* 0 is defined as $f(0) = g(\infty)$. A C^∞ differential form ω on $]0, i\infty[$ is called *admissible at* 0 if ω^σ is admissible at ∞; for such a form we set

$$\int_0^\tau \omega = -\int_{-1/\tau}^\infty \omega^\sigma,$$

which is well-defined since ω^σ is admissible at ∞.

Lemma 6. *Let ω be a differential form on $]0, i\infty[$ which is admissible at* 0. *Then $\int_0^\tau \omega$ is the unique primitive of ω whose regularised value at* 0 *is zero.*

Proof. By Lemma 4 applied to ω^σ, we know that $\mathrm{d}\left(\int_\tau^\infty \omega^\sigma\right) = -\omega^\sigma$. Pulling back by $\sigma\colon \tau \mapsto -1/\tau$ gives the desired identity. The statement about the regularised value at 0 follows from the definition and Lemma 4. ☐

Finally, a function or differential form on $]0, i\infty[$ is called *admissible* if it is admissible at both 0 and ∞. Lemmas 2 and 4 also hold for admissibility at 0, and thus for general admissibility.

Lemma 7. *Let ω be an admissible differential form on $]0, i\infty[$, and $f(\tau)$ any primitive of ω. Then f is admissible.*

The only polynomials in τ which are admissible are the constants. If f is a modular form of weight $k \geq 2$ (on a finite-index subgroup of Γ), then $f(\tau)\tau^{m-1}\mathrm{d}\tau$ is admissible for any $m \in \{1, \dots, k-1\}$; if f is a cusp form, then $f(\tau)\tau^{m-1}\mathrm{d}\tau$ is admissible for any $m \in \mathbb{Z}$. Furthermore, the regularised integral $\int_0^\infty f(\tau)\tau^{m-1}\,\mathrm{d}\tau$ recovers the modular value $i^{-m}\Lambda(f, m)$.

4. Iterated integrals

A general situation is as follows. Given $\omega_1, \dots, \omega_r$ smooth 1-forms on a differentiable manifold M and a piecewise smooth path $\gamma\colon [0,1] \to M$, the iterated integral of $\omega_1, \dots, \omega_r$ along γ is

$$\int_\gamma \omega_1 \cdots \omega_r = \int\cdots\int_{0<z_1<\cdots<z_r<1} \gamma^*(\omega_1)(z_1)\cdots\gamma^*(\omega_r)(z_r);$$

the empty iterated integral (when $r = 0$) is defined to be 1. Then Chen's composition-of-paths formula reads

$$\int_{\gamma\gamma'} \omega_1 \cdots \omega_r = \sum_{j=0}^r \int_\gamma \omega_1 \cdots \omega_j \int_{\gamma'} \omega_{j+1} \cdots \omega_r \tag{6}$$

whenever $\gamma(1)=\gamma'(0)$ and where $\gamma\gamma'$ denotes the path γ followed by γ'. The shuffle-product formula states that iterated integration along a path is a homomorphism for the shuffle product, which after extending the definition by linearity becomes

$$\int_\gamma \omega_1\cdots\omega_r \int_\gamma \omega_1'\cdots\omega_l' = \int_\gamma \omega_1\cdots\omega_r \sqcup\!\sqcup\, \omega_1'\cdots\omega_l',$$

while the reversal-of-path formula states that

$$\int_{\gamma^{-1}} \omega_1\cdots\omega_r = (-1)^r \int_\gamma \omega_r\cdots\omega_1$$

where γ^{-1} denotes the reversed path $z\mapsto\gamma(1-z)$. In many situations it is convenient to work with generating series of iterated integrals indexed by non-commuting variables, however we will not pursue the topic in this generality.

We will now extend Brown's regularisation in our practical form to the case of path $]0,i\infty[$ and admissible differential forms.

For differential forms $\omega_1,\dots,\omega_r$ on $]0,i\infty[$ which are admissible at ∞, define

$$\int_\tau^\infty \omega_1\cdots\omega_r = \lim_{y\to+\infty}\sum_{j=0}^r \int_\tau^{iy} \omega_1\cdots\omega_j \times \int_{iy}^0 \omega_{j+1}^\infty\cdots\omega_r^\infty. \tag{7}$$

The convergence in (7) will be justified below.

Classically, the iterated integral is understood as a succession of one-variable integrals, and applications of regularised iterated integrals also require such interpretations. For that purpose, we introduce the following 'naïve' iteration:

$$\int_\tau^{\infty,*} \omega_1\cdots\omega_r = \int_\tau^\infty \omega_1(\tau_1)\int_{\tau_1}^\infty \omega_2(\tau_2)\cdots\int_{\tau_{r-1}}^\infty \omega_r(\tau_r), \tag{8}$$

where the right-hand integrals are understood as (4).

Lemma 8. *The naïve iterated integral $\int_\tau^{\infty,*}\omega_1\cdots\omega_r$ is well-defined and is admissible at ∞ as a function of τ. Its regularised value at ∞ is zero.*

Proof. This follows from inductive application of Lemmas 2 and 4. □

Proposition 9. *Let $\omega_1,\dots,\omega_r$ be differential forms which are admissible at ∞. Then*

$$\int_\tau^\infty \omega_1\cdots\omega_r = \int_\tau^{\infty,*} \omega_1\cdots\omega_r.$$

To prove this, we need the following lemma.

Lemma 10. *The polynomial part of the naïve iterated integral is given by*

$$\biggl(\int_\tau^{\infty,*} \omega_1\cdots\omega_r\biggr)^\infty = \int_\tau^0 \omega_1^\infty\cdots\omega_r^\infty,$$

where the right-hand side is the usual (absolutely convergent) iterated integral.

Proof. We proceed by induction on r. The case $r=1$ follows from Lemma 3. For $r\ge2$, we have

$$\int_\tau^{\infty,*} \omega_1\cdots\omega_r = \int_\tau^{\infty,*} \omega_1(\tau_1)\int_{\tau_1}^{\infty,*} \omega_2\cdots\omega_r.$$

By the induction hypothesis applied to $\omega_2\cdots\omega_r$, we have

$$\biggl(\omega_1(\tau_1)\int_{\tau_1}^{\infty,*} \omega_2\cdots\omega_r\biggr)^\infty = \omega_1^\infty(\tau_1)\biggl(\int_{\tau_1}^{\infty,*} \omega_2\cdots\omega_r\biggr)^\infty = \omega_1^\infty(\tau_1)\int_{\tau_1}^0 \omega_2^\infty\cdots\omega_r^\infty.$$

Therefore, using Lemma 3,

$$\int_\tau^{\infty,*} \omega_1 \cdots \omega_r = \int_\tau^\infty \left(\omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \cdots \omega_r\right)^0 + \int_\tau^0 \omega_1^\infty(\tau_1) \int_{\tau_1}^0 \omega_2^\infty \cdots \omega_r^\infty. \tag{9}$$

The first term in (9) decays exponentially as $\tau \to \infty$, and the second term is a polynomial in τ, which finishes the proof. □

Proposition 9 is now a consequence of the following finer result, which controls the convergence as $y \to +\infty$.

Proposition 11. *We have*

$$\sum_{j=0}^r \int_\tau^{iy} \omega_1 \cdots \omega_j \times \int_{iy}^0 \omega_{j+1}^\infty \cdots \omega_r^\infty = \int_\tau^{\infty,*} \omega_1 \cdots \omega_r + O_{y\to+\infty}(c^y), \tag{10}$$

the constant c, $0 < c < 1$, being uniform with respect to τ on domains of the form $\{\operatorname{Im}\tau \geq y_0 > 0\}$.

Proof. We proceed by induction on r. The case $r = 1$ follows from Lemma 3. Let $r \geq 2$. Using the induction hypothesis to $\omega_2 \cdots \omega_r$, the left-hand side of (10) can be written as

$$\begin{aligned}
&\int_\tau^p \omega_1(\tau_1)\left(\sum_{j=1}^r \int_{\tau_1}^{iy} \omega_2 \cdots \omega_j \times \int_{iy}^0 \omega_{j+1}^\infty \cdots \omega_r^\infty\right) + \int_{iy}^0 \omega_1^\infty \cdots \omega_r^\infty \\
&= \int_\tau^{iy} \omega_1(\tau_1)\left(\int_{\tau_1}^{\infty,*} \omega_2 \cdots \omega_r + O_{y\to+\infty}(c^y)\right) + \int_{iy}^0 \omega_1^\infty \cdots \omega_r^\infty \\
&= \int_\tau^{iy} \omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \cdots \omega_r + \left(\int_\tau^{iy} \omega_1(\tau_1)\right) O_{y\to+\infty}(c^y) + \int_{iy}^0 \omega_1^\infty \cdots \omega_r^\infty \\
&= \int_\tau^{iy} \omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \cdots \omega_r + \int_{iy}^0 \omega_1^\infty \cdots \omega_r^\infty + O_{y\to+\infty}(c_2^y).
\end{aligned} \tag{11}$$

Consider the differential form

$$\alpha(\tau_1) = \omega_1(\tau_1) \int_{\tau_1}^{\infty,*} \omega_2 \cdots \omega_r.$$

Applying Lemma 10 to $\omega_2 \cdots \omega_r$, the polynomial part of α is

$$\alpha^\infty(\tau_1) = \omega_1^\infty(\tau_1) \int_{\tau_1}^0 \omega_2^\infty \cdots \omega_r^\infty.$$

Therefore,

$$\begin{aligned}
(11) &= \int_\tau^{iy} \alpha(\tau_1) + \int_{iy}^0 \omega_1^\infty(\tau_1) \int_{\tau_1}^0 \omega_2^\infty \cdots \omega_r^\infty + O_{y\to+\infty}(c_2^y) \\
&= \int_\tau^{iy} \alpha(\tau_1) + \int_{iy}^0 \alpha^\infty(\tau_1) + O_{y\to+\infty}(c_2^y) \\
&= \int_\tau^{\infty,*} \alpha + O_{y\to+\infty}(c_3^y).
\end{aligned}$$

□

Proposition 9 and Lemma 4 have the following consequence.

Lemma 12. *For any differential forms $\omega_1, \ldots, \omega_r$ which are admissible at ∞, we have*

$$\mathrm{d}\left(\int_\tau^\infty \omega_1 \cdots \omega_r\right) = -\omega_1(\tau) \int_\tau^\infty \omega_2 \cdots \omega_r.$$

We now proceed with defining the regularised iterated integral from 0 to τ of differential forms $\omega_1,\dots,\omega_r$ which are admissible at 0. We set

$$\int_0^\tau \omega_1\cdots\omega_r=\int_\infty^{-1/\tau}\omega_1^\sigma\cdots\omega_r^\sigma=(-1)^r\int_{-1/\tau}^\infty \omega_r^\sigma\cdots\omega_1^\sigma,$$

where the last equality is an application of the reversal-of-path formula.

We have the following analogues of Lemmas 8 and 12.

Lemma 13. *The integral $\int_0^\tau \omega_1\cdots\omega_r$ is admissible at 0 as a function of τ, and its regularised value at 0 is zero.*

Proof. This follows from Lemma 8 applied to $\omega_r^\sigma\cdots\omega_1^\sigma$. □

Lemma 14. *We have*

$$\mathrm{d}\biggl(\int_0^\tau \omega_1\cdots\omega_r\biggr)=\omega_r(\tau)\int_0^\tau \omega_1\cdots\omega_{r-1}.$$

Proof. By Lemma 12, we have

$$\mathrm{d}\biggl(\int_\tau^\infty \omega_r^\sigma\cdots\omega_1^\sigma\biggr)=-\omega_r^\sigma(\tau)\int_\tau^\infty \omega_{r-1}^\sigma\cdots\omega_1^\sigma.$$

Applying σ^* to this identity gives

$$\mathrm{d}\biggl(\int_{-1/\tau}^\infty \omega_r^\sigma\cdots\omega_1^\sigma\biggr)=-\omega_r(\tau)\int_{-1/\tau}^\infty \omega_{r-1}^\sigma\cdots\omega_1^\sigma=(-1)^r\omega_r(\tau)\int_0^\tau \omega_1\cdots\omega_{r-1}.$$ □

Finally, if $\omega_1,\dots,\omega_r$ are admissible differential forms on $]0,i\infty[$, define

$$\int_0^\infty \omega_1\cdots\omega_r=\sum_{j=0}^r\int_0^\tau \omega_1\cdots\omega_j\times\int_\tau^\infty \omega_{j+1}\cdots\omega_r. \tag{12}$$

Lemma 15. *The definition (12) does not depend on τ.*

Proof. Using Lemmas 12 and 14, the differential of the right-hand side of (12) is

$$\sum_{j=0}^{r-1}\int_0^\tau \omega_1\cdots\omega_j\times\bigl(-\omega_{j+1}(\tau)\bigr)\int_\tau^\infty \omega_{j+2}\cdots\omega_r$$
$$+\sum_{j=1}^{n}\omega_j(\tau)\int_0^\tau \omega_1\cdots\omega_{j-1}\times\int_\tau^\infty \omega_{j+1}\cdots\omega_r$$

which vanishes by changing $k\to k+1$ in the second sum. □

The last lemma concerns with the use of the regularised iterated integral as an iterated integral.

Proposition 16. *Let $\omega_1,\dots,\omega_r$ be admissible differential forms on $]0,i\infty[$. Then*

$$\int_\tau^\infty \omega_1\cdots\omega_r$$

is admissible at 0 as a function of τ, and its regularised value at 0 is $\int_0^\infty \omega_1\cdots\omega_r$. Moreover,

$$\int_0^\infty \omega_1\cdots\omega_r=\int_0^\infty \omega_1(\tau_1)\int_{\tau_1}^\infty \omega_2(\tau_2)\cdots\int_{\tau_{r-1}}^\infty \omega_r(\tau_r), \tag{13}$$

where the right-hand side of (13) is understood as successive one-variable regularisations.

Proof. For the first part of the proposition, we proceed by induction on r. The case $r = 1$ follows from $\int_0^\infty \omega_1 = \int_0^\tau \omega_1 + \int_\tau^\infty \omega_1$ and Lemma 6. For $r \geq 2$, we can write

$$\int_\tau^\infty \omega_1 \cdots \omega_r = \int_0^\infty \omega_1 \cdots \omega_r - \sum_{j=1}^r \int_0^\tau \omega_1 \cdots \omega_j \times \int_\tau^\infty \omega_{j+1} \cdots \omega_r.$$

By the induction hypothesis and Lemma 13, the right-hand side is admissible at 0. Moreover, the regularised value at 0 of the product

$$\int_0^\tau \omega_1 \cdots \omega_j \times \int_\tau^\infty \omega_{j+1} \cdots \omega_r$$

is the product of the regularised values, hence it is zero by Lemma 13.

Finally, (13) follows formally by using the case $r = 1$ with the form $\omega_1(\tau_1) \int_{\tau_1}^\infty \omega_2 \cdots \omega_r$. □

An important feature of all the regularisations of iterated integrals we have discussed, $\int_0^\infty$ as well as $\int_0^\tau$ and $\int_\tau^\infty$, is that they satisfy the shuffle relations. One also has the Newton–Leibniz formula $\int_0^\infty \mathrm{d}f = f(\infty) - f(0)$ generalising the result of Lemma 5, which itself is a particular instance of integration by parts

$$\begin{aligned}\int_0^\infty & \omega_1(\tau_1) \cdots \frac{\mathrm{d}f}{\mathrm{d}\tau}(\tau_p)\, \mathrm{d}\tau_p \cdots \omega_r(\tau_r) \\ &= \int_0^\infty \omega_1(\tau_1) \cdots \omega_{p-1}(\tau_{p-1})\, f(\tau_{p+1}) \omega_{p+1}(\tau_{p+1})\, \omega_{p+2}(\tau_{p+2}) \cdots \omega_r(\tau_r) \\ &\quad - \int_0^\infty \omega_1(\tau_1) \cdots \omega_{p-2}(\tau_{p-2})\, f(\tau_{p-1}) \omega_{p-1}(\tau_{p-1})\, \omega_{p+1}(\tau_{p+1}) \cdots \omega_r(\tau_r),\end{aligned}$$

in which the first summand is interpreted as

$$f(\infty) \int_0^\infty \omega_1(\tau_1) \cdots \omega_{r-1}(\tau_{r-1})$$

when $p = r$, while the second summand is

$$-f(0) \int_0^\infty \omega_2(\tau_2) \cdots \omega_r(\tau_r)$$

when $p = 1$. Furthermore, one can also differentiate the regularised (iterated) integral when a differential form depends smoothly on a *real* parameter. Details of the corresponding statement as well as proofs of all these facts can be found in our joint work with Brunault [6, Section 2].

5. Multiple modular values

For modular forms $f_1, \dots, f_r$ of respective weights $k_1, \dots, k_r \geq 2$, and any integers $m_1, \dots, m_r$ with $1 \leq m_j \leq k_j - 1$, the regularised iterated integral

$$\begin{aligned}\Lambda(f_1, \dots, f_r; m_1, \dots, m_r) &= \int_0^\infty f_1(\tau)\tau^{m_1 - 1}\mathrm{d}\tau \cdots f_r(\tau)\tau^{m_r - 1}\mathrm{d}\tau \\ &= \int_0^\infty f_1(\tau_1)\tau_1^{m_1-1}\mathrm{d}\tau_1 \int_{\tau_1}^\infty f_2(\tau_2)\tau_2^{m_2-1}\mathrm{d}\tau_2 \cdots \int_{\tau_{r-1}}^\infty f_r(\tau_r)\tau_r^{m_r-1}\mathrm{d}\tau_r\end{aligned} \tag{14}$$

is called a (totally holomorphic) *multiple modular value* (MMV) of length (or depth) r. In the case all m_j are equal to 1, we simply write $\Lambda(f_1, \dots, f_r) = \Lambda(f_1, \dots, f_r; 1, \dots, 1)$. Sometimes the normalised version $(2\pi)^{k_1 + \cdots + k_r - r} \Lambda(f_1, \dots, f_r; m_1, \dots, m_r)$ is considered as a totally holomorphic MMV; when $m_1 = \cdots = m_r = 1$ the two normalisations agree.

In the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, the multiple modular values are periods of the relative completion of the fundamental group of $\mathcal{M}_{1,1}$, the moduli space of elliptic curves; they do not exhaust all periods of the latter. Applications to the integrals of regulator maps deal with the principal congruence subgroup $\Gamma(N)$ and with f_j being Eisenstein series of weight ≥ 2 on $\Gamma(N)$. In this case (14) is also called a *multiple Eisenstein value* (MEV). The case where all f_j are cuspidal was considered by Manin, and the integral is an ordinary integral from 0 to ∞, so regularisation is not required.

It follows from the general properties of iterated integrals that the numbers (14) satisfy a reflection symmetry and shuffle product relations; in particular, they generate an algebra. The values $\Lambda(f_1,\dots,f_r;m_1,\dots,m_r)$ for fixed f_j and varying $0 < m_j < k_j$ satisfy non-abelian cocycle relations, which are realised as functional equations of generalised period polynomials.

There is a mechanism of *transference* of periods between integrals of different sets of modular forms. It can be viewed as a higher analogue of the Petersson inner product. It generates relations between multiple modular values

$$\Lambda(f_1,\dots,f_p;\,\cdot\,)\Lambda(f_{p+1},\dots,f_r;\,\cdot\,) \quad \text{for } 1 \le p < r$$

where $f_1,\dots,f_r$ are any fixed set of modular forms. The transference principle roots in applications of the Rankin–Selberg method (but is more general!); it can be viewed as a manifestation of a certain convolution on modular forms. For example, it implies a relation between multiple modular values of the form $\Lambda(f_1,f_2)\Lambda(f_3)$ and those of the form $\Lambda(f_1)\Lambda(f_2,f_3)$. It follows that the multiple modular values $\Lambda(f_1,f_2;m_1,m_2)$ for any two modular forms f_1, f_2 pick up information from *a priori* unrelated modular forms f_3.

In the level 1 case, certain linear combinations of iterated Eisenstein integrals are multiple zeta values. This is a consequence of the fact that the relative completion of the fundamental group of $\mathcal{M}_{1,1}$ has a monodromy representation on the pro-unipotent fundamental group of the punctured Tate curve, whose periods are multiple zeta values.

Another mysterious source of linear relations between MMVs is Beilinson's conjectures on extensions of motives; such relations presently have no direct proof.

6. Multiple modular values of level 1

In the case of modular forms on the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, multiple modular values of length 1 are generated by powers of $2\pi i$, odd zeta values, periods of cusp forms (all of which are totally holomorphic) and quasi-periods of cusp forms (which are not totally holomorphic).

For f a modular form of even weight k, the totally holomorphic values (14) were already discussed as modular values in Section 2. The factorisation $L(E_k,s) = \zeta(s)\zeta(s-k+1)$ of the L-function of the Eisenstein series $E_k(\tau)$ implies that the even values satisfy

$$\Lambda(E_k;2j) = \frac{(-1)^j}{2}\,\frac{B_{2j}}{2j}\,\frac{B_{k-2j}}{k-2j}.$$

The values $\Lambda(E_k;m)$ vanish for odd values of $3 \le m \le k-3$, while

$$(2\pi)^{k-1}\Lambda(E_k;k-1) = -\frac{(k-2)!}{2}\,\zeta(k-1) \quad \text{and} \quad \Lambda(E_k;1) = (-1)^{k/2}\Lambda(E_k;k-1).$$

For a general modular form f of weight k set

$$P_f(y) = \sum_{m=1}^{k-1} i^{k-m-1}\binom{k-2}{m-1}\Lambda(f;m)y^{m-1}. \tag{15}$$

The function P_f can be interpreted as the value of a canonical cocycle C_f in a certain cochain complex. The cocycle relations imply functional relations for P_f. When f is cuspidal, these are equivalent to the period polynomial equations

$$P_f(y) + y^{k-2} P_f(-y^{-1}) = 0,$$
$$P_f(y) + (1-y)^{k-2} P_f\left(\frac{1}{1-y}\right) + y^{k-2} P_f\left(\frac{y-1}{y}\right) = 0.$$

The first equation follows from the functional equation of Λ. A variant of these equations is satisfied for an Eisenstein series $f = E_k$. The 'extra relation' of Kohnen and Zagier, which expresses orthogonality of cusp forms to Eisenstein series, is a consequence of transference equations. Manin showed that when f is a Hecke eigenform, the function P_f is an eigenfunction for a certain action of Hecke operators. From this he deduced that

$$P_f = \omega_f^+ P_{f,+} + i\omega_f^- P_{f,-} \tag{16}$$

where $\omega_f^+, \omega_f^- \in \mathbb{R}$ and $P_{f,+}, P_{f,-} \in K_f[y]$ where K_f is the field generated by the Fourier coefficients of f.

There is a generalisation of period polynomials to higher length, which encode cocycle relations but also shuffle relations but it fails to see many other relations. In length 2, for example, the period polynomial is defined by

$$P_{f_1,f_2}(y_1, y_2) = \sum_{m_1=1}^{k_1-1} i^{k_1-m_1-1} \binom{k_1-2}{m_1-1} \sum_{m_2=1}^{k_2-1} i^{k_2-m_2-1} \binom{k_2-2}{m_2-1} \Lambda(f_1, f_2; m_1, m_2) y_1^{m_1-1} y_2^{m_2-1}$$

and the shuffle product implies that

$$P_{f_1,f_2}(y_1, y_2) + P_{f_2,f_1}(y_2, y_1) = P_{f_1}(y_1) P_{f_2}(y_2).$$

Details about cocycle relations can be found in Brown's paper [3].

In length 2 and higher, one mostly look for the multiple Eisenstein values, as they seem to be 'more structural.' For example, a theorem of Saad states that every multiple zeta value of depth r and weight k can be expressed as a rational linear combination of MEVs

$$(2\pi)^k \Lambda(E_{k_1}, \dots, E_{k_r}; m_1, \dots, m_r), \quad \text{where } 1 \le m_j \le k_j - 1 \text{ and } k_1 + \dots + k_r - r = k.$$

On the other hand, one conjecture predicts that modulo lower-depth MZVs we have

$$(2\pi)^{k_1+\dots+k_r-r} \Lambda(E_{k_1}, \dots, E_{k_r}; 1, \dots, 1) \equiv (-1)^{(k_1+\dots+k_r)/2-r} \frac{(k_1-1)! \cdots (k_r-1)!}{2^r} \zeta_{k_1-1,\dots,k_r-1},$$

where $\zeta_{k_1-1,\dots,k_r-1}$ is a different indexation of MZVs associated to a certain inverse problem for periods of a mixed Tate motive over $\mathbb{Z}$.

One can also produce explicit evaluations of MEVs of length 2 (and 3) for lower-weight Eisenstein series. MEVs $\pi^6 \Lambda(E_4, E_4; m_1, m_2)$ are polynomials in $\pi, \zeta(3), \zeta(5)$. MEVs

$$\pi^8 \Lambda(E_4, E_6; m_1, m_2) = \pi^8 \Lambda(E_6, E_4; 6 - m_2, 4 - m_1)$$

make the appearance of the MZV $\zeta(3, 5)$ and otherwise of π and odd zeta values. One needs to go to weight $k_1 + \dots + k_r - r \ge 12$ to encounter modular periods. For example, two new periods which are not expected to be multiple zeta values, are

$$\pi^{-1} \Lambda(\Delta; 12) = 600\Lambda(E_4, E_{10}; 2, 5) + 480\Lambda(E_4, E_{10}; 3, 4)$$

(provable using the Rankin–Selberg method) which is a *non-critical* value of the completed L-function of Δ, and

$$c(\Delta; 12) = 70\Lambda(E_4, E_{10}; 3, 5) = 0.00022512654819026299916898101\,5\dots$$

which is a new number only well-defined modulo $\mathbb{Q}$: one might equally well have taken the quantity

$$\Lambda(E_4, E_{10}; 2, 6) = \frac{13}{2^{12}3^5 5^2 7{\cdot}11} + \Lambda(E_4, E_{10}; 3, 5)$$

as a representative. We further expect

$$\Lambda(E_4, E_{10}; 1, 1) \overset{?}{=} \frac{2^2 3^2}{691} c(\Delta; 12) - \frac{3^2 5{\cdot}7}{2^6} \frac{\zeta_{3,9}}{\pi^{12}}, \tag{17}$$

where

$$\begin{aligned}\zeta_{3,9} = \frac{1}{19{\cdot}691}\biggl(& 2^4 3^2 \zeta(5,3,2,2) - \frac{3^3 5{\cdot}179}{2{\cdot}7}\zeta(5,7) - 2{\cdot}3^3 29\zeta(7,5) \\ & - 3{\cdot}7^2\zeta(3)\zeta(9) + 2^4 3\zeta(3)^4 + 2^5 3^3 11\zeta(3,7)\zeta(2) + 2^5 3^2 31\zeta(7,3)\zeta(2) \\ & - 2^4 3^4 \zeta(3,5)\zeta(4) - 2^5 3^2 \zeta(5,3)\zeta(4) - 2^3 3{\cdot}5^2\zeta(3)^2\zeta(6) + \frac{3{\cdot}128583229}{2^4 7{\cdot}691}\zeta(12)\biggr)\end{aligned}$$

is the multiple zeta value of weight 12 and (presumably) depth 4; this is consistent with the conjecture mentioned above.

7. Multiple Eisenstein values and regulator integrals

I will use the notation $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$, so that $q = e(\tau)$ for $\tau \in H$. For any $\alpha \in \mathbb{R}$, write also $q^\alpha = e(\alpha\tau)$. The Eisenstein series of weight $k \ge 1$ (even or odd) to consider are indexed by the coordinates $\boldsymbol{x} = (x_1, x_2) \in (\mathbb{R}/\mathbb{Z})^2$ on the lattice $\mathbb{Z} + \tau\mathbb{Z}$ (so that $x_1 + x_2\tau \in \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ is the elliptic variable), with $\boldsymbol{x} \ne \boldsymbol{0}$ in the case $k = 2$. The series are defined via their Fourier expansion as follows:

$$E_{\boldsymbol{x}}^{(k)}(\tau) = a_0(E_{\boldsymbol{x}}^{(k)}) - \sum_{\substack{m\ge1\\ n\in\mathbb{R}_{>0}\\ n\equiv x_1 \bmod 1}} e(mx_2) n^{k-1} q^{mn} + (-1)^{k+1} \sum_{\substack{m\ge1\\ n\in\mathbb{R}_{>0}\\ n\equiv -x_1 \bmod 1}} e(-mx_2) n^{k-1} q^{mn}, \tag{18}$$

with

$$a_0(E_{\boldsymbol{x}}^{(k)}) = \frac{B_k(\{x_1\})}{k} \quad \text{for } k \ge 2,$$

where $B_k(t)$ is the k-th Bernoulli polynomial and $\{\,\cdot\,\}$ stands for the fractional part. One can also give a formula for $a_0(E_{\boldsymbol{x}}^{(k)})$ when $k = 1$; note however that there is no range for m to have the form $E_{\boldsymbol{x}}^{(k)}(\tau)\tau^{m-1}\,\mathrm{d}\tau$ admissible when $k = 1$.

Lemma 17. *Let $k \ge 1$ be an integer, and $\boldsymbol{x} \in (\mathbb{R}/\mathbb{Z})^2$. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, we have*

$$E_{\boldsymbol{x}}^{(k)}(\gamma\tau) = (c\tau + d)^k E_{\boldsymbol{x}\gamma}^{(k)}(\tau),$$

where $\boldsymbol{x}\gamma$ means the right multiplication by γ on the row vector $\boldsymbol{x}$.

Taking $\gamma = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ in Lemma 17, we see that $E_{-\boldsymbol{x}}^{(k)} = (-1)^k E_{\boldsymbol{x}}^{(k)}$. Lemma 17 also shows that if $\boldsymbol{x}$ is N-torsion in $(\mathbb{R}/\mathbb{Z})^2$, then $E_{\boldsymbol{x}}^{(k)}$ is a modular form of weight k on $\Gamma(N)$, except when $k = 2$ and $\boldsymbol{x} = \boldsymbol{0}$ (in which case $E_{\boldsymbol{0}}^{(2)}$ is not holomorphic).

The Eisenstein series $E_{\boldsymbol{x}}^{(2)}$ are related to the so-called Siegel units as follows. For $\boldsymbol{x} = (x_1, x_2) \in (\mathbb{R}/\mathbb{Z})^2$, $\boldsymbol{x} \ne \boldsymbol{0}$, consider the following function on H:

$$g_{\boldsymbol{x}}(\tau) = q^{B_2(\{x_1\})/2} \prod_{\substack{n\in\mathbb{R}_{\ge0}\\ n\equiv x_1 \bmod 1}} (1 - q^n e(x_2)) \prod_{\substack{n\in\mathbb{R}_{>0}\\ n\equiv -x_1 \bmod 1}} (1 - q^n e(-x_2)). \tag{19}$$

For $a, b \in \mathbb{Z}$, $(a,b) \not\equiv (0,0) \bmod N$, the function $g_{a/N,b/N}$ is none other than the classical Siegel unit $g_{\bar{a},\bar{b}}$. This function is a $(12N)$-th root of a unit on the modular curve $Y(N)$ over $\mathbb{Q}$, thus defining an element of $\mathcal{O}(Y(N))^\times \otimes \mathbb{Q}$.

We also define, for $\boldsymbol{x} \in (\mathbb{R}/\mathbb{Z})^2$, $\boldsymbol{x} \neq \mathbf{0}$, the logarithm of $g_{\boldsymbol{x}}$ by taking the logarithm of the infinite product (19) and specifying the branch:

$$\log g_{\boldsymbol{x}}(\tau) = \pi i B_2(\{x_1\})\tau + \log(1-e(x_2))\cdot\mathbf{1}_{x_1=0} - \sum_{\substack{m\geq 1\\ n\in\mathbb{R}_{>0}\\ n\equiv x_1 \bmod 1}} \frac{e(mx_2)}{m} q^{mn} - \sum_{\substack{m\geq 1\\ n\in\mathbb{R}_{>0}\\ n\equiv -x_1 \bmod 1}} \frac{e(-mx_2)}{m} q^{mn}, \tag{20}$$

where

$$\log(1-e(x_2)) = -\hat{\zeta}(x_2,1) = \log|1-e(x_2)| + \pi i\Big(\{x_2\} - \frac{1}{2}\Big).$$

Lemma 18. *For any $\boldsymbol{x} \in (\mathbb{R}/\mathbb{Z})^2$, $\boldsymbol{x} \neq \mathbf{0}$, we have $\mathrm{d}\log g_{\boldsymbol{x}}(\tau) = 2\pi i\, E_{\boldsymbol{x}}^{(2)}(\tau)\,\mathrm{d}\tau$.*

Proof. This follows from comparing the Fourier expansions (18) and (20). □

The series $E_{\boldsymbol{x}}^{(k)}$ satisfies a differential relation with respect to both elliptic and modular parameters, and this plays an important role in reduction of length of iterated integrals.

Lemma 19. *For $k \geq 1$, the function $\boldsymbol{x} \mapsto E_{\boldsymbol{x}}^{(k)}(\tau)$ is smooth on the domain $(\mathbb{R}/\mathbb{Z})^2 \setminus \{\mathbf{0}\}$. Moreover, we have*

$$\frac{\partial}{\partial x_2} E_{\boldsymbol{x}}^{(k+1)}(\tau) = \frac{\partial}{\partial \tau} E_{\boldsymbol{x}}^{(k)}(\tau). \tag{21}$$

Proof. The Fourier expansion (18) shows that $\boldsymbol{x} \mapsto E_{\boldsymbol{x}}^{(k)}(\tau)$ is smooth on the domain $\{x_1 \neq 0\}$. Using Lemma 17 with $\gamma = \sigma$, the function is also smooth on $\{x_2 \neq 0\}$, whence the claim.

The identity (21) follows either by inspecting the Fourier expansions of both sides. □

Finally, for $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r \in (\mathbb{R}/\mathbb{Z})^2$, define

$$\begin{aligned}\Lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r) &= (2\pi i)^r \Lambda(E_{\boldsymbol{x}_1}^{(2)}, \ldots, E_{\boldsymbol{x}_r}^{(2)}) \\ &= \int_0^\infty \mathrm{d}\log g_{\boldsymbol{x}_1}\, \mathrm{d}\log g_{\boldsymbol{x}_2} \cdots \mathrm{d}\log g_{\boldsymbol{x}_r}.\end{aligned}$$

These are general examples of (totally holomorphic) MEVs of length r. In general, we expect $\Lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r)$ to be a period only when the parameters $\boldsymbol{x}_j$ belong to $(\mathbb{Q}/\mathbb{Z})^2$. In the latter case, we can implicitly identify $(\mathbb{Z}/N\mathbb{Z})^2$ with a subgroup of $(\mathbb{R}/\mathbb{Z})^2$ by mapping a pair $(\overline{x}_1, \overline{x}_2)$ to the class of $(x_1/N, x_2/N)$. In this way $\Lambda(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r)$ makes sense for arguments $\boldsymbol{x}_j$ in $(\mathbb{Z}/N\mathbb{Z})^2$.

The *single* modular values are essentially the critical L-values of a modular form. In the particular case of an Eisenstein series, these values are computed classically in terms of Bernoulli polynomials.

As an exercise, one can check that the single modular value $\Lambda(\boldsymbol{x})$ can be evaluated in a closed form:

$$\Lambda(\boldsymbol{x}) = \begin{cases} 2\pi i \left(\{x_1\} - \frac{1}{2}\right)\left(\{x_2\} - \frac{1}{2}\right) & \text{if } x_1, x_2 \neq 0, \\ \log|1-e(x_2)| & \text{if } x_1 = 0,\ x_2 \neq 0, \\ -\log|1-e(x_1)| & \text{if } x_1 \neq 0,\ x_2 = 0. \end{cases}$$

Note that the function $\boldsymbol{x} \mapsto \Lambda(\boldsymbol{x})$ has discontinuities at $\{x_1 = 0\} \cup \{x_2 = 0\}$.

Let $Y(N)$ be the modular curve over $\mathbb{Q}$ of level $N \ge 1$. The cup-products $\{g_{\boldsymbol{a}}, g_{\boldsymbol{b}}\}$ of two Siegel units $g_{\boldsymbol{a}}$ and $g_{\boldsymbol{b}}$ provide important elements in the K-group $K_2^{(2)}(Y(N))$; the Beilinson regulator of $\{g_{\boldsymbol{a}}, g_{\boldsymbol{b}}\}$ is represented by the differential form $i\,\eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}})$ on $Y(N)(\mathbb{C})$, where

$$\eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}}) = \log|g_{\boldsymbol{a}}|\,\mathrm{d}\arg g_{\boldsymbol{b}} - \log|g_{\boldsymbol{b}}|\,\mathrm{d}\arg g_{\boldsymbol{a}}.$$

Here $\log|g_{\boldsymbol{a}}|$ and $\arg g_{\boldsymbol{a}}$ are the real and imaginary parts of the admissible form $\log g_{\boldsymbol{a}}$. The following statement is more-or-less tautological.

Proposition 20. *Let all the coordinates $\boldsymbol{a}, \boldsymbol{b} \in (\mathbb{Z}/N\mathbb{Z})^2$ be different from 0. Then for the regulator integral of $\eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}})$ along the modular symbol $\{0, i\infty\}$ we have*

$$\int_0^\infty \eta(g_{\boldsymbol{a}}, g_{\boldsymbol{b}}) = \operatorname{Im}\Lambda(\boldsymbol{a}, \boldsymbol{b}).$$

The very same regulator integral can be computed in terms of L-values at $s = 0$ of modular forms of weight 2 and level $\Gamma(N)$. Beilinson proved this using the Rankin–Selberg method. A version of Beilinson result with the corresponding modular form of weight 2 provided explicitly was given by Brunault [4] based on a method developed earlier in my joint works with Mat Rogers [11, 12]. In our latest work [6] with Brunault we compute the (Goncharov) regulator integral $\mathcal{G}(\boldsymbol{a}, \boldsymbol{b})$ of special classes in the K-group $K_4^{(3)}(Y(N))$ (these are constructed in [5]) which are similarly indexed by $\boldsymbol{a}, \boldsymbol{b} \in (\mathbb{Z}/N\mathbb{Z})^2$. The story is quite involved (already the expression of the Goncharov regulator map in terms of dilogarithm and logarithm is much more complex than that of the Beilinson regulator η) and the final expression of the regulator integral is as follows (the statement reflects a simplification obtained recently by Brunault's student Boisan; the original formulation in [6] is less economical).

Theorem 21. *Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in (\mathbb{Z}/N\mathbb{Z})^2$ such that $\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c} = \boldsymbol{0}$. Assume that all the coordinates of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-zero. Then*

$$\mathcal{G}(\boldsymbol{a}, \boldsymbol{b}) = \operatorname{Re}\big(\Lambda(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{b}) - \Lambda(\boldsymbol{c}, \boldsymbol{b}, \boldsymbol{b}) + \Lambda(\boldsymbol{b}, \boldsymbol{a}, \boldsymbol{a}) - \Lambda(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{a}) + \Lambda(\boldsymbol{c}, \boldsymbol{b}, \boldsymbol{a}) + \Lambda(\boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b})\big).$$

The resulting MEV expression for the regulator integral is interpreted in terms of interpolated Eisenstein series, when each $\boldsymbol{a} \in (\mathbb{Z}/N\mathbb{Z})^2$ is rescaled to $\boldsymbol{a}/N \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$ and the latter interpolates to a function of $\boldsymbol{a}$ on $(\mathbb{R}/\mathbb{Z})^2$. This gives us access to differentiating with respect to a_2. The differentiation of $\mathcal{G}(\boldsymbol{a}, \boldsymbol{b})$ is preceded by derivation of auxiliary Borisov–Gunnells relations [1] for pairwise products of Eisenstein series; the resulting expression of $\frac{\partial}{\partial a_2}\mathcal{G}(\boldsymbol{a}, \boldsymbol{b})$ is then reduced to a single modular value using the Rogers–Zudilin method. In turn, our proof of the Borisov–Gunnells relations requires the level N structure to be used, so that we make several switches between interpolated and non-interpolated Eisenstein series. Finally, we deduce a single modular value expression for $\mathcal{G}(\boldsymbol{a}, \boldsymbol{b})$ by integrating its a_2-derivative.

Theorem 22. *For any $\boldsymbol{a} = (a_1, a_2)$, $\boldsymbol{b} = (b_1, b_2)$ in $(\mathbb{Z}/N\mathbb{Z})^2$ such that the coordinates of $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a}+\boldsymbol{b}$ are non-zero, we have*

$$\begin{aligned}\mathcal{G}(\boldsymbol{a}, \boldsymbol{b}) = \frac{3\pi^2}{N} L'\big(f_{\boldsymbol{a},\boldsymbol{b}}, -1\big) \\ - \frac{\zeta(3)}{4}\big(B_2(\{\tfrac{a_1}{N}\}) + B_2(\{\tfrac{b_1}{N}\}) + 4B_1(\{\tfrac{a_1}{N}\})B_1(\{\tfrac{b_1}{N}\}) \\ - B_2(\{\tfrac{a_2}{N}\}) - B_2(\{\tfrac{b_2}{N}\}) - 4B_1(\{\tfrac{a_2}{N}\})B_1(\{\tfrac{b_2}{N}\})\big),\end{aligned}$$

with a modular form $f_{\boldsymbol{a},\boldsymbol{b}}$ given explicitly as a linear combination of products of weight 1 Eisenstein series.

Note that $L'(f_{\boldsymbol{a},\boldsymbol{b}}, -1)$ connects to $L(f_{\boldsymbol{a},\boldsymbol{b}}, 3)$ via the functional equation for the L-function.

References

[1] L. A. Borisov and P. E. Gunnells, Toric modular forms of higher weight, *J. Reine Angew. Math.* **560** (2003), 43–64.

[2] F. Brown, Multiple Modular Values and the relative completion of the fundamental group of $M_{1,1}$, arXiv:1407.5167v4 [math.NT].

[3] F. Brown, From the Deligne–Ihara conjecture to Multiple Modular Values, arXiv:1904.00179 [math.AG].

[4] F. Brunault, Regulators of Siegel units and applications, *J. Number Theory* **163** (2016), 542–569.

[5] F. Brunault, On the K_4 group of modular curves, arXiv:2009.07614v2 [math.NT].

[6] F. Brunault and W. Zudilin, Modular regulators and multiple Eisenstein values, arXiv:2303.15554 [math.NT].

[7] M. E. Hoffman, Multiple harmonic series, *Pacific J. Math.* **152** (1992), no. 2, 275–290.

[8] M. E. Hoffman, The algebra of multiple harmonic series, *J. Algebra* **194** (1997), no. 2, 477–495.

[9] Y. I. Manin, Iterated Shimura integrals, *Moscow Math. J.* **5** (2005), no. 4, 869–881.

[10] Y. I. Manin, Iterated integrals of modular forms and noncommutative modular symbols, in: *Algebraic geometry and number theory*, Progr. Math. **253** (Birkhäuser Boston, Boston, MA, 2006), pp. 565–597.

[11] M. Rogers and W. Zudilin, From L-series of elliptic curves to Mahler measures, *Compositio Math.* **148** (2012), 385–414.

[12] M. Rogers and W. Zudilin, On the Mahler measure of $1 + X + 1/X + Y + 1/Y$, *Intern. Math. Res. Not.* **2014** (2014), 2305–2326.

[13] D. Zagier, Values of zeta functions and their applications, in *First European Congress of Mathematics*, Vol. II (Paris, 1992), Progress in Math. **120** (Birkhäuser, Basel, 1994), 497–512.

Institute for Mathematics, Astrophysics and Particle Physics, Radboud University, PO Box 9010, 6500 GL Nijmegen, Netherlands

URL: `http://www.math.ru.nl/~wzudilin/`