A journey through generalised symmetric Freud weights

Ana Loureiro

collaborators:
Peter Clarkson (U Kent) & Kerstin Jordaan (U South Africa)
Generalised Sextic Freud weights:

\[ w(x; t) = |x|^{\rho} \exp(-x^6 + \tau x^4 + t x^2) \, , \, x \in (-\infty, \infty) \]

with \( \rho > -1 \) and \( t, \tau \in \mathbb{R} \).

Let \( (P_n)_{n \geq 0} \) be the corresponding monic Orthogonal Polynomial Sequence (OPS).

So, we have

\[ xP_n(x) = P_{n+1}(x) + \beta_n P_{n-1}(x), \text{ with } P_0(x) = 1 \text{ and } P_1(x) = x. \]

**AIM:** to describe the recurrence coefficients \( \beta_n \)
Let $\left( P_n \right)_{n \geq 0}$ be the monic Orthogonal Polynomial Sequence with respect to the positive symmetric weight $w(x)$ on $\mathbb{R}$, such that

$$\int_{-\infty}^{+\infty} P_n(x)P_k(x)w(x)dx = h_n \delta_{n,m} \quad \text{with} \quad h_n > 0.$$ 

So, we have $P_n(-x) = (-1)^n P_n(x)$ and

$$nP_n(x) = P_{n+1}(x) + \beta_n P_{n-1}(x),$$

with $P_0(x) = 1$ and $P_{-1}(x) = 0,$

where

$$\beta_n = \frac{1}{h_{n-1}} \int_{-\infty}^{+\infty} xP_{n-1}(x)P_n(x)w(x)dx.$$
Monic symmetric Orthogonal Polynomial Sequence

The coefficient $\beta_n$ in

$$xP_n(x) = P_{n+1}(x) + \beta_n P_{n-1}(x)$$

can also be expressed in terms of Hankel determinants

$$\beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2},$$

where

$$\Delta_n = \begin{vmatrix}
\mu_0 & \mu_1 & \cdots & \mu_{n-1} \\
\mu_1 & \mu_2 & \cdots & \mu_n \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{n-1} & \mu_n & \cdots & \mu_{2n-2}
\end{vmatrix},$$

with $\mu_n = \int_{-\infty}^{+\infty} x^n w(x) \, dx$ the moments of the weight function $w(x)$. 
Further properties

The Hankel determinant \( \Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-1} \\ \mu_1 & \mu_2 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-2} \end{vmatrix} \),

also has the integral representation due to Heine (1878)

\[
\Delta_n = \frac{1}{n!} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{\ell=1}^{n} w(x_{\ell}) \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 d x_1 \cdot d x_n
\]

which is the partition function in random matrix theory.

Furthermore, \( P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_n \\ \mu_1 & \mu_2 & \ldots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-1} \\ 1 & x & \ldots & x^n \end{vmatrix} \)
Lemma. Let \( w_0(x) \) be a symmetric positive function on \((-\infty, + \infty)\) for which all the moments exist and are finite and
\[
w(x; t) = \exp(tx^2)w_0(x)
\]
with \( t \in \mathbb{R} \) is a weight for which all moments \( \mu_n(t) = \int_{-\infty}^{\infty} x^n w(x; t) \, dx < \infty \).

Then
\[
\mathcal{A}_n = \mathcal{W} \left( \mu_0, \frac{d\mu_0(t)}{dt}, \ldots, \frac{d^{n-1} \mu_0(t)}{dt} \right), \quad \mathcal{B}_n = \mathcal{W} \left( \frac{d\mu_0(t)}{dt}, \frac{d^2 \mu_0(t)}{dt^2}, \ldots, \frac{d^n \mu_0(t)}{dt^n} \right),
\]
and the recurrence coefficients \( \beta_n := \beta_n(t) \) satisfy the Volterra lattice equation
\[
\frac{d\beta_n}{dt} = \beta_n (\beta_{n+1} - \beta_{n-1})
\]

Remark. also known as the
discrete KdV equation ; Kac-van Moerbeke lattice ; Langmuir lattice
Freud weights – some background

• The relationship between semi-classical orthogonal polynomials and integrable equations dates back to Shohat (1939) and Freud (1976).

• Fokas, Its & Kitaev (1991, 1992) identified these integrable equations as discrete Painlevé equations.

• Magnus (1995) considered the Freud weight \( w(x; t) = \exp(-x^4 + tx^2), x \in \mathbb{R} \), and showed that the coefficients in the three-term recurrence relation can be expressed in terms of solutions of the string equation - Gross&Migdal(1990), Periwal&Shevitz(1990)

\[
q_n(q_{n+1} + q_n + q_{n-1} + 2t) = n
\]

as shown by Bonan&Nevai’1984 and

\[
\frac{d^2 q_n}{dt^2} = \frac{1}{2q_n} \left( \frac{dq_n}{dt} \right)^2 + \frac{3}{2} q_n^3 + 4tq_n^2 + 2 \left( t^2 + \frac{n}{2} \right) q_n - \frac{n^2}{2q_n}
\]

which is \( P_{IV} \) with \( \alpha = -\frac{n}{2} \) and \( \beta = -\frac{n^2}{2} \).

• Connection between Freud weight and solutions of \( dP_I \) and \( P_{IV} \) is due to Kitaev’1988
Consider

$$\omega(x; t, \lambda) = |x|^{2\lambda+1} \exp(-x^{2m} + tx^2), \quad x \in \mathbb{R}$$

with parameters $\lambda > -1$, $t \in \mathbb{R}$ and $m = 2, 3, \ldots$
**Proposition.** (Clarkson, Jordaan & L’ 23) For $\lambda > -1$, $t \in \mathbb{R}$ and $m = 2, 3, \ldots$ consider the weight
\[
\omega(x; t, \lambda) = |x|^{2\lambda + 1} \exp(-x^{2m} + tx^2), \quad x \in \mathbb{R}
\]
whose moments are
\[
\mu_n(t; \lambda) = \int_{-\infty}^{\infty} |x|^{2\lambda + 1} \exp(tx^2 - x^{2m}) \, dx = \frac{1}{m} \sum_{n=0}^{\infty} \frac{t^n}{n!} \Gamma\left(\frac{\lambda + n + 1}{m}\right)
\]
\[
= \frac{1}{m} \sum_{k=1}^{m} \frac{t^{k-1}}{(k-1)!} \Gamma\left(\frac{\lambda + k}{m}\right) \binom{\lambda}{k} \frac{\lambda + k}{m} \left(\frac{t}{m}\right)^k \binom{\lambda + (k-1)m}{k-1} \Gamma\left(\frac{\lambda + k}{m}\right)
\]
and one has
\[
\mu_{2k}(t; \lambda, m) = \frac{d^k}{dt^k} \mu_0(t; \lambda, m), \quad \mu_{2k}(t; \lambda, m) = \mu_0(t; \lambda + k, m)
\]
and the first moment $\mu_0(t; \lambda, m)$ satisfies the differential equation
\[
m \frac{d^m \varphi}{dt^m} - t \frac{d\varphi}{dt} - (\lambda + 1) \varphi = 0
\]
Lemma. (Clarkson, Jordaan & L 2023)

For the weight $\omega(x; t, \lambda) = |x|^{2\lambda+1} \exp(-x^{2m} + tx^2), x \in \mathbb{R}$, the corresponding orthogonal polynomials

$$P_{n+1}(x) = xP_n(x) - \beta_n(t; \lambda)P_{n-1}(x), \quad n = 0, 1, 2, \ldots,$$

with $P_{-1}(x) = 0$ and $P_0(x) = 1$, where

$$\beta_{2n}(t; \lambda) = \frac{A_{n+1}(t; \lambda)A_{n-1}(t; \lambda + 1)}{A_n(t; \lambda)A_n(t; \lambda + 1)} = \frac{d}{dt} \ln \frac{A_n(t; \lambda + 1)}{A_n(t; \lambda)},$$

$$\beta_{2n+1}(t; \lambda) = \frac{A_n(t; \lambda)A_{n+1}(t; \lambda + 1)}{A_{n+1}(t; \lambda)A_n(t; \lambda + 1)} = \frac{d}{dt} \ln \frac{A_{n+1}(t; \lambda)}{A_n(t; \lambda + 1)}.$$

where $A_n(t; \lambda)$ is the Wronskian given by

$$A_n(t; \lambda) = \mathcal{W} \left( \mu_0, \frac{d\mu_0}{dt}, \ldots, \frac{d^{n-1}\mu_0}{dt^{n-1}} \right),$$

with

$$\mu_0(t; \lambda, m) = \int_{-\infty}^{\infty} |x|^{2\lambda+1} \exp(-x^{2m} + tx^2) \, dx$$

$$= \frac{1}{m} \sum_{k=1}^{m} \frac{t^{k-1}}{(k-1)!} \Gamma \left( \frac{\lambda + k}{m} \right) \, _2F_m \left( \frac{\lambda + k}{m}, 1; \frac{k}{m}, \frac{k + 1}{m}, \ldots, \frac{m + k - 1}{m}; \left( \frac{t}{m} \right)^m \right).$$
The weight function \( w(x, t, \lambda) \) satisfies
\[
\frac{d}{dx} \left( xw(x) \right) - 2(tx^2 - mx^{2m} + \lambda + 1)w(x) = 0
\]

Therefore
\[
x \frac{d}{dx} P_n(x) = \sum_{\ell = 0}^{m} \rho_{n,2\ell} P_{n-2\ell}(x), \quad \text{for} \quad n \geq 0,
\]

where
\[
\rho_{n,2\ell} = \begin{cases} 
\frac{2m}{h_n} \int_{-\infty}^{\infty} x^{2m} P_n^2(x) w(x) \, dx - 2t(\beta_n + \beta_{n-1}) - 2 \left( \lambda + 1 + \frac{n}{2} \right) & \text{if } \ell = 0 \\
\frac{2m}{h_{n-2}} \int_{-\infty}^{\infty} x^{2m} P_{n-2}(x) P_n(x) w(x) \, dx - 2t \beta_n \beta_{n-1} & \text{if } \ell = 1 \\
\frac{2m}{h_{n-2\ell}} \int_{-\infty}^{\infty} x^{2m} P_{n-2\ell}(x) P_n(x) w(x) \, dx & \text{if } 2 \leq \ell \leq m - 1 \\
\frac{2m}{h_{n-2m}} h_n & \text{if } \ell = m \\
0 & \text{if } \ell \geq \min\{m + 1, \left\lfloor \frac{n}{2} \right\rfloor \} \text{ or } \ell < 0.
\end{cases}
\]
Equations for the recurrence coefficients

For \( m = 2 \) the discrete equation is

\[
4\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) - 2t\beta_n = n + (2\lambda + 1) \frac{[1 - (-1)^n]}{2}
\]

which is \( dP_1 \).

For \( m = 3 \) the discrete equation is

\[
6\beta_n \left( \beta_{n-2}\beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} + \beta_{n-1}^2 + 2\beta_n\beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2} - \frac{t}{3} \right)
\]

\[
= n + (2\lambda + 1) \frac{[1 - (-1)^n]}{2},
\]

which is a special case of \( dP_1^{(2)} \), the second member of the discrete Painlevé I hierarchy
- see the works of Cresswell and Joshi’99.
The recurrence coefficient $\beta_n$ for the generalised higher-order Freud weight

$$\omega(x; t, \lambda) = |x|^{2\lambda+1} \exp \left(-x^{2m} + tx^2\right), \quad x \in \mathbb{R}$$

satisfies the discrete equation (Benassia&Moro’20 and Bonora&Martellini&Xiong’92)

$$2mV_n^{(2m)} - 2t\beta_n = n + (2\lambda + 1) \frac{[1 - (-1)^n]}{2},$$

where $V_n^{(2m)} = \sqrt{\beta_n} \left(L^{2m-1}\right)_{n,n+1}$ and $L = \begin{pmatrix}
0 & \sqrt{\beta_1} & 0 & 0 & \cdots \\
\sqrt{\beta_1} & 0 & \sqrt{\beta_2} & 0 & \cdots \\
0 & \sqrt{\beta_2} & 0 & \sqrt{\beta_3} & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}.$
The Volterra lattice hierarchy...

is given by

\[
\frac{\partial \beta_n}{\partial t_{2k}} = \beta_n \left( V_{n+1}^{(2k)} - V_{n-1}^{(2k)} \right), \quad k = 1, 2, \ldots
\]

where \( V_{n}^{(2k)} \) is a nonlinear combination of \( \beta_n \) evaluated at different points of the lattice.

The first are

\[
V_{n}^{(2)} = \beta_n, \quad V_{n}^{(4)} = V_{n}^{(2)} \left( V_{n-1}^{(2)} + V_{n}^{(2)} + V_{n+1}^{(2)} \right) = \beta_n (\beta_{n-1} + \beta_n + \beta_{n+1}),
\]

\[
V_{n}^{(6)} = V_{n}^{(2)} \left( V_{n-1}^{(2)} V_{n+1}^{(2)} + V_{n-1}^{(4)} + V_{n}^{(4)} + V_{n+1}^{(4)} \right)
\]

\[
= \beta_n (\beta_{n-2} \beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1} \beta_n + \beta_{n-1} \beta_{n+1} + \beta_n^2 + 2\beta_n \beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1} \beta_{n+2}),
\]

Note that the discrete equation satisfied by \( \beta_n \) can be written as

\[
6V_{n}^{(6)} - 4\tau V_{n}^{(4)} - 2tV_{n}^{(2)} = n
\]

and

\[
\frac{\partial \beta_n}{\partial t} = \beta_n \left( V_{n+1}^{(2)} - V_{n-1}^{(2)} \right), \quad \frac{\partial \beta_n}{\partial \tau} = \beta_n \left( V_{n+1}^{(4)} - V_{n-1}^{(4)} \right)
\]
In particular

\[ V_n^{(2)} = \beta_n, \quad V_n^{(4)} = V_n^{(2)} \left( V_{n-1}^{(2)} + V_n^{(2)} + V_{n+1}^{(2)} \right) = \beta_n (\beta_{n-1} + \beta_n + \beta_{n+1}), \]

\[ V_n^{(6)} = V_n^{(2)} \left( V_{n-1}^{(2)} V_{n+1} + V_{n-1}^{(4)} + V_n^{(4)} + V_{n+1}^{(4)} \right) \]

\[ = \beta_n (\beta_{n-2} \beta_{n-1} + \beta_{n-1}^2 + 2 \beta_{n-1} \beta_n + \beta_{n-1} \beta_{n+1} + \beta_n^2 + 2 \beta_n \beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1} \beta_{n+2}), \]

\[ V_n^{(8)} = V_n^{(2)} \left( V_{n+1}^{(6)} + V_n^{(6)} + V_{n-1}^{(6)} \right) + V_n^{(4)} V_{n+1}^{(2)} V_{n-1}^{(2)} + V_{n+1}^{(2)} V_n^{(2)} V_{n-1}^{(2)} \left( V_{n+2}^{(2)} + V_{n-2}^{(2)} \right), \]

\[ V_n^{(10)} = V_n^{(2)} \left( V_{n+1}^{(8)} + V_n^{(8)} + V_{n-1}^{(8)} \right) + V_n^{(6)} V_{n+1}^{(2)} V_{n-1}^{(2)} + V_{n+1}^{(2)} V_n^{(2)} V_{n-1}^{(2)} \left( V_{n+2}^{(4)} + V_{n-2}^{(4)} \right) \]

\[ + V_{n+1}^{(2)} V_n^{(2)} V_{n-1}^{(2)} \left( V_n^{(2)} + V_{n-1}^{(2)} \right) V_{n+2}^{(2)} + \left( V_{n+1}^{(2)} + V_n^{(2)} \right) V_{n-2}^{(2)} + V_{n+2}^{(2)} V_{n-2}^{(2)} \right\}. \]
Asymptotic behaviour

**Theorem (Freud’s conjecture ‘76).** (Saff, Lubinski, Mhaskar 1988)
For the generalised higher order Freud weight \( \omega(x; t, \lambda) = |x|^{2\lambda+1} \exp \left( -x^{2m} + tx^2 \right) \), the recurrence coefficients \( \beta_n \) associated with this weight satisfy

\[
\lim_{n \to \infty} \frac{\beta_n(t; \lambda)}{n^{1/m}} = \frac{1}{4} \left( \frac{(m - 1)!}{\left( \frac{1}{2} \right)_m} \right)^{1/m}
\]

**Theorem.** (Kuijlaars, Van Assche 1999) Let \( \phi(n) = n^{1/(2m)} \) and assume that \( n, N \) tend to infinity in such a way that the ratio \( n/N \to \ell \). Then, the asymptotic zero distribution as \( n \to \infty \) for \( P_{n,N}(x) = (\phi(N))^{-n}P_n(\phi(N)x) \), has density

\[
a_m(\ell) = \frac{2m}{c\pi(2m - 1)} \left( 1 - x^2/c^2 \right)^{1/2} _2F_1 \left( 1, 1 - m; \frac{3 - 2m}{2}; x^2/c^2 \right)
\]

where \( c = 2a\ell^{1/(2m)} \) with \( a = \frac{1}{2} \left( \frac{(m - 1)!}{\left( \frac{1}{2} \right)_m} \right)^{1/(2m)} \) for \( x \in (-2a\ell^{1/(2m)}, 2a\ell^{1/(2m)}) \).
The zeros of $P_{n,N}(x)$ for $\lambda = 0.5$, $t = 1$, $m = 3$, $n = N = 10$ and $\ell = 1$ with the corresponding limiting distribution $a_m(\ell') = \frac{2m}{c\pi(2m - 1)} \left(1 - \frac{x^2}{c^2}\right)^{1/2} {}_2F_1\left(1, 1 - m; \frac{3 - 2m}{2}; \frac{x^2}{c^2}\right)$

and endpoints $(-2a,0)$ and $(2a,0)$. 

Asymptotic zero distribution
Weight: \(w(x, \tau, t) = \exp(-x^6 + \tau x^4 + tx^2)\)
**Weight:** \( w(x, \tau, t) = \exp(-x^6 + \tau x^4 - \kappa \tau^2 x^2) \)

(i), \( \kappa > \frac{1}{4}, \tau > 0 \)  
(ii), \( \kappa < \frac{1}{4}, \tau > 0 \)  
(iii), \( \kappa = \frac{1}{4}, \tau > 0 \)  
(iv), \( \tau < 0 \)  
(v), \( \tau = 0, \ t \neq 0 \)  
(vi), \( \tau \neq 0, \ t = 0 \)
Case analysis for the weight $w(x) = \exp(-U(x))$

Observe that

$$U(x) = x^2 \left( \left( x^2 - \frac{\tau}{2} \right)^2 + \left( \kappa - \frac{1}{4} \right) \tau^2 \right) = - \left( x^2 - \frac{\tau}{3} \right)^3 + \frac{(1 - 3\kappa)\tau^2}{3} x^2 - \frac{\tau^3}{27}$$

where $\kappa = -t/\tau^2$

Case (i) $\kappa > \frac{1}{4}$ and $\tau > 0$, then $U(x)$ has 4 complex zeros

Case (ii) $\kappa = \frac{1}{4}$ and $\tau > 0$, then $U(x) = x^2 \left( x^2 - \frac{\tau}{2} \right)^2$

Case (iii) $0 < \kappa < \frac{1}{4}$ and $\tau > 0$, then $U(x)$ has 4 real zeros

Case (iv) $\kappa = 0$ and $|\tau| > 0$, then $U(x) = x^4(x^2 - \tau)$

Case (v) $\kappa < 0$ and $\tau > 0$, then $U(x)$ has two real, two purely imaginary and a double zero

Case (vi) $\tau = 0$ and $|t| > 0$

Case (vii) $\tau < 0$ and $|t| > 0$

Case (viii) $\tau = t = 0$
**Weight:**  \( w(x, t, \tau, \rho) = \exp(-x^6 + \tau x^4 + tx^2) \)

**Lemma.** (Clarkson, Jordaan & L - ongoing) The first moment

\[
\mu_0(\tau, t) = \int_{-\infty}^{+\infty} \exp(-x^6 + \tau x^4 - tx^2)dx \quad \text{is a solution to}
\]

\[
\frac{\partial^3 \phi}{\partial t^3} - \frac{2}{3} \tau \frac{\partial^2 \phi}{\partial t^2} - \frac{1}{3} t \frac{\partial \phi}{\partial t} - \frac{1}{6} \phi = 0
\]

Moreover,

\[
\mu_0(\tau, t) = \frac{1}{3} \sum_{j=0}^{+\infty} \frac{\tau^j}{j!} \left\{ \Gamma \left( \frac{2}{3} j + \frac{1}{3} n + \frac{1}{6} \right) \right\} _1 F_2 \left( \begin{array}{c} \frac{2}{3} j + \frac{1}{3} n + \frac{1}{6} \\ \frac{1}{3}, \frac{2}{3}, \frac{t^3}{27} \end{array} \right)
\]

\[
+ t \left( \frac{2}{6} j + \frac{1}{3} n + \frac{1}{2} \right) _1 F_2 \left( \begin{array}{c} \frac{2}{3} j + \frac{1}{3} n + \frac{1}{2} \\ \frac{2}{3}, \frac{4}{3}, \frac{t^3}{27} \end{array} \right)
\]

\[
+ \frac{1}{2} t^2 \left( \frac{2}{3} j + \frac{1}{3} n + \frac{5}{6} \right) _1 F_2 \left( \begin{array}{c} \frac{2}{3} j + \frac{1}{3} n + \frac{5}{6} \\ \frac{4}{3}, \frac{5}{3}, \frac{t^3}{27} \end{array} \right)
\]
About the moments

The moment sequence \((\mu_n)_{n\geq 0}\) defined by 
\[
\mu_n(\tau, t) = \int_{-\infty}^{+\infty} x^n \exp(-x^6 + \tau x^4 - \kappa \tau^2 x^2)dx
\]
satisfies the recurrence relation
\[
3\mu_{2n+6} - 2\tau\mu_{2n+4} + \kappa \tau^2 \mu_{2n+2} - \left(n + \frac{1}{2}\right) \mu_{2n} = 0.
\]

Moreover,
\[
\partial^2_\tau \mu_0 - (4\kappa^2 - 3\kappa + \frac{4}{9})\tau^2 \partial_\tau \mu_0 + \frac{1}{9}(6\kappa - 1)\tau \mu_0 = \frac{1}{6}(4\kappa - 1)[4\kappa(3\kappa - 1)\tau^3 - 3] \mu_2,
\]

And
\[
\partial^2_\tau \mu_{2n} - (4\kappa^2 - 3\kappa + \frac{4}{9})\tau^2 \partial_\tau \mu_{2n} + \frac{1}{9}(2n + 1)(6\kappa - 1)\tau \mu_{2n} = \left\{\frac{1}{6}(4\kappa - 1)[4\kappa(3\kappa - 1)\tau^3 - 3] + \frac{1}{9}n\right\} \mu_{2n+2}.
\]

\[
\frac{d^3 \mu_0}{d\tau^3} + \left\{\frac{2(9\kappa - 2)\tau^2}{9} - \frac{12\kappa(3\kappa - 1)\tau^2}{4\kappa(3\kappa - 1)\tau^3 - 3}\right\} \frac{d^2 \mu_0}{d\tau^2} + \left\{\frac{(4\kappa - 1)^2 \tau^4}{3} - \frac{(36\kappa^2 - 27\kappa + 4)\tau}{4\kappa(3\kappa - 1)\tau^3 - 3}\right\} \frac{d\mu_0}{d\tau} + \left\{\frac{(4\kappa - 1)\kappa^2 \tau^3}{3} - \kappa + \frac{5}{36} + \frac{1 - 6\kappa}{4\kappa(3\kappa - 1)\tau^3 - 3}\right\} \mu_0 = 0.
\]

\[
\frac{d^3 \mu_0}{d\tau^3} + \frac{(4\kappa - 1)\kappa^2 \tau^3}{3} - \kappa + \frac{5}{36} + \frac{1 - 6\kappa}{4\kappa(3\kappa - 1)\tau^3 - 3} \mu_0 = 0.
\]
About the moments – particular cases

The moment sequence \((\mu_n)_{n \geq 0}\) defined by
\[
\mu_n(\tau, t) = \int_{-\infty}^{+\infty} x^n \exp(-x^6 + \tau x^4 - \kappa \tau^2 x^2)dx
\]
satisfies the recurrence relation
\[
3\mu_{2n+6} - 2\tau \mu_{2n+4} + \kappa \tau^2 \mu_{2n+2} - \left(n + \frac{1}{2}\right) \mu_{2n} = 0.
\]

For \(\kappa = \frac{1}{4}\), then
\[
\mu_0(\tau, \frac{1}{4}) = \frac{\pi \sqrt{6\tau}}{9} \left\{ I_{1/6} \left( \frac{\tau^3}{108} \right) + I_{-1/6} \left( \frac{\tau^3}{108} \right) \right\} \exp \left(-\frac{\tau^3}{108}\right).
\]

For \(\kappa = \frac{1}{3}\), then
\[
\mu_0(\tau, \frac{1}{3}) = \left\{ \frac{1}{3} \Gamma \left( \frac{1}{6} \right) \ _2F_2 \left( \frac{1}{6}, \frac{1}{2}; \frac{1}{3}, \frac{2}{3}; \frac{\tau^3}{27} \right) + \frac{1}{3} \tau \Gamma \left( \frac{5}{6} \right) \ _2F_2 \left( \frac{1}{2}, \frac{5}{6}; \frac{2}{3}, \frac{4}{3}; \frac{\tau^3}{27} \right) \right. \\
\left. - \frac{\tau^2 \sqrt{\pi}}{36} \ _2F_2 \left( \frac{5}{6}, \frac{7}{6}; \frac{4}{3}, \frac{5}{3}; \frac{\tau^3}{27} \right) \right\} \exp \left(-\frac{\tau^3}{27}\right),
\]
Asymptotics for $\beta_1$

The moment sequence $(\mu_n)_{n \geq 0}$ defined by

$$\mu_n(\tau, t) = \int_{-\infty}^{+\infty} x^n \exp(-x^6 + \tau x^4 - tx^2) dx$$

satisfies the recurrence relation

$$3\mu_{2n+6} - 2\tau \mu_{2n+4} - t \mu_{2n+2} - \left(n + \frac{1}{2}\right) \mu_{2n} = 0$$

with $\mu_{2n+1} = 0$.

**Theorem.** For fixed $\kappa > \frac{1}{4}$ and $\tau > 0$, then for all $n \geq 0$

$$\beta_1 \sim \frac{1}{8\tau^2 \left(\kappa - \frac{1}{4}\right)^2}, \quad \text{as} \quad \tau \to +\infty.$$ 

For fixed $0 < \kappa < \frac{1}{4}$ and $\tau > 0$, then for all $n \geq 0$

$$\beta_1 \sim \frac{1}{2} \tau \left(1 + \sqrt{1 - 3\kappa}\right), \quad \text{as} \quad \tau \to +\infty.$$
About this weight

Analysis of

\[ w(x; \tau, t) = \exp(-U(x; \tau, t)) \text{ with } U(x; \tau, t) = x^6 - \tau x^4 - t x^2 \]

where \( \tau, t \in \mathbb{R} \),

and the recurrence coefficients satisfy

\[
6\beta_n (\beta_{n+1} \beta_{n-1} + \beta_{n-1} (\beta_{n-2} + \beta_{n-1} + \beta_n) + \beta_n (\beta_{n-1} + \beta_n + \beta_{n+1}) + \beta_{n+1} (\beta_n + \beta_{n+1} + \beta_{n+2})) \\
-4\tau \beta_n (\beta_{n-1} + \beta_n + \beta_{n+1}) - 2t \beta_n = n
\]
Lemma. Let $w_0(x)$ be a symmetric positive weight on the real line and suppose that

$$w(x; t, \tau) = \exp(tx^2 + \tau x^4) w_0(x), \quad x \in \mathbb{R} \text{ with } t, \tau \in \mathbb{R},$$

is a weight such that all the moments of exist.

Then the recurrence coefficient $\beta_n(t, \tau)$ satisfies the Volterra, or the Langmuir lattice, equation

$$\partial_t \beta_n = \beta_n(\beta_{n+1} - \beta_{n-1})$$

and the differential–difference equation

$$\partial_\tau \beta_n = \beta_n \left( (\beta_{n+2} + \beta_{n+1} + \beta_n)\beta_{n+1} - (\beta_n + \beta_{n-1} + \beta_{n-2})\beta_{n-1} \right).$$
The recurrence coefficients $\beta_n(\tau, t)$ satisfy the recurrence relation

$$6\beta_n(\beta_{n-1} (\beta_{n-2} + \beta_{n-1} + \beta_n + \beta_{n+1}) + \beta_n (\beta_{n-1} + \beta_n + \beta_{n+1}) + \beta_{n+1} (\beta_n + \beta_{n+1} + \beta_{n+2}))$$

$$-4\tau \beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) - 2t \beta_n = n$$

**Remark.** This equation is:

**I.** A special case of $dP_1^{(2)}$, the 2nd member of the discrete Painlevé I hierarchy. Cresswell & Joshi showed that its continuum limit is equivalent to

$$\frac{d^4w}{dz^4} = 10w \frac{d^2w}{dz^2} + 5 \left( \frac{dw}{dz} \right)^2 - 10w^3 + z$$

which is $P_1^{(2)}$.

**II.** Also known as the “string equation” and arises in physical applications such as 2-dimensional quantum gravity.
Consider the weight
\[ w(x; \tau, t) = \exp \left( -x^6 + \tau x^4 + tx^2 \right) \]
which is equivalent to the weight
\[ W(z) = \exp \{-NV(x)\}, \quad V(x) = g_2 x^2 + g_4 x^4 + g_6 x^6 \]
with \( N, g_2, g_4 \) and \( g_6 \) parameters.

- “Chaotic behavior in one matrix models” ([Jurkiewicz [1991]])
- “Chaos in the Hermitian one-matrix model” ([Sénéchal (1992)])
Benassi & Moro (2020) and Dell’Atti (2022) considered the weight

\[ W(x; T_2, T_4, T_6, N) = \exp \left( N \left[ T_6 x^6 + T_4 x^4 + (T_2 - \frac{1}{2}) x^2 \right] \right) \]

with \( T_2, T_4, T_6 \) and \( N \) parameters.

They interpreted the Jurkiewicz’s “chaotic phase” as a dispersive shock propagating through the chain in the continuum/thermodynamic limit and explained the complexity of its phase diagram in the context of dispersive hydrodynamics.

The recurrence coefficients satisfy the discrete equation

\[
\begin{align*}
6T_6 (u_{n-2}u_{n-1} + u_{n-1}^2 + 2u_{n-1}u_n + u_{n-1}u_{n+1} + u_n^2 + 2u_n u_{n+1} + u_{n+1}^2 + u_{n+1}u_{n+2}) \\
+4T_4 (u_{n-1} + u_n + u_{n+1}) + (2T_2 - 1) = -\frac{n}{N}
\end{align*}
\]

and the associated cubic equation is

\[
60T_6 u^3 + 12T_4 u^2 + (2T_2 - 1)u + \frac{n}{N} = 0
\]
A “Limiting curve”? 

Asymptotic behaviour: 

\[ \beta_n \sim \beta(n), \quad \text{as} \quad n \to \infty, \]

where \( \beta(n) \) is the \( \beta \)-curve

\[ 60\beta^3 - 12\tau\beta^2 + 2\kappa\tau^2\beta = n. \]
Case (i) \( \kappa > 1/4 + \epsilon \) and \( \tau = 20 \) 

"one-branch case"
Case (i) $\kappa > 2/5$ and $\tau = 20$  

“one-branch case”

\[
\beta_n \text{ for } 0 \leq n \leq 400 \quad (\kappa = 0.425)
\]

\[
\beta_n - \beta(n) \text{ for } 0 \leq n \leq 400 \quad (\kappa = 0.425)
\]
Case (i) \( \frac{1}{4} + \epsilon \leq \kappa < \frac{2}{5} \) and \( \tau = 20 \) 

“one-branch case”

\[ \tau = 50, \; \kappa = 0.275 \]

\[ \tau = 50, \; \kappa = 0.3 \]

\[ \tau = 50, \; \kappa = 0.325 \]

\[ \tau = 50, \; \kappa = 0.35 \]
Case (ii): $\tau > 0$ and $\kappa = \frac{1}{4}$

In this case the weight is

$$\omega(x; \tau) = \exp \left\{ -x^2(x^2 - \frac{1}{2}\tau)^2 \right\}$$

Plotting $\beta_{3n}, \beta_{3n+1}, \beta_{3n+2}$

\[\tau = 12\]  \[\tau = 15\]  \[\tau = 18\]
Case (ii): \( \tau > 0 \) and \( \kappa = \frac{1}{4} \)

Setting \( \beta_{3n} = u \) and \( \beta_{3n \pm 1} = v \) in

\[
6\beta_n \left( \beta_{n-2}\beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} + \beta_n^2 + 2\beta_n\beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2} \right) \\
- 4\tau \beta_n (\beta_{n-1} + \beta_{n} + \beta_{n+1}) + \frac{1}{2} \tau^2 \beta_n = n
\]

gives

\[
6u(u^2 + 4uv + 5v^2) - 4\tau u(u + 2v) + \frac{1}{2} \tau^2 u = n \\
6v(u^2 + 5uv + 4v^2) - 4\tau v(u + 2v) + \frac{1}{2} \tau^2 v = n
\]

and then it can be shown that \( u \) and \( v \) satisfy the cubics

\[
12u^3 - 12\tau u^2 + 3\tau^2 u - 8n = 0 \\
12v^3 - 3\tau v^2 + n = 0
\]

Setting \( \beta_n = \beta \) gives

\[
60\beta^3 - 12\tau \beta^2 + \frac{1}{2} \tau^2 \beta = n
\]

All three cubics meet at the point

\[
\left( \frac{\tau^3}{36}, \frac{\tau}{6} \right)
\]
Case (ii): \( \tau > 0 \) and \( \kappa = \frac{1}{4} \)

\[
\begin{align*}
\beta_{3n} & \quad 12u^3 - 12\tau u^2 + 3\tau^2 u - 8n = 0 \\
\beta_{3n+1}, \beta_{3n+2} & \quad 12v^3 - 3\tau v^2 + n = 0 \\
& \quad 60\beta^3 - 12\tau \beta^2 + \frac{1}{2}\tau^2 \beta = n
\end{align*}
\]
Case (ii): $\tau > 0$ and $\kappa = \frac{1}{4}$
Case (iii) \( 0 < \kappa \leq \frac{1}{4} - \epsilon \) and \( \tau = 20 \) “two-branch case”

Setting \( \beta_{2n} = u, \beta_{2n+1} = v \) and \( t = -\kappa \tau^2 \) in

\[
6\beta_n (\beta_{n-2} \beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1} \beta_n + \beta_{n-1} \beta_{n+1} + \beta_n^2 + 2\beta_n \beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1} \beta_{n+2})
- 4\tau \beta_n (\beta_{n-1} + \beta_n + \beta_{n+1}) - 2t \beta_n = n
\]

gives the system

\[
6u(u^2 + 6uv + 3v^2) - 4\tau u(u + 2v) + 2\kappa \tau^2 u = n
\]
\[
6v(3u^2 + 6uv + v^2) - 4\tau v(2u + v) + 2\kappa \tau^2 v = n
\]

Letting \( u = \xi - \eta \) and \( v = \xi + \eta \) gives

\[
144\xi^3 - 72\tau \xi^2 + 4(2 + 3\kappa)\tau^2 \xi - 2\kappa \tau^3 + 3n = 0
\]
\[
\eta = (\frac{1}{3} \xi^2 - \frac{2}{3} \tau \xi + \frac{1}{6} \kappa \tau^2)^{1/2}
\]
Case (iii) \(0 < \kappa \leq \frac{1}{4} - \epsilon\) and \(\tau = 20\)

\[
\begin{align*}
6u (u^2 + 6uv + 3v^2) - 4\tau u(u + 2v) + 2\kappa \tau^2 u &= n \\
6v (3u^2 + 6uv + v^2) - 4\tau v(2u + v) + 2\kappa \tau^2 v &= n
\end{align*}
\]

\[n_1 = \frac{4}{9} \left(\frac{1}{3} - \kappa\right)^{3/2} \tau^3\]

\[n_2 = \frac{\left[2(4 - 27\kappa) + (4 - 18\kappa)^{3/2}\right]}{243} \tau^3\]

Note that \(n_2 = 0\) \(\Rightarrow\) \(\kappa = \frac{1}{6}\)

Also \(n_1 = n_2\) when

\[
\frac{4}{9} \left(\frac{1}{3} - \kappa\right)^{3/2} = \frac{\left[2(4 - 27\kappa) + (4 - 18\kappa)^{3/2}\right]}{243} \quad \Rightarrow \quad \kappa = -\frac{2}{3}
\]
Case(iii): Evolution of $0 \leq \kappa < \frac{1}{4} - \epsilon$ and $\tau = 25$
Evolution of $\left| \kappa - \frac{1}{4} \right| \leq \epsilon$ and $\tau = 25$
Evolution of $0 \leq \kappa \leq \frac{1}{4} + \epsilon$ and $\tau = 25$

$\kappa = 0$ and $\tau = 25$
Case (i)-(iii): \( \tau > 0 \) and \( \kappa - \frac{1}{4} < \epsilon \)

\[ \tau = 25, \quad 0.2499 \leq \kappa \leq 0.2501, \quad \beta_{3n}, \beta_{3n+1}, \beta_{3n+2} \]

\[ \kappa = 0.2501 \quad \kappa = 0.25001 \quad \kappa = 0.24996 \]

\[ \kappa = 0.25007 \quad \kappa = 0.25 \quad \kappa = 0.24993 \]

\[ \kappa = 0.25004 \quad \kappa = 0.24999 \quad \kappa = 0.2499 \]
Case (v): $\kappa < 0$ and $\tau > 0$. Critical value $\kappa = -\frac{2}{3}$.
Summary

\[ U(x) = x^6 - \tau x^4 - tx^2, \quad \kappa = -t/\tau^2 \]

\begin{align*}
A: \quad & \tau > 0, \quad \frac{1}{4} < \kappa < \frac{2}{5} \\
B: \quad & \tau > 0, \quad 0 < \kappa < \frac{1}{4} \\
C: \quad & \tau > 0, \quad -\frac{2}{3} < \kappa < 0 \\
D: \quad & \tau > 0, \quad \kappa < -\frac{2}{3} \\
E: \quad & \tau > 0, \quad \kappa > \frac{2}{5} \\
\end{align*}

\[ \kappa < -\frac{2}{3} \quad -\frac{2}{3} < \kappa < 0 \quad 0 < \kappa < \frac{1}{4} \quad \frac{1}{4} < \kappa < \frac{2}{5} \quad \kappa > \frac{2}{5} \]
Some references


P A Clarkson, K Jordaan and A Loureiro, Generalized higher-order Freud weights, Proc. R. Soc. A, 479 (2023) 20220788

P A Clarkson, K Jordaan and A Loureiro, Symmetric Sextic Freud Weight, in prep.

Thanks!