A journey through generalised symmetric Freud weights

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This talk is about...

Generalised Sextic Freud weights:

$$w(x;t) = |x|^\rho \exp(-x^6 + \tau x^4 + t\,x^2)\ ,\ x \in (-\infty, \infty)$$

with $\rho > -1$ and $t, \tau \in \mathbb{R}$.

Let $(P_n)_{n\geq 0}$ be the corresponding monic Orthogonal Polynomial Sequence (OPS).

So, we have

$$xP_n(x) = P_{n+1}(x) + \beta_n P_{n-1}(x), \text{ with } P_0(x) = 1 \text{ and } P_1(x) = x.$$

AIM: to describe the recurrence coefficients β_n

Monic symmetric Orthogonal Polynomial Sequence

Let $(P_n)_{n\geq 0}$ be the monic Orthogonal Polynomial Sequence with respect to the positive symmetric weight $w(x)$ on $\mathbb{R}$, such that

$$\int_{-\infty}^{+\infty} P_n(x)P_k(x)w(x)\mathrm{d}x = h_n\delta_{n,m} \quad \text{with } h_n > 0.$$

So, we have $P_n(-x) = (-1)^n P_n(x)$ and

$$xP_n(x) = P_{n+1}(x) + \beta_n P_{n-1}(x),$$

with $P_0(x) = 1$ and $P_{-1}(x) = 0$,

where

$$\beta_n = \frac{1}{h_{n-1}} \int_{-\infty}^{+\infty} xP_{n-1}(x)P_n(x)\, w(x)\, \mathrm{d}x.$$

Monic symmetric Orthogonal Polynomial Sequence

The coefficient β_n in

$$xP_n(x) = P_{n+1}(x) + \beta_n P_{n-1}(x)$$

can also be expressed in terms of Hankel determinants

$$\beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2},$$

where

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix},$$

with $\mu_n = \displaystyle\int_{-\infty}^{+\infty} x^n\, w(x)\,\mathrm{d}x$ the **moments** of the weight function $w(x)$.

Further properties

The Hankel determinant $\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}$,

also has the integral representation due to Heine (1878)

$$\Delta_n = \frac{1}{n!} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{\ell=1}^{n} w(x_\ell) \prod_{1 \le j < k \le n} (x_j - x_k)^2 \mathrm{d}x_1 \cdot \mathrm{d}x_n$$

which is the partition function in random matrix theory.

Furthermore, $P_n(x) = \dfrac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ 1 & x & \ldots & x^n \end{vmatrix}$.

Lemma. Let $w_0(x)$ be a symmetric positive function on $(-\infty, +\infty)$ for which all the moments exist and are finite and

$$w(x;t) = \exp(tx^2)\, w_0(x)$$

with $t \in \mathbb{R}$ is a weight for which all moments $\mu_n(t) = \displaystyle\int_{-\infty}^{\infty} x^n w(x;t)\mathrm{d}x < \infty$.

Then

$$\mathscr{A}_n = \mathscr{W}\left(\mu_0, \frac{\mathrm{d}\mu_0(t)}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0(t)}{\mathrm{d}^{n-1}t}\right), \qquad \mathscr{B}_n = \mathscr{W}\left(\frac{\mathrm{d}\mu_0(t)}{\mathrm{d}t}, \frac{\mathrm{d}^2\mu_0(t)}{\mathrm{d}^2t}, \ldots, \frac{\mathrm{d}^n\mu_0(t)}{\mathrm{d}^nt}\right),$$

and the recurrence coefficients $\beta_n := \beta_n(t)$ satisfy the **Volterra lattice equation**

$$\frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n\left(\beta_{n+1} - \beta_{n-1}\right)$$

Remark. also known as the

discrete KdV equation ; Kac-van Moerbeke lattice ; Langmuir lattice

Freud weights - some background

- The relationship between semi-classical orthogonal polynomials and integrable equations dates back to Shohat (1939) and Freud (1976).

- Fokas, Its & Kitaev (1991, 1992) identified these integrable equations as discrete Painlevé equations.

- Magnus (1995) considered the Freud weight $w(x;t) = \exp(-x^4 + tx^2)$, $x \in \mathbb{R}$, and showed that the coefficients in the three-term recurrence relation can be expressed in terms of solutions of the string equation - Gross&Migdal(1990), Periwal&Shevitz(1990)

$$q_n(q_{n+1} + q_n + q_{n-1} + 2t) = n$$

as shown by Bonan&Nevai'1984 and

$$\frac{d^2q_n}{dt^2} = \frac{1}{2q_n}\left(\frac{dq_n}{dt}\right)^2 + \frac{3}{2}q_n^3 + 4tq_n^2 + 2\left(t^2+\frac{n}{2}\right)q_n - \frac{n^2}{2q_n}$$

which is P_{IV} with $\alpha = -\frac{n}{2}$ and $\beta = -\frac{n^2}{2}$.

- Connection between Freud weight and solutions of dP_I and P_{IV} is due to Kitaev'1988

Higher order Freud weights

Consider

$$\omega(x; t, \lambda) = |x|^{2\lambda+1} \exp\left(-x^{2m} + tx^2\right), \qquad x \in \mathbb{R}$$

with parameters $\lambda > -1$, $t \in \mathbb{R}$ and $m = 2,3,\ldots$

Higher order Freud weights

Proposition. (Clarkson, Jordaan & L′ 23) For $\lambda > -1$, $t \in \mathbb{R}$ and $m = 2,3,\ldots$ consider the weight

$$\omega(x;t,\lambda) = |x|^{2\lambda+1}\exp\left(-x^{2m}+tx^2\right), \qquad x \in \mathbb{R}$$

whose **moments** are

$$\mu_n(t;\lambda) = \int_{-\infty}^{\infty} |x|^{2\lambda+1}\exp(tx^2 - x^{2m})\,\mathrm{d}x = \frac{1}{m}\sum_{n=0}^{\infty}\frac{t^n}{n!}\Gamma\left(\frac{\lambda+n+1}{m}\right)$$

$$= \frac{1}{m}\sum_{k=1}^{m}\frac{t^{k-1}}{(k-1)!}\Gamma\left(\frac{\lambda+k}{m}\right){}_2F_m\left(\begin{matrix}\frac{\lambda+k}{m}, & 1\\ \frac{k}{m}, \frac{k+1}{m}, \ldots, & \frac{m+k-1}{m}\end{matrix}; \left(\frac{t}{m}\right)^m\right)$$

and one has

$$\mu_{2k}(t;\lambda,m) = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mu_0(t;\lambda,m), \qquad \mu_{2k}(t;\lambda,m) = \mu_0(t;\lambda+k,m)$$

and the first moment $\mu_0(t;\lambda,m)$ satisfies the differential equation

$$m\frac{\mathrm{d}^m\varphi}{\mathrm{d}t^m} - t\frac{\mathrm{d}\varphi}{\mathrm{d}t} - (\lambda+1)\,\varphi = 0$$

Lemma. (Clarkson, Jordaan & L 2023)

For the weight $\omega(x;t,\lambda)=|x|^{2\lambda+1}\exp\left(-x^{2m}+tx^2\right)$, $x\in\mathbb{R}$, the corresponding orthogonal polynomials

$$P_{n+1}(x)=xP_n(x)-\beta_n(t;\lambda)P_{n-1}(x),\qquad n=0,1,2,\ldots\ ,$$

with $P_{-1}(x)=0$ *and* $P_0(x)=1$*, where*

$$\beta_{2n}(t;\lambda)=\frac{\mathcal{A}_{n+1}(t;\lambda)\mathcal{A}_{n-1}(t;\lambda+1)}{\mathcal{A}_n(t;\lambda)\mathcal{A}_n(t;\lambda+1)}=\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\mathcal{A}_n(t;\lambda+1)}{\mathcal{A}_n(t;\lambda)},$$
$$\beta_{2n+1}(t;\lambda)=\frac{\mathcal{A}_n(t;\lambda)\mathcal{A}_{n+1}(t;\lambda+1)}{\mathcal{A}_{n+1}(t;\lambda)\mathcal{A}_n(t;\lambda+1)}=\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\mathcal{A}_{n+1}(t;\lambda)}{\mathcal{A}_n(t;\lambda+1)}.$$

where $\mathcal{A}_n(t;\lambda)$ *is the Wronskian given by*

$$\mathcal{A}_n(t;\lambda)=\mathcal{W}\left(\mu_0,\frac{\mathrm{d}\mu_0}{\mathrm{d}t},\ldots,\frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right),$$

with

$$\mu_0(t;\lambda,m)=\int_{-\infty}^{\infty}|x|^{2\lambda+1}\exp(-x^{2m}+tx^2)\,\mathrm{d}x$$

$$=\frac{1}{m}\sum_{k=1}^{m}\frac{t^{k-1}}{(k-1)!}\Gamma\left(\frac{\lambda+k}{m}\right)\ {}_2F_m\left(\frac{\lambda+k}{m},1;\frac{k}{m},\frac{k+1}{m},\ldots,\frac{m+k-1}{m};\left(\frac{t}{m}\right)^m\right)$$

Equations for the recurrence coefficients - Part II

The weight function $w(x,t,\lambda)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(xw(x)\big) - 2(tx^2 - mx^{2m} + \lambda + 1)w(x) = 0$$

Therefore $\quad x\frac{\mathrm{d}}{\mathrm{d}x}P_n(x) = \sum_{\ell=0}^{m} \rho_{n,2\ell}\, P_{n-2\ell}(x), \quad$ for $\quad n \geq 0,$

where

$$\rho_{n,2\ell} = \begin{cases} \dfrac{2m}{h_n}\displaystyle\int_{-\infty}^{\infty} x^{2m} P_n^2(x)\, w(x)\, \mathrm{d}x - 2t(\beta_n + \beta_{n-1}) - 2\left(\lambda + 1 + \tfrac{n}{2}\right) & \text{if} \quad \ell = 0 \\ \dfrac{2m}{h_{n-2}}\displaystyle\int_{-\infty}^{\infty} x^{2m} P_{n-2}(x) P_n(x)\, w(x)\, \mathrm{d}x - 2t\,\beta_n\beta_{n-1} & \text{if} \quad \ell = 1 \\ \dfrac{2m}{h_{n-2\ell}}\displaystyle\int_{-\infty}^{\infty} x^{2m} P_{n-2\ell}(x) P_n(x) w(x)\, \mathrm{d}x & \text{if} \quad 2 \leq \ell \leq m-1 \\ \dfrac{2m}{h_{n-2m}}\, h_n & \text{if} \quad \ell = m \\ 0 & \text{if} \quad \ell \geq \min\{m+1, \lfloor \tfrac{n}{2} \rfloor\} \text{ or } \ell < 0. \end{cases}$$

Equations for the recurrence coefficients

For $m = 2$ the discrete equation is

$$4\beta_n\left(\beta_{n-1} + \beta_n + \beta_{n+1}\right) - 2t\beta_n = n + (2\lambda + 1)\frac{[1 - (-1)^n)]}{2}$$

which is $\mathrm{dP_I}$.

For $m = 3$ the discrete equation is

$$6\beta_n\left(\beta_{n-2}\beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} + \beta_n^2 + 2\beta_n\beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2} - \frac{t}{3}\right)$$
$$= n + (2\lambda + 1)\frac{[1 - (-1)^n)]}{2},$$

which is a special case of $\mathrm{dP_I^{(2)}}$, the second member of the discrete Painlevé I hierarchy

- see the works of Cresswell and Joshi'99.

Equations for the recurrence coefficients

The recurrence coefficient β_n for the generalised higher-order Freud weight

$$\omega(x;t,\lambda) = |x|^{2\lambda+1} \exp\left(-x^{2m} + tx^2\right), \qquad x \in \mathbb{R}$$

satisfies the discrete equation (Benassia&Moro'20 and Bonora&Martellini&Xiong'92)

$$2mV_n^{(2m)} - 2t\beta_n = n + (2\lambda+1)\frac{[1-(-1)^n)]}{2},$$

where $V_n^{(2m)} = \sqrt{\beta_n}\left(\mathbf{L}^{2m-1}\right)_{n,n+1}$ and $\mathbf{L} = \begin{pmatrix} 0 & \sqrt{\beta_1} & 0 & 0 & \dots \\ \sqrt{\beta_1} & 0 & \sqrt{\beta_2} & 0 & \dots \\ 0 & \sqrt{\beta_2} & 0 & \sqrt{\beta_3} & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$

The Volterra lattice hierarchy…

is given by

$$\frac{\partial \beta_n}{\partial t_{2k}} = \beta_n \left(V_{n+1}^{(2k)} - V_{n-1}^{(2k)} \right) \quad , \quad k = 1,2,\ldots$$

where $V_n^{(2k)}$ is a nonlinear combination of β_n evaluated at different points of the lattice.

The first are

$$V_n^{(2)} = \beta_n, \qquad V_n^{(4)} = V_n^{(2)} \left(V_{n-1}^{(2)} + V_n^{(2)} + V_{n+1}^{(2)} \right) = \beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}),$$

$$V_n^{(6)} = V_n^{(2)} \left(V_{n-1}^{(2)} V_{n+1}^{(2)} + V_{n-1}^{(4)} + V_n^{(4)} + V_{n+1}^{(4)} \right)$$
$$= \beta_n(\beta_{n-2}\beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} + \beta_n^2 + 2\beta_n\beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2}),$$

Note that the discrete equation satisfied by β_n can be written as

$$6V_n^{(6)} - 4\tau V_n^{(4)} - 2t V_n^{(2)} = n$$

and

$$\frac{\partial \beta_n}{\partial t} = \beta_n \left(V_{n+1}^{(2)} - V_{n-1}^{(2)} \right), \qquad \frac{\partial \beta_n}{\partial \tau} = \beta_n \left(V_{n+1}^{(4)} - V_{n-1}^{(4)} \right)$$

The Volterra lattice hierarchy (cont'd)

In particular

$$V_n^{(2)} = \beta_n, \qquad V_n^{(4)} = V_n^{(2)}\left(V_{n-1}^{(2)} + V_n^{(2)} + V_{n+1}^{(2)}\right) = \beta_n\big(\beta_{n-1} + \beta_n + \beta_{n+1}\big),$$

$$\begin{aligned} V_n^{(6)} &= V_n^{(2)}\left(V_{n-1}^{(2)}V_{n+1}^{(2)} + V_{n-1}^{(4)} + V_n^{(4)} + V_{n+1}^{(4)}\right) \\ &= \beta_n\big(\beta_{n-2}\beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} + \beta_n^2 + 2\beta_n\beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2}\big), \end{aligned}$$

$$V_n^{(8)} = V_n^{(2)}\left(V_{n+1}^{(6)} + V_n^{(6)} + V_{n-1}^{(6)}\right) + V_n^{(4)}V_{n+1}^{(2)}V_{n-1}^{(2)} + V_{n+1}^{(2)}V_n^{(2)}V_{n-1}^{(2)}\left(V_{n+2}^{(2)} + V_{n-2}^{(2)}\right),$$

$$\begin{aligned} V_n^{(10)} &= V_n^{(2)}\left(V_{n+1}^{(8)} + V_n^{(8)} + V_{n-1}^{(8)}\right) + V_n^{(6)}V_{n+1}^{(2)}V_{n-1}^{(2)} + V_{n+1}^{(2)}V_n^{(2)}V_{n-1}^{(2)}\left(V_{n+2}^{(4)} + V_{n-2}^{(4)}\right) \\ &\quad + V_{n+1}^{(2)}V_n^{(2)}V_{n-1}^{(2)}\left\{\left(V_n^{(2)} + V_{n-1}^{(2)}\right)V_{n+2}^{(2)} + \left(V_{n+1}^{(2)} + V_n^{(2)}\right)V_{n-2}^{(2)} + V_{n+2}^{(2)}V_{n-2}^{(2)}\right\}. \end{aligned}$$

Asymptotic behaviour

Theorem (Freud's conjecture'76). (Saff, Lubinski, Mhaskar 1988)
For the generalised higher order Freud weight $\omega(x;t,\lambda) = |x|^{2\lambda+1}\exp\left(-x^{2m}+tx^2\right)$, the recurrence coefficients β_n associated with this weight satisfy

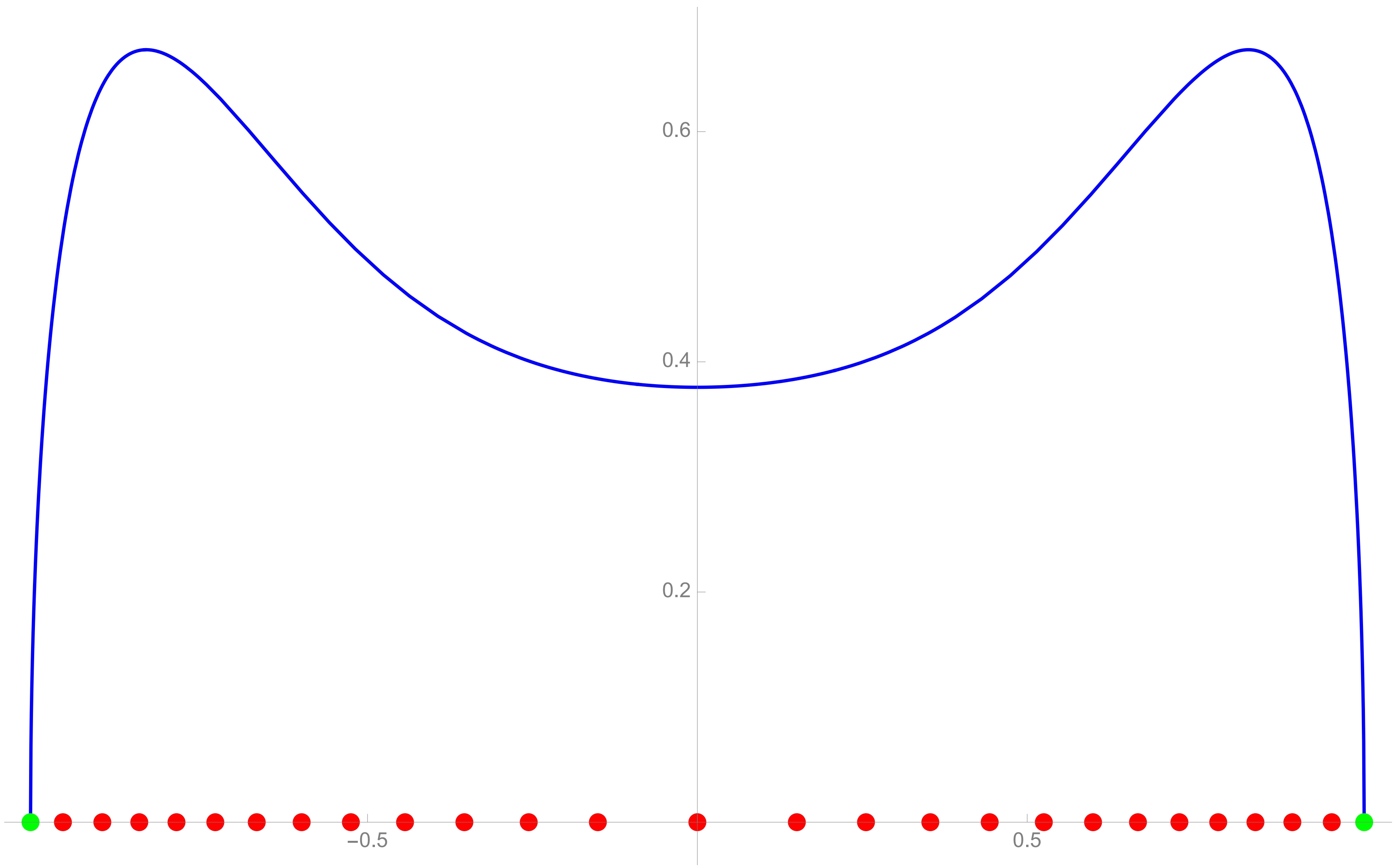
$$\lim_{n\to\infty}\frac{\beta_n(t;\lambda)}{n^{1/m}} = \frac{1}{4}\left(\frac{(m-1)!}{\left(\frac{1}{2}\right)_m}\right)^{1/m}$$

Theorem. (Kuijlaars, Van Assche 1999) Let $\phi(n) = n^{1/(2m)}$ and assume that n, N tend to infinity in such a way that the ratio $n/N \to \ell$. Then, the asymptotic zero distribution as $n \to \infty$ for $P_{n,N}(x) = (\phi(N))^{-n}P_n(\phi(N)x)$, has density

$$a_m(\ell) = \frac{2m}{c\pi(2m-1)}\left(1-x^2/c^2\right)^{1/2}{}_2F_1\left(1\,,1-m\,;\,\frac{3-2m}{2}\,;\,x^2/c^2\right)$$

where $c = 2a\ell^{1/(2m)}$ with $a = \frac{1}{2}\left(\frac{(m-1)!}{(\frac{1}{2})_m}\right)^{1/(2m)}$ for $x \in (-2a\ell^{1/(2m)}, 2a\ell^{1/(2m)})$.

Asymptotic zero distribution

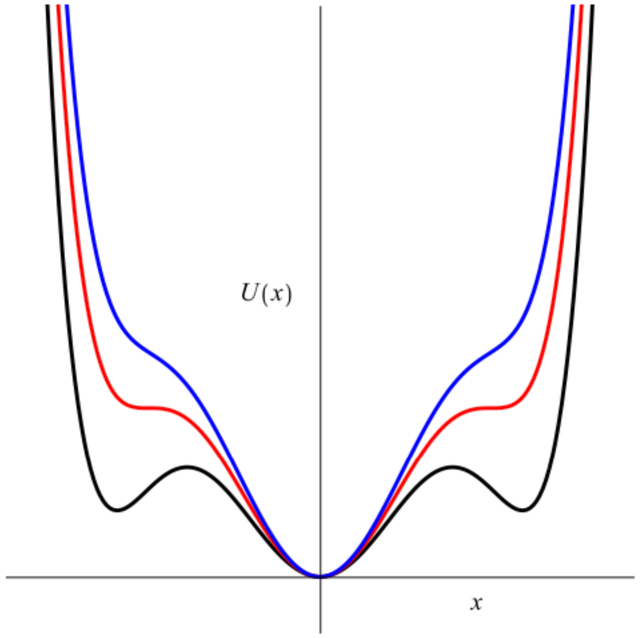


The zeros of $P_{n,N}(x)$ for $\lambda = 0.5$, $t = 1$, $m = 3$, $n = N = 10$ and $\ell = 1$ with the corresponding limiting distribution $a_m(\ell) = \dfrac{2m}{c\pi(2m-1)}\left(1 - x^2/c^2\right)^{1/2} {}_2F_1\left(1\,,1-m\,;\,\frac{3-2m}{2}\,;\,x^2/c^2\right)$

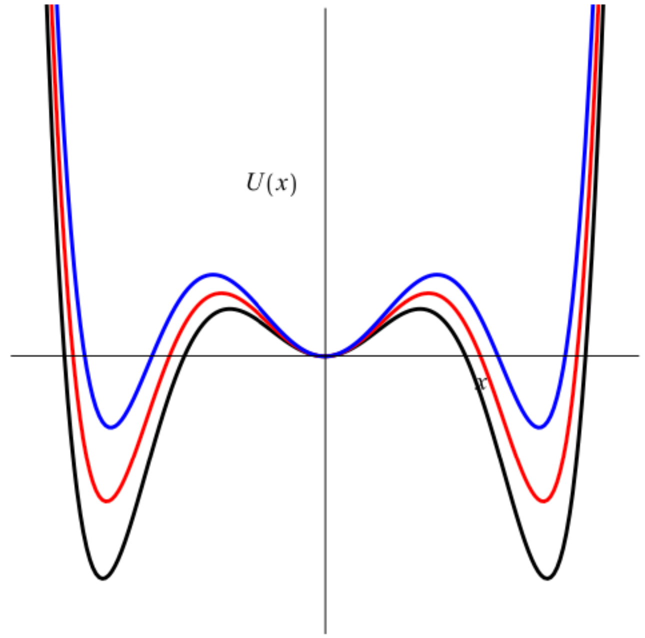
and endpoints $(-2a,0)$ and $(2a,0)$.

Weight: $w(x, \tau, t) = \exp(-x^6 + \tau x^4 + tx^2)$

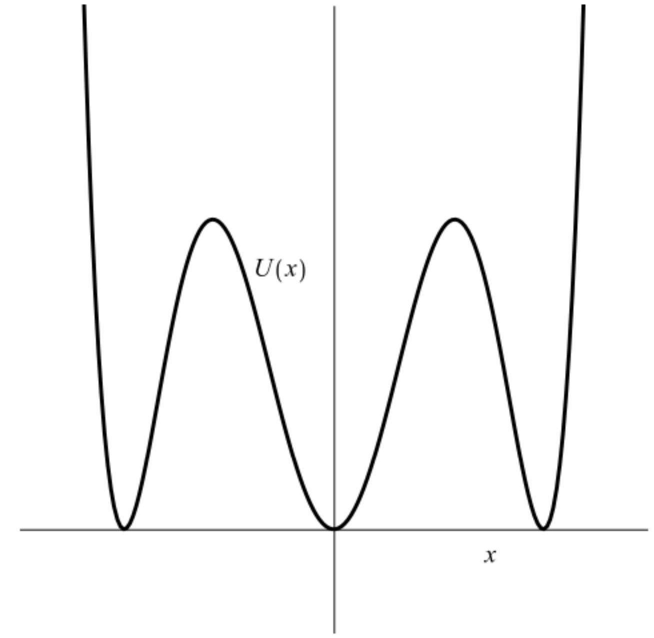
Weight: $w(x,\tau,t) = \exp(-x^6 + \tau x^4 - \kappa\tau^2 x^2)$



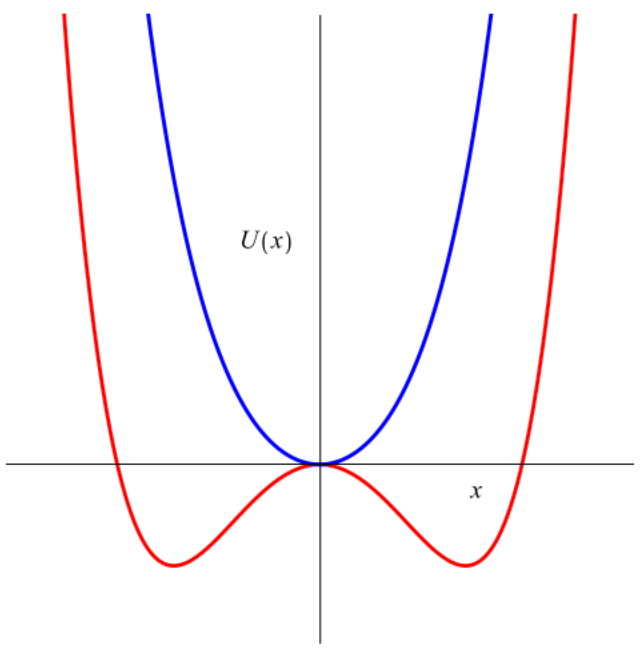
(i), $\kappa > \frac{1}{4},\ \ \tau > 0$



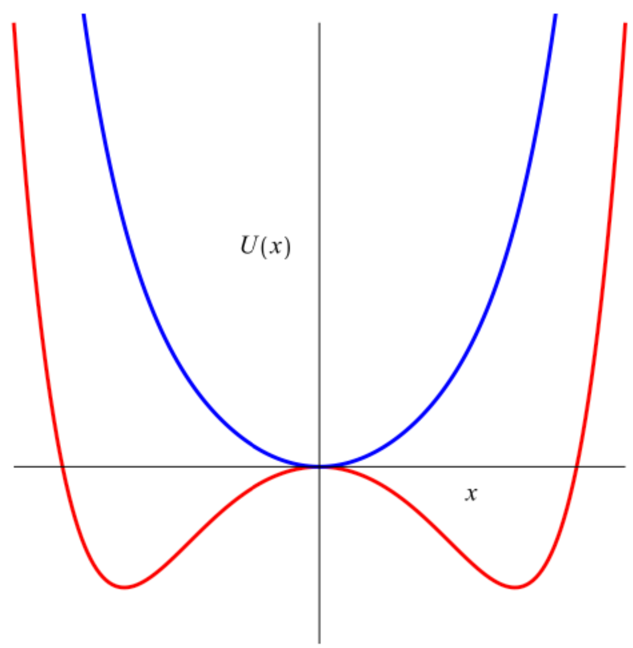
(ii), $\kappa < \frac{1}{4},\ \ \tau > 0$



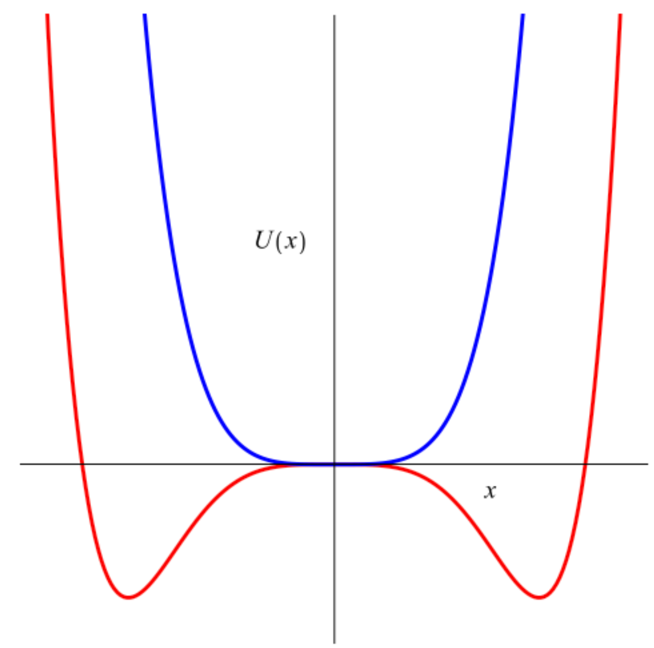
(iii), $\kappa = \frac{1}{4},\ \ \tau > 0$



(iv), $\tau < 0$



(v), $\tau = 0,\ \ t \neq 0$



(vi), $\tau \neq 0,\ \ t = 0$

Case analysis for the weight $w(x) = \exp(-U(x))$

Observe that

$$U(x) = x^2\left(\left(x^2 - \frac{\tau}{2}\right)^2 + \left(\kappa - \frac{1}{4}\right)\tau^2\right) = -\left(x^2 - \frac{\tau}{3}\right)^3 + \frac{(1-3\kappa)\tau^2}{3}x^2 - \frac{\tau^3}{27}$$

where $\kappa = -t/\tau^2$

Case (i) $\kappa > \frac{1}{4}$ and $\tau > 0$, then $U(x)$ has 4 complex zeros

Case (ii) $\kappa = \frac{1}{4}$ and $\tau > 0$, then $U(x) = x^2\left(x^2 - \frac{\tau}{2}\right)^2$

Case (iii) $0 < \kappa < \frac{1}{4}$ and $\tau > 0$, then $U(x)$ has 4 real zeros

Case (iv) $\kappa = 0$ and $|\tau| > 0$, then $U(x) = x^4(x^2 - \tau)$

Case (v) $\kappa < 0$ and $\tau > 0$, then $U(x)$ has two real, two purely imaginary and a double zero

Case (vi) $\tau = 0$ and $|t| > 0$

Case (vii) $\tau < 0$ and $|t| > 0$

Case (viii) $\tau = t = 0$

Lemma. (Clarkson, Jordaan & L - ongoing) The first moment

$\mu_0(\tau,t) = \int_{-\infty}^{+\infty} \exp(-x^6 + \tau x^4 - tx^2)\mathrm{d}x$ is a solution to

$$\frac{\partial^3\varphi}{\partial t^3} - \tfrac{2}{3}\tau\frac{\partial^2\varphi}{\partial t^2} - \tfrac{1}{3}t\frac{\partial\varphi}{\partial t} - \tfrac{1}{6}\varphi = 0$$

Moreover,

$$\mu_0(\tau,t) = \frac{1}{3}\sum_{j=0}^{+\infty}\frac{\tau^j}{j!}\Bigg\{\Gamma\left(\tfrac{2}{3}j+\tfrac{1}{3}n+\tfrac{1}{6}\right)\,{}_1F_2\left(\begin{matrix}\frac{2}{3}j+\frac{1}{3}n+\frac{1}{6} \\ \frac{1}{3},\ \frac{2}{3}\end{matrix};\frac{t^3}{27}\right)$$

$$+t\,\Gamma\left(\tfrac{2}{6}j+\tfrac{1}{3}n+\tfrac{1}{2}\right)\,{}_1F_2\left(\begin{matrix}\frac{2}{3}j+\frac{1}{3}n+\frac{1}{2} \\ \frac{2}{3},\ \frac{4}{3}\end{matrix};\frac{t^3}{27}\right)$$

$$+\tfrac{1}{2}t^2\,\Gamma\left(\tfrac{2}{3}j+\tfrac{1}{3}n+\tfrac{5}{6}\right)\,{}_1F_2\left(\begin{matrix}\frac{2}{3}j+\frac{1}{3}n+\frac{5}{6} \\ \frac{4}{3},\ \frac{5}{3}\end{matrix};\frac{t^3}{27}\right)\Bigg\}$$

About the moments

The moment sequence $(\mu_n)_{n\geq 0}$ defined by $\mu_n(\tau,t) = \displaystyle\int_{-\infty}^{+\infty} x^n \exp(-x^6 + \tau x^4 - \kappa\tau^2 x^2)\mathrm{d}x$

satisfies the recurrence relation

$$3\mu_{2n+6} - 2\tau\mu_{2n+4} + \kappa\tau^2\mu_{2n+2} - \left(n+\tfrac{1}{2}\right)\mu_{2n} = 0.$$

Moreover,

$$\partial_\tau^2\mu_0 - (4\kappa^2 - 3\kappa+\tfrac{4}{9})\tau^2\,\partial_\tau\mu_0 \;+\; \tfrac{1}{9}(6\kappa-1)\tau\mu_0 = \tfrac{1}{6}(4\kappa-1)\left[4\kappa(3\kappa-1)\tau^3 - 3\right]\mu_2,$$

And

$$\partial_\tau^2\mu_{2n} - (4\kappa^2 - 3\kappa+\tfrac{4}{9})\tau^2\partial_\tau\mu_{2n}+\tfrac{1}{9}(2n+1)(6\kappa-1)\tau\mu_{2n} = \left\{\tfrac{1}{6}(4\kappa-1)[4\kappa(3\kappa-1)\tau^3-3]+\tfrac{1}{9}n\right\}\mu_{2n+2}\,.$$

$$\begin{aligned}\frac{\mathrm{d}^3\mu_0}{\mathrm{d}\tau^3} &+ \left\{\frac{2(9\kappa-2)\tau^2}{9} - \frac{12\kappa(3\kappa-1)\tau^2}{4\kappa(3\kappa-1)\tau^3-3}\right\}\frac{\mathrm{d}^2\mu_0}{\mathrm{d}\tau^2} + \left\{\frac{(4\kappa-1)\kappa^2\tau^4}{3} + \frac{(36\kappa^2-27\kappa+4)\tau}{4\kappa(3\kappa-1)\tau^3-3}\right\}\frac{\mathrm{d}\mu_0}{\mathrm{d}\tau}\\ &+ \left\{\frac{(4\kappa-1)\kappa^2\tau^3}{3} - \kappa + \frac{5}{36} + \frac{1-6\kappa}{4\kappa(3\kappa-1)\tau^3-3}\right\}\mu_0 = 0.\end{aligned}$$

About the moments - particular cases

The moment sequence $(\mu_n)_{n\geq 0}$ defined by $\mu_n(\tau,t) = \displaystyle\int_{-\infty}^{+\infty} x^n \exp(-x^6 + \tau x^4 - \kappa\tau^2 x^2)\mathrm{d}x$

satisfies the recurrence relation

$$3\mu_{2n+6} - 2\tau\mu_{2n+4} + \kappa\tau^2\mu_{2n+2} - \left(n+\tfrac{1}{2}\right)\mu_{2n} = 0.$$

For $\kappa = \dfrac{1}{4}$, then

$$\mu_0(\tau,\tfrac{1}{4}) = \frac{\pi\sqrt{6\tau}}{9}\left\{I_{1/6}\left(\frac{\tau^3}{108}\right) + I_{-1/6}\left(\frac{\tau^3}{108}\right)\right\}\exp\left(-\frac{\tau^3}{108}\right).$$

For $\kappa = \dfrac{1}{3}$, then

$$\mu_0(\tau,\tfrac{1}{3}) = \left\{\tfrac{1}{3}\Gamma\left(\tfrac{1}{6}\right)\,{}_2F_2\left(\tfrac{1}{6},\tfrac{1}{2};\tfrac{1}{3},\tfrac{2}{3};\frac{\tau^3}{27}\right) + \tfrac{1}{3}\tau\Gamma\left(\tfrac{5}{6}\right)\,{}_2F_2\left(\tfrac{1}{2},\tfrac{5}{6};\tfrac{2}{3},\tfrac{4}{3};\frac{\tau^3}{27}\right)\right.$$
$$\left. - \frac{\tau^2\sqrt{\pi}}{36}\,{}_2F_2\left(\tfrac{5}{6},\tfrac{7}{6};\tfrac{4}{3},\tfrac{5}{3};\frac{\tau^3}{27}\right)\right\}\exp\left(-\frac{\tau^3}{27}\right),$$

Asymptotics for β_1

The moment sequence $(\mu_n)_{n\geq 0}$ defined by $\mu_n(\tau,t) = \displaystyle\int_{-\infty}^{+\infty} x^n \exp(-x^6 + \tau x^4 - t x^2)\mathrm{d}x$

satisfies the recurrence relation

$$3\mu_{2n+6} - 2\tau\mu_{2n+4} - t\mu_{2n+2} - \left(n+\tfrac{1}{2}\right)\mu_{2n} = 0$$

with $\mu_{2n+1} = 0$.

Theorem. For fixed $\kappa > \frac{1}{4}$ and $\tau > 0$, then for all $n \geq 0$

$$\beta_1 \sim \frac{1}{8\tau^2\left(\kappa - \frac{1}{4}\right)^2}, \quad \text{as} \quad \tau \to +\infty\,.$$

For fixed $0 < \kappa < \frac{1}{4}$ and $\tau > 0$, then for all $n \geq 0$

$$\beta_1 \sim \tfrac{1}{2}\tau\left(1 + \sqrt{1-3\kappa}\right), \quad \text{as} \quad \tau \to +\infty\,.$$

About this weight

Analysis of

$$w(x;\tau,t) = \exp(-U(x;\tau,t)) \quad \text{with} \quad U(x;\tau,t) = x^6 - \tau\, x^4 - t\, x^2$$

where $\tau,\ t \in \mathbb{R}$,

and the recurrence coefficients satisfy

$$6\beta_n\big(\beta_{n+1}\beta_{n-1} + \beta_{n-1}\left(\beta_{n-2} + \beta_{n-1} + \beta_n\right) + \beta_n\left(\beta_{n-1} + \beta_n + \beta_{n+1}\right) + \beta_{n+1}\left(\beta_n + \beta_{n+1} + \beta_{n+2}\right)\big)$$
$$-4\tau\, \beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) - 2t\, \beta_n = n$$

The recurrence coefficients

Lemma. Let $w_0(x)$ be a symmetric positive weight on the real line and suppose that $w(x;t,\tau) = \exp(tx^2 + \tau x^4)\, w_0(x), \quad x \in \mathbb{R}$ with $t, \tau \in \mathbb{R}$, is a weight such that all the moments of exist.
Then the recurrence coefficient $\beta_n(t,\tau)$ satisfies the Volterra, or the Langmuir lattice, equation

$$\partial_t \beta_n = \beta_n(\beta_{n+1} - \beta_{n-1})$$

and the differential-difference equation

$$\partial_\tau \beta_n = \beta_n\Big((\beta_{n+2} + \beta_{n+1} + \beta_n)\beta_{n+1} - (\beta_n + \beta_{n-1} + \beta_{n-2})\beta_{n-1}\Big).$$

The recurrence coefficients $\beta_n(\tau,t)$ satisfy the recurrence relation

$$6\beta_n\big(\beta_{n-1}\left(\beta_{n-2}+\beta_{n-1}+\beta_n+\beta_{n+1}\right)+\beta_n\left(\beta_{n-1}+\beta_n+\beta_{n+1}\right)+\beta_{n+1}\left(\beta_n+\beta_{n+1}+\beta_{n+2}\right)\big)$$
$$-4\tau\,\beta_n(\beta_{n-1}+\beta_n+\beta_{n+1})-2t\,\beta_n=n$$

Remark. This equation is:

I. A special case of $\mathrm{dP}_{\mathrm{I}}^{(2)}$, the 2nd member of the discrete Painlevé I hierarchy. Cresswell & Joshi showed that its continuum limit is equivalent to

$$\frac{\mathrm{d}^4w}{\mathrm{d}z^4}=10w\frac{\mathrm{d}^2w}{\mathrm{d}z^2}+5\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2-10w^3+z$$

which is $\mathrm{P}_{\mathrm{I}}^{(2)}$.

II. Also known as the "string equation" and arises in physical applications such as 2-dimensional quantum gravity.

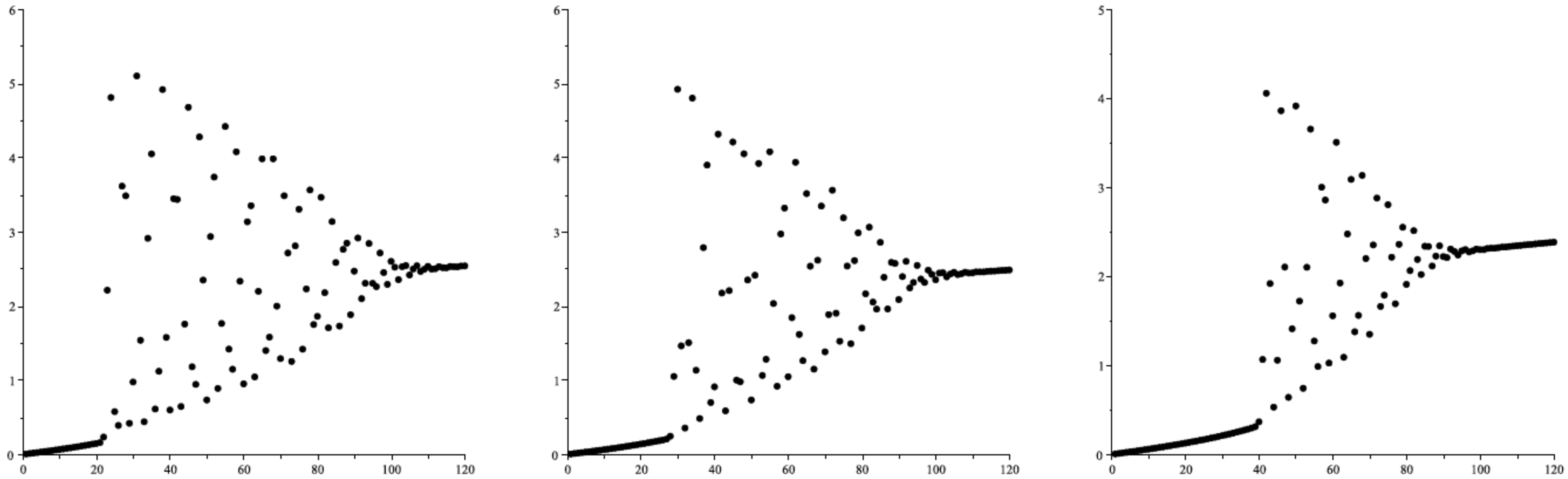
Consider the weight

$$w(x;\tau,t) = \exp\left(-x^6 + \tau x^4 + tx^2\right)$$

which is equivalent to the weight

$$W(z) = \exp\{-NV(x)\}, \qquad V(x) = g_2 x^2 + g_4 x^4 + g_6 x^6$$

with N, g_2, g_4 and g_6 parameters.



- "Chaotic behavior in one matrix models" (**Jurkiewicz [1991]**)
- "Chaos in the Hermitian one-matrix model" (**Sénéchal (1992]**)

Some historical remarks & applications

Benassi & Moro (2020) and Dell'Atti (2022) considered the weight

$$W(x;T_2,T_4,T_6,N) = \exp\left(N\left[T_6x^6 + T_4x^4 + (T_2-\tfrac{1}{2})x^2\right]\right)$$ with T_2, T_4, T_6 and N parameters.

They interpreted the Jurkiewicz's "chaotic phase" as a dispersive shock propagating through the chain in the continuum/thermodynamic limit and explained the complexity of its phase diagram in the context of dispersive hydrodynamics.

The recurrence coefficients satisfy the discrete equation

$$u_n\Big\{6T_6(u_{n-2}u_{n-1} + u_{n-1}^2 + 2u_{n-1}u_n + u_{n-1}u_{n+1} + u_n^2 + 2u_nu_{n+1} + u_{n+1}^2 + u_{n+1}u_{n+2}) \\ +4T_4(u_{n-1} + u_n + u_{n+1}) + (2T_2 - 1)\Big\} = -\frac{n}{N}$$

and the associated cubic equation is

$$60T_6u^3 + 12T_4u^2 + (2T_2-1)u + \frac{n}{N} = 0$$

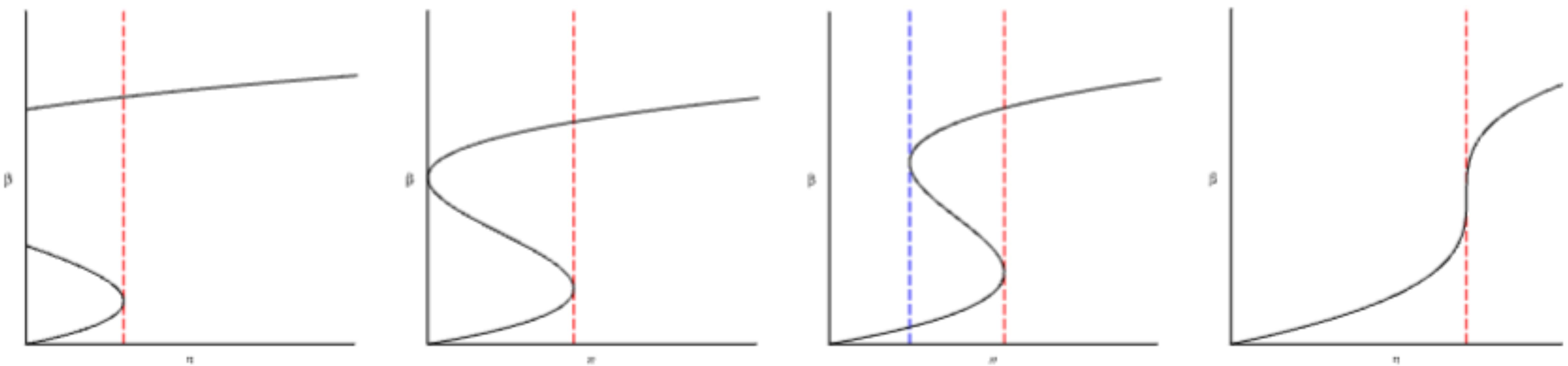
$$6\beta_n\left(\beta_{n-1}\left(\beta_{n-2}+\beta_{n-1}+\beta_n+\beta_{n+1}\right)+\beta_n\left(\beta_{n-1}+\beta_n+\beta_{n+1}\right)+\beta_{n+1}\left(\beta_n+\beta_{n+1}+\beta_{n+2}\right)\right)$$
$$-4\tau\,\beta_n(\beta_{n-1}+\beta_n+\beta_{n+1})-2t\,\beta_n=n$$

Asymptotic behaviour:

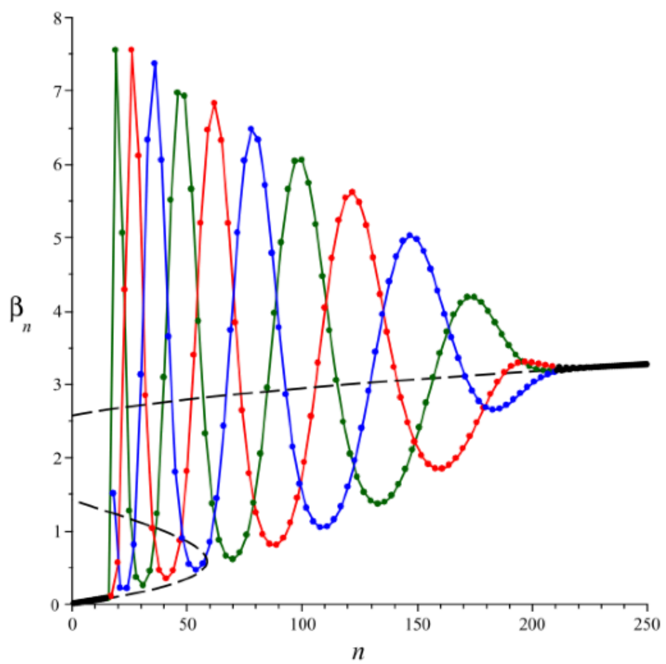
$$\beta_n \sim \beta(n), \qquad \text{as} \qquad n \to \infty,$$

where $\beta(n)$ is the β-curve

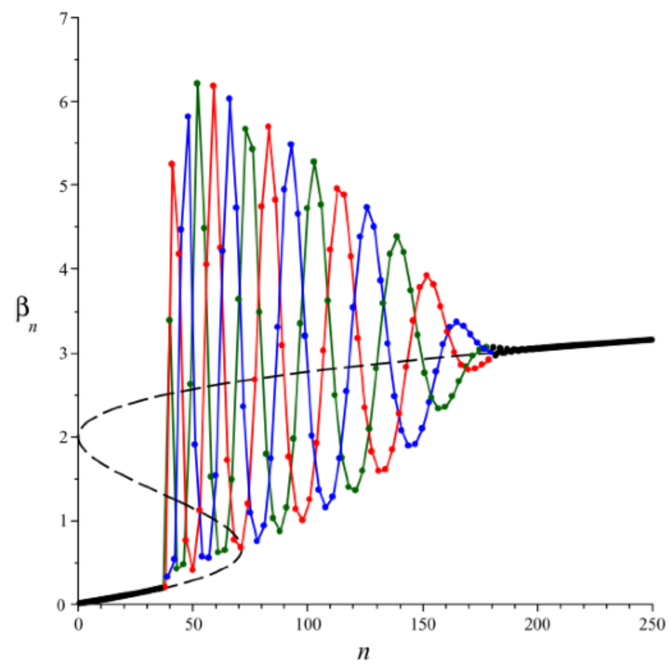
$$60\beta^3-12\tau\beta^2+2\kappa\tau^2\beta=n.$$



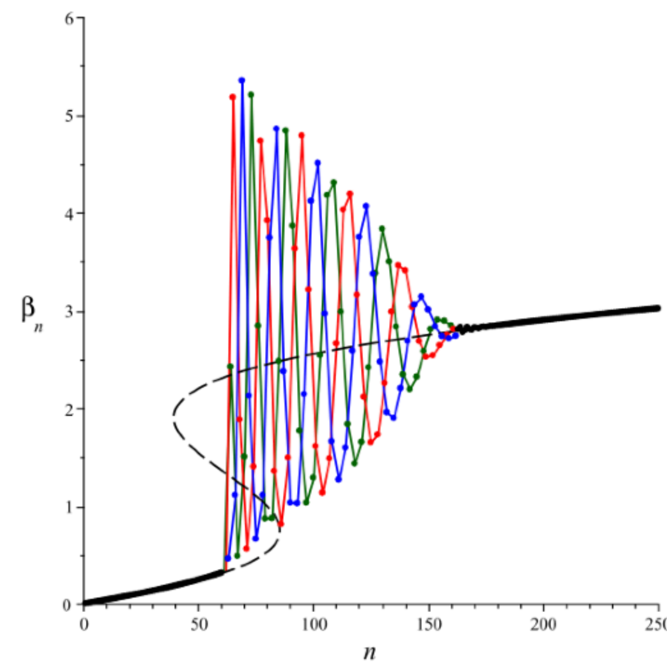
$\kappa=\frac{1}{4}$ $\kappa=\frac{3}{10}$ $\kappa=\frac{1}{3}$ $\kappa=\frac{2}{5}$



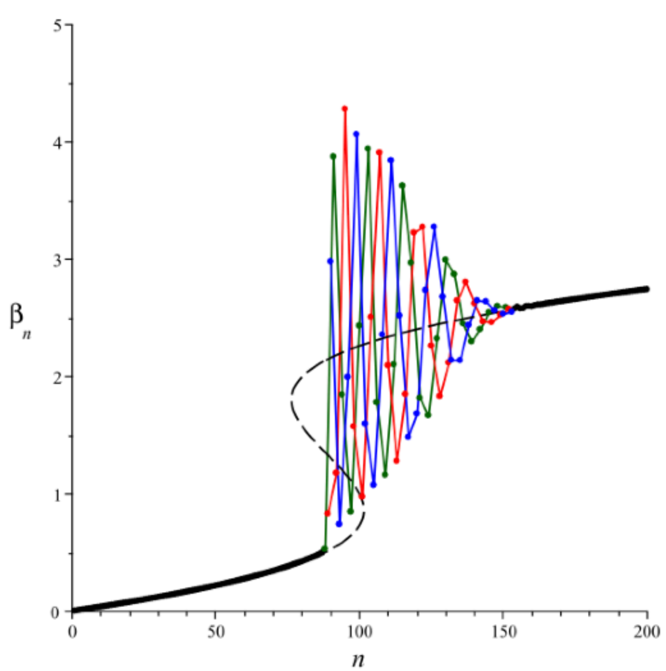
$\kappa = 0.275$



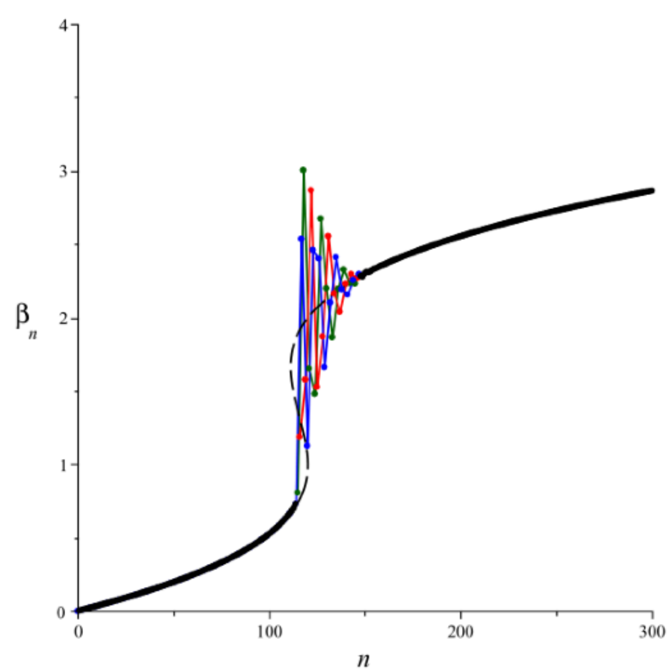
$\kappa = 0.3$



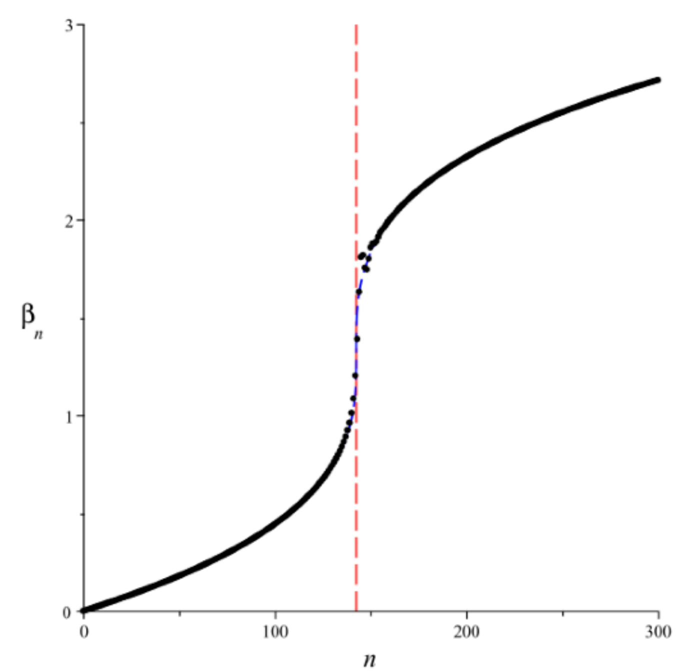
$\kappa = 0.325$



$\kappa = 0.35$

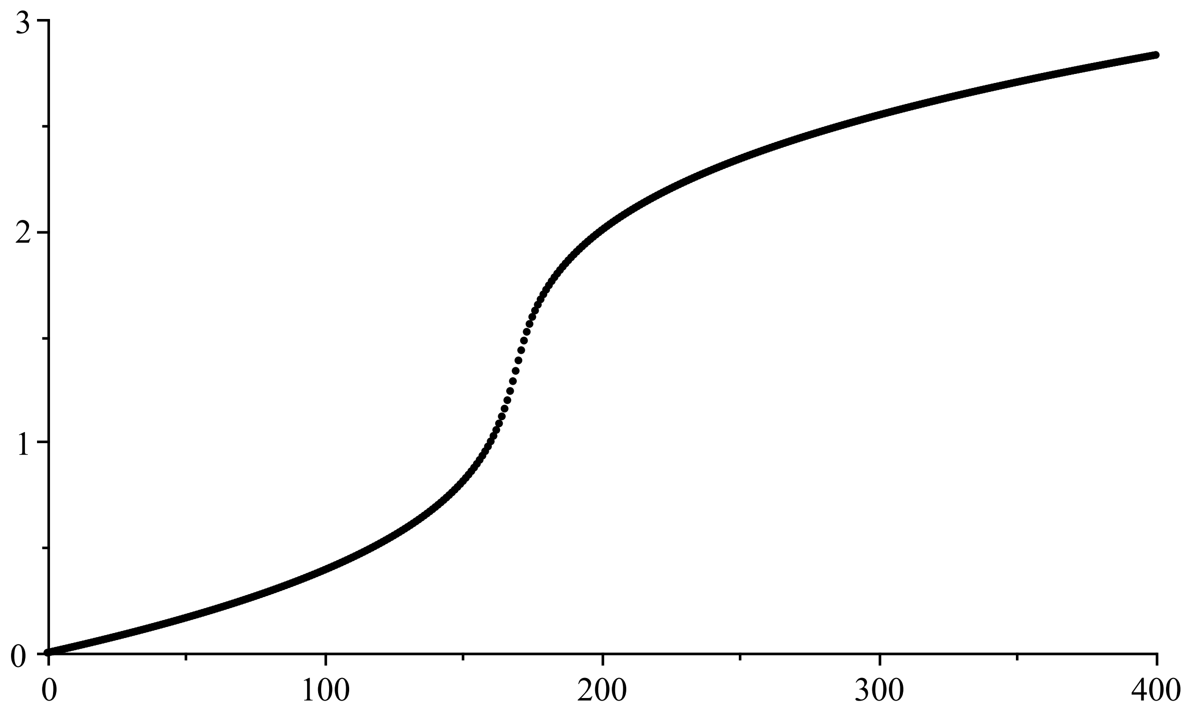


$\kappa = 0.375$

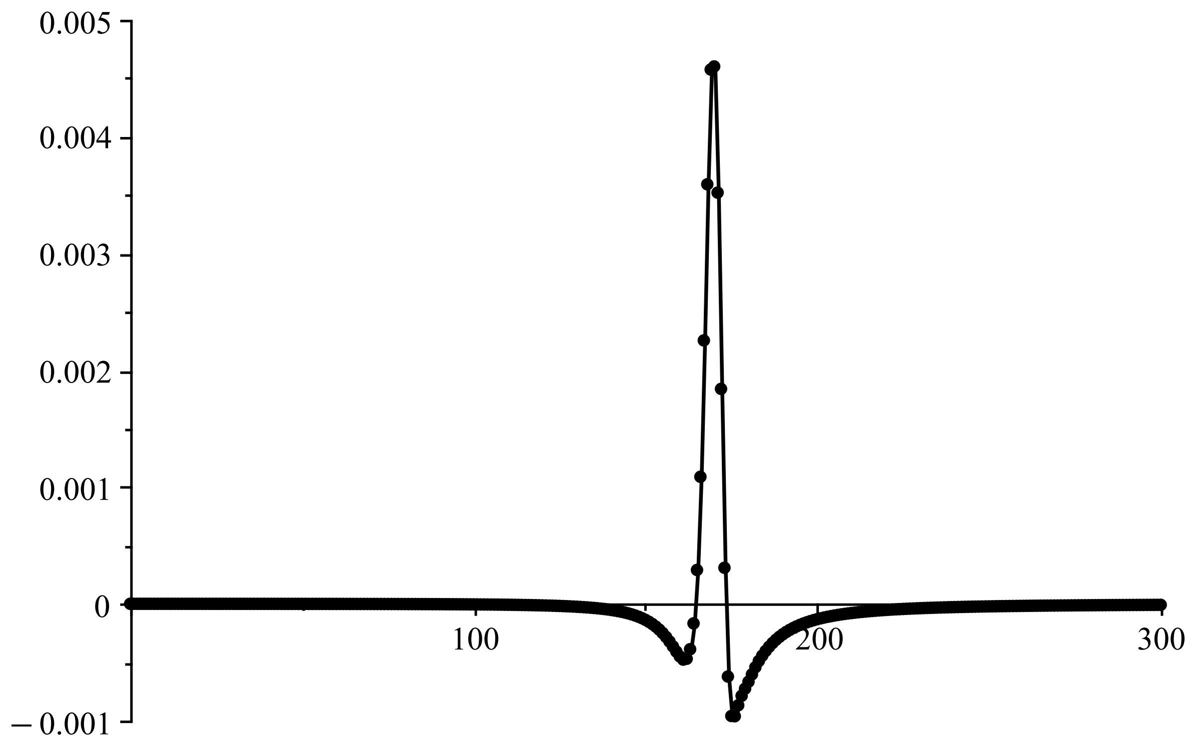


$\kappa = 0.4$

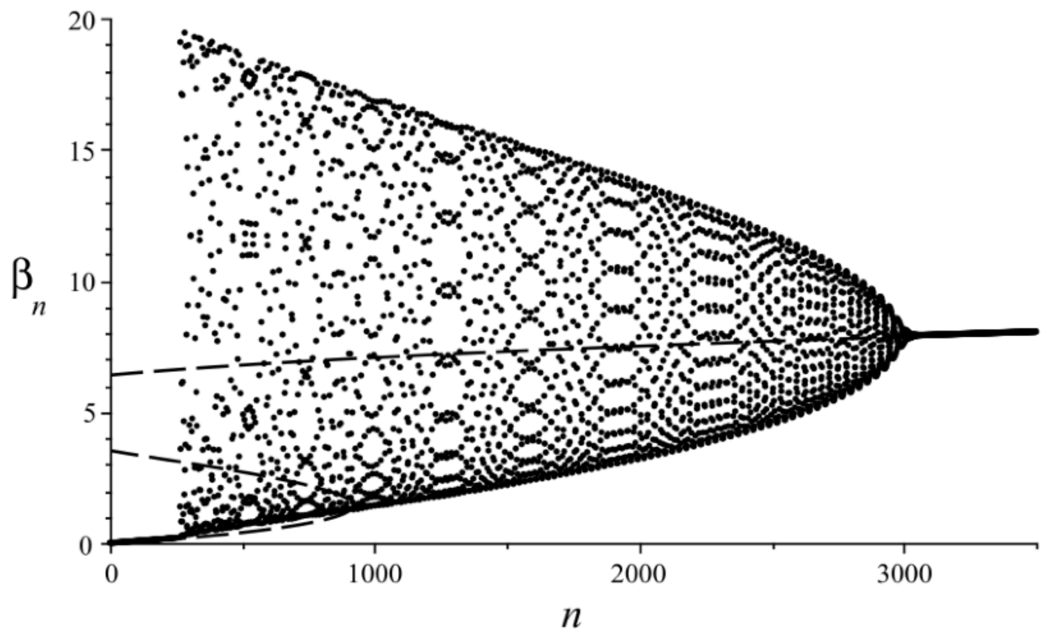
Case (i) $\kappa > 2/5$ and $\tau = 20$ "one-branch case"



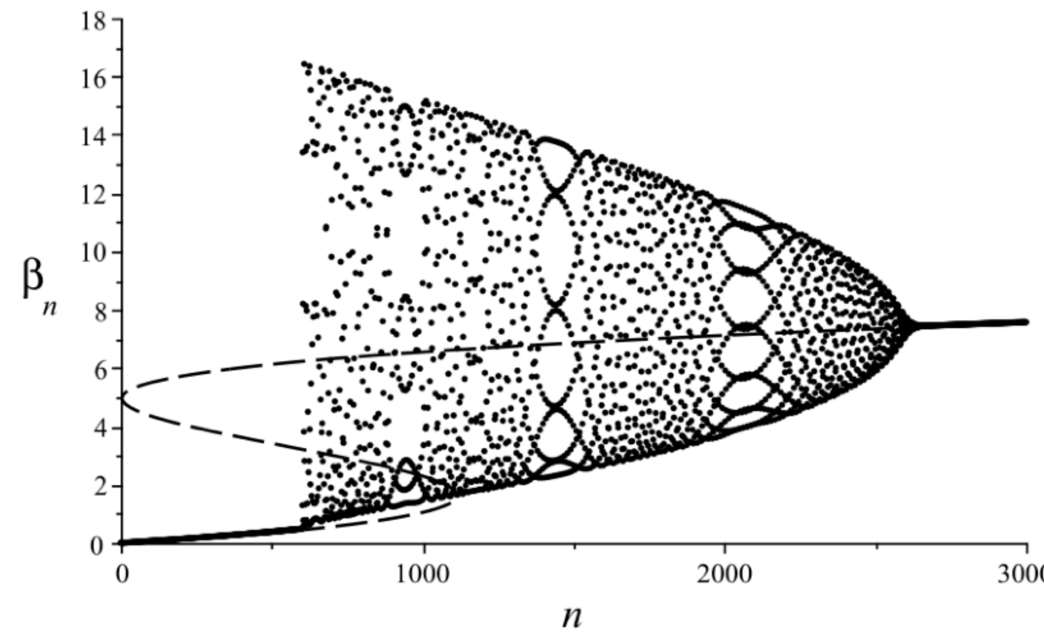
β_n for $0 \le n \le 400$ $(\kappa = 0.425)$



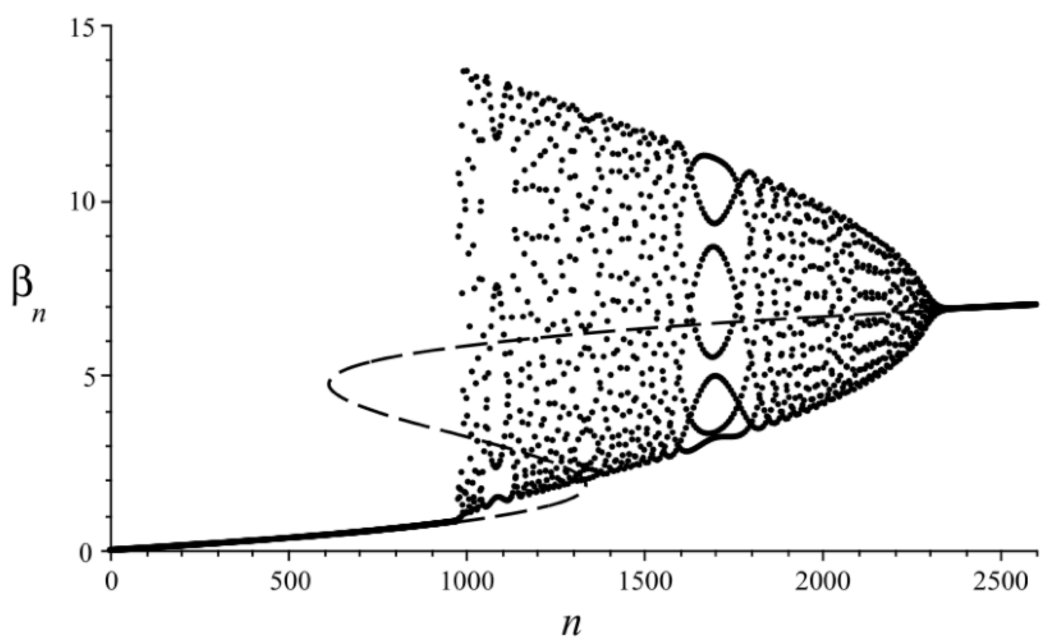
$\beta_n - \beta(n)$ for $0 \le n \le 400$
$(\kappa = 0.425)$



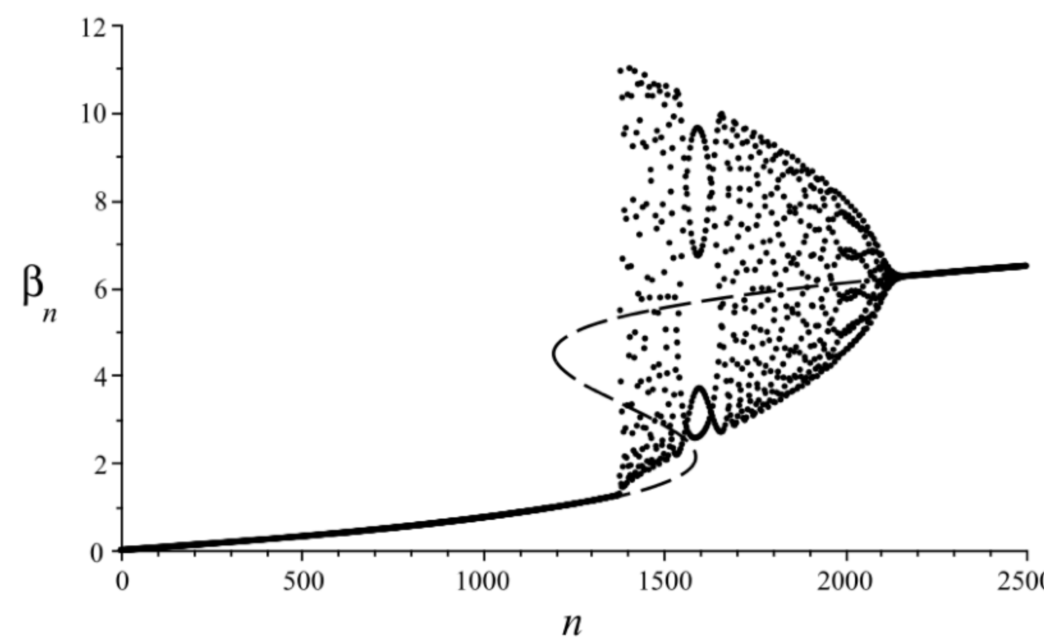
$\tau = 50, \quad \kappa = 0.275$



$\tau = 50, \quad \kappa = 0.3$



$\tau = 50, \quad \kappa = 0.325$

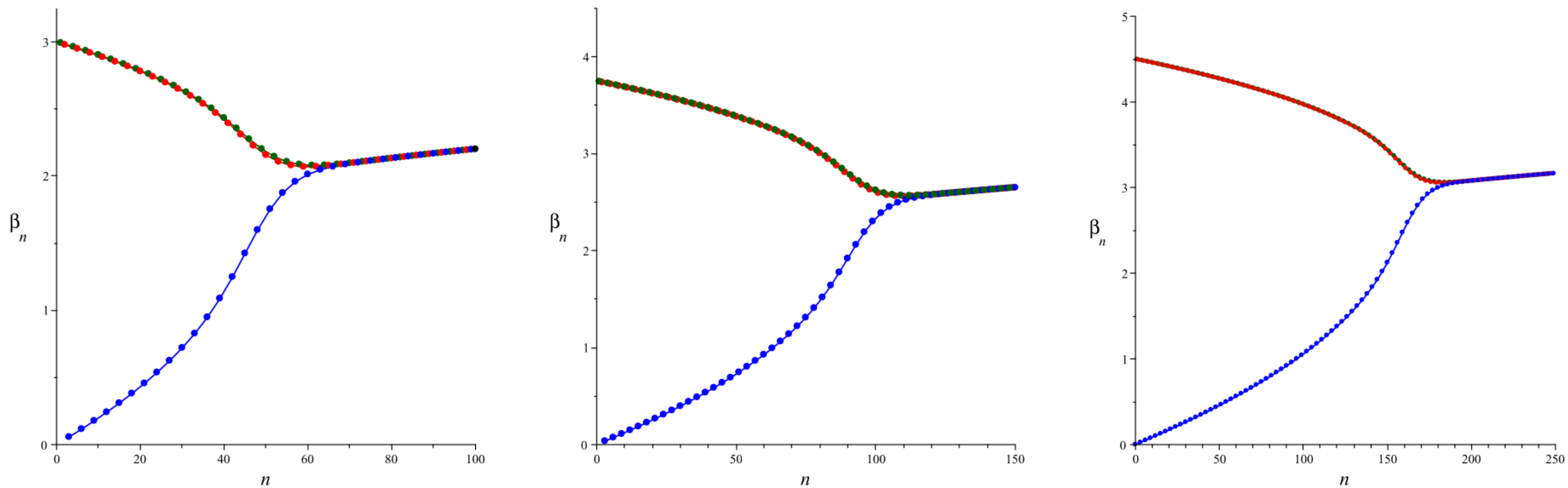


$\tau = 50, \quad \kappa = 0.35$

In this case the weight is

$$\omega(x;\tau) = \exp\left\{-x^2(x^2 - \tfrac{1}{2}\tau)^2\right\}$$

Plotting β_{3n}, β_{3n+1}, β_{3n+2}



$\tau = 12$ $\tau = 15$ $\tau = 18$

Case (ii): $\tau > 0$ and $\kappa = \frac{1}{4}$

Setting $\beta_{3n} = u$ and $\beta_{3n\pm1} = v$ in

$$6\beta_n\big(\beta_{n-2}\beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} + \beta_n^2 + 2\beta_n\beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2}\big) - 4\tau\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) + \tfrac{1}{2}\tau^2\beta_n = n$$

gives

$$6u\big(u^2 + 4uv + 5v^2\big) - 4\tau u(u + 2v) + \tfrac{1}{2}\tau^2 u = n$$
$$6v\big(u^2 + 5uv + 4v^2\big) - 4\tau v(u + 2v) + \tfrac{1}{2}\tau^2 v = n$$

and then it can be shown that u and v satisfy the cubics

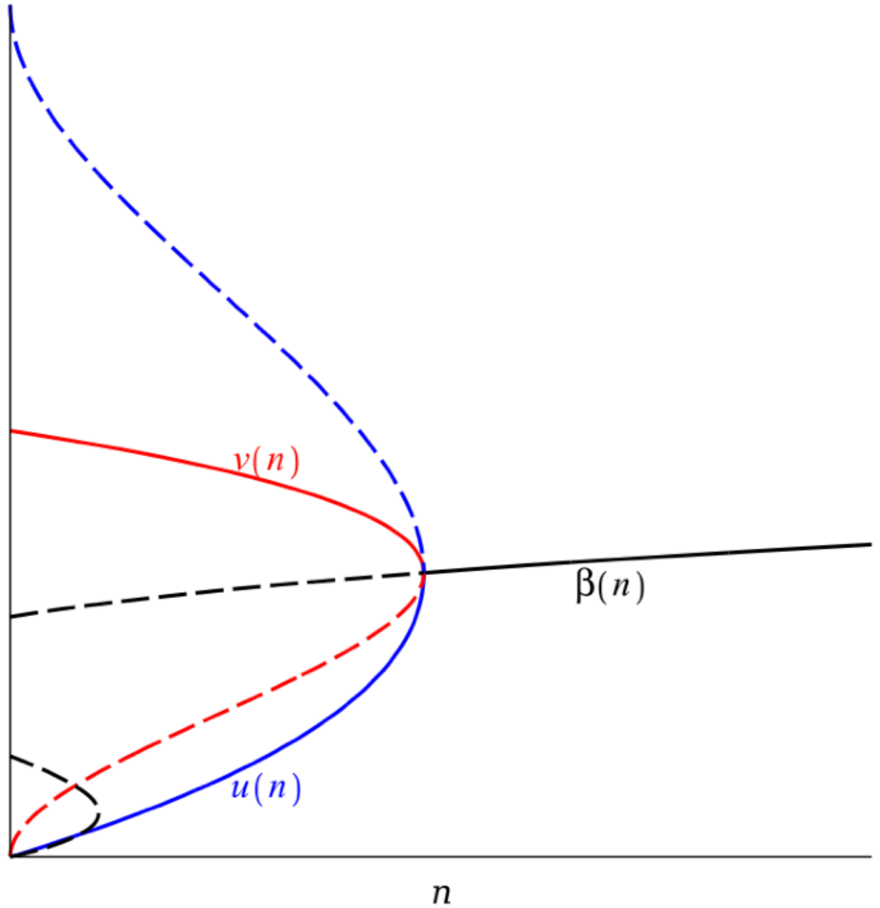
$$12u^3 - 12\tau u^2 + 3\tau^2 u - 8n = 0$$
$$12v^3 - 3\tau v^2 + n = 0$$

Setting $\beta_n = \beta$ gives

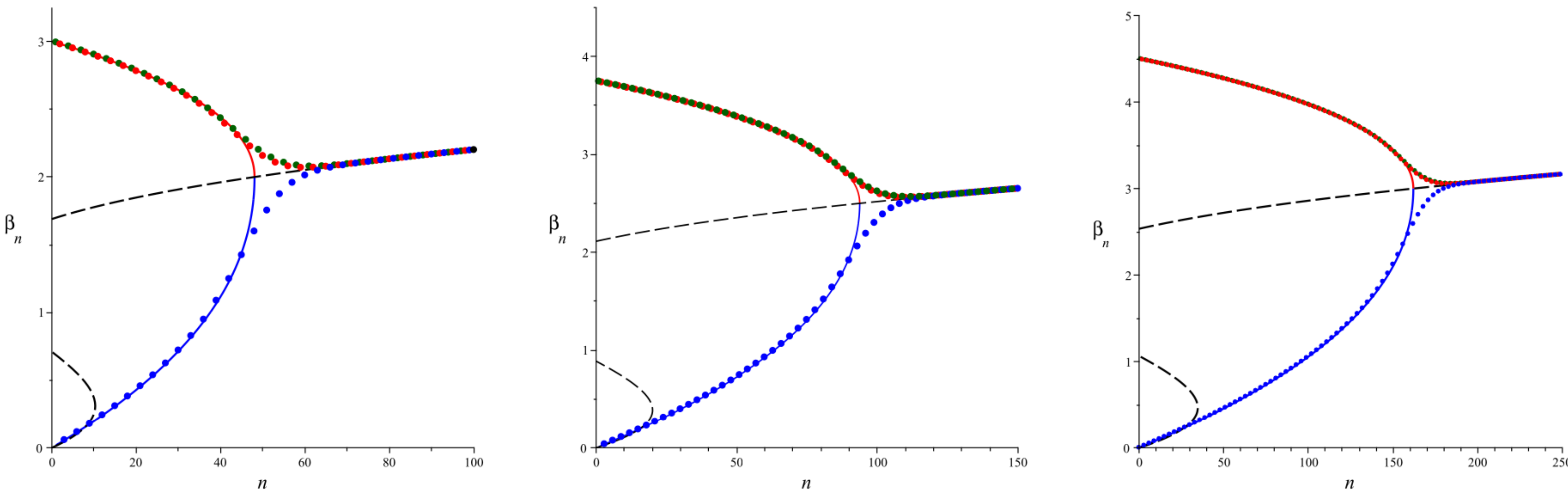
$$60\beta^3 - 12\tau\beta^2 + \tfrac{1}{2}\tau^2\beta = n$$

All three cubics meet at the point

$$\left(\frac{\tau^3}{36}, \frac{\tau}{6}\right)$$



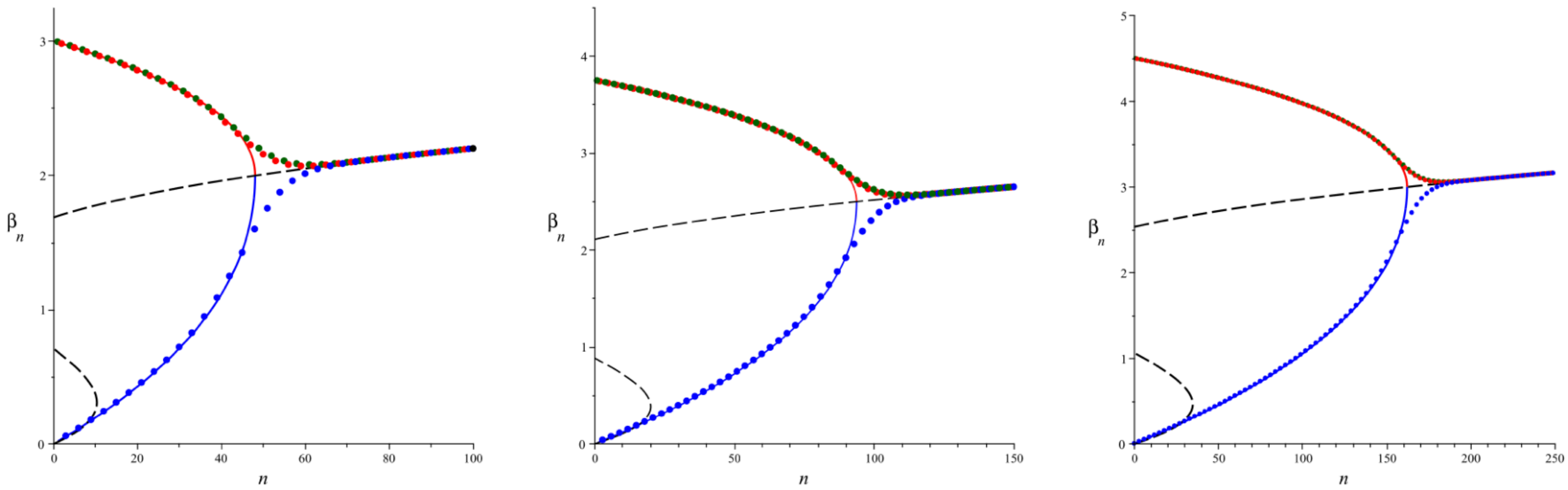
Case (ii): $\tau > 0$ and $\kappa = \frac{1}{4}$



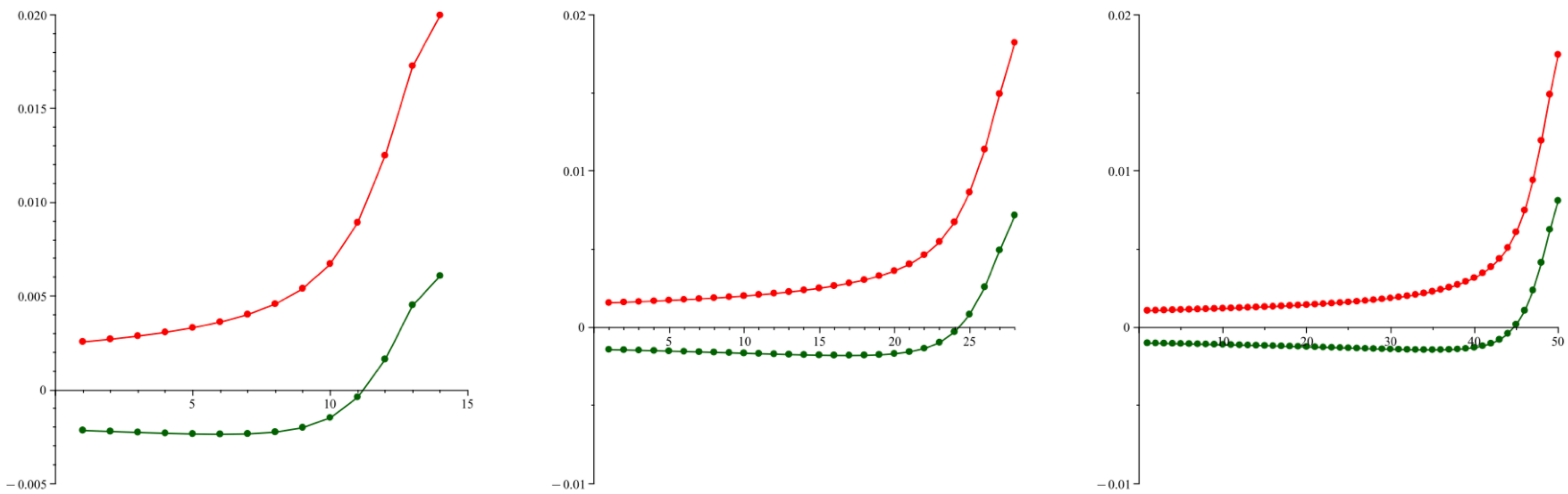
$\tau = 12$ $\tau = 15$ $\tau = 18$

β_{3n}	$12u^3 - 12\tau u^2 + 3\tau^2 u - 8n = 0$
$\beta_{3n+1}, \beta_{3n+2}$	$12v^3 - 3\tau v^2 + n = 0$
	$60\beta^3 - 12\tau\beta^2 + \frac{1}{2}\tau^2\beta = n$

Case (ii): $\tau > 0$ and $\kappa = \frac{1}{4}$



$\tau = 12$ $\tau = 15$ $\tau = 18$



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Setting $\beta_{2n} = u$, $\beta_{2n+1} = v$ and $t = -\kappa\tau^2$ in

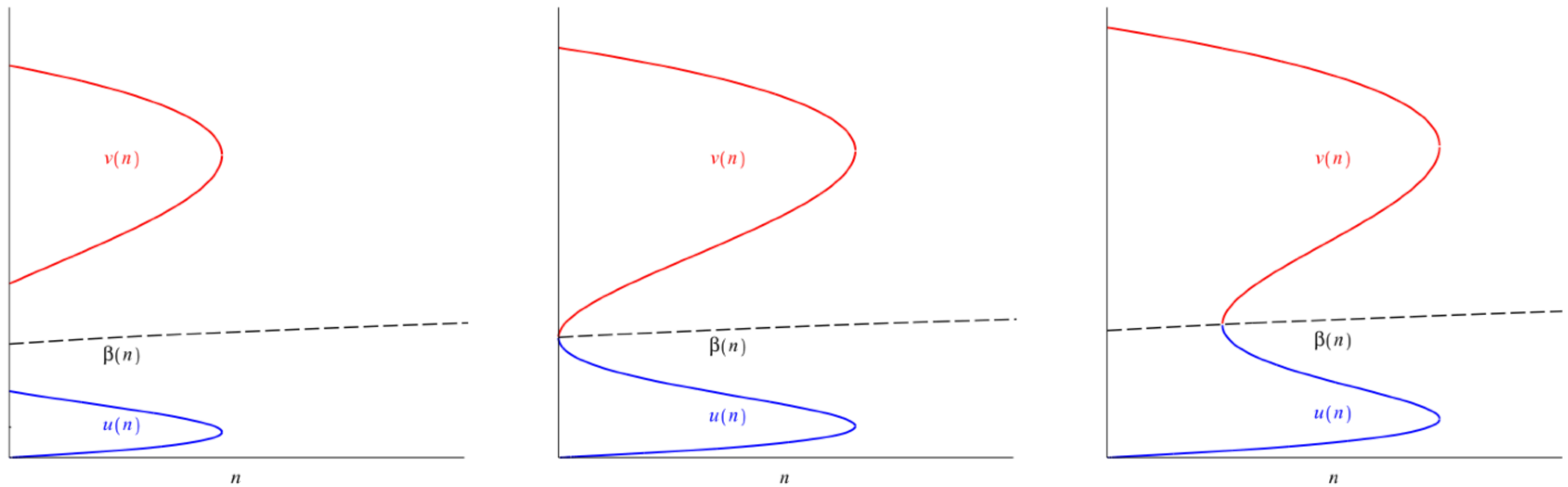
$$6\beta_n\left(\beta_{n-2}\beta_{n-1} + \beta_{n-1}^2 + 2\beta_{n-1}\beta_n + \beta_{n-1}\beta_{n+1} + \beta_n^2 + 2\beta_n\beta_{n+1} + \beta_{n+1}^2 + \beta_{n+1}\beta_{n+2}\right) \\ - 4\tau\beta_n(\beta_{n-1} + \beta_n + \beta_{n+1}) - 2t\beta_n = n$$

gives the system

$$6u\left(u^2 + 6uv + 3v^2\right) - 4\tau u(u + 2v) + 2\kappa\tau^2 u = n$$
$$6v\left(3u^2 + 6uv + v^2\right) - 4\tau v(2u + v) + 2\kappa\tau^2 v = n$$

Letting $u = \xi - \eta$ and $v = \xi + \eta$ gives

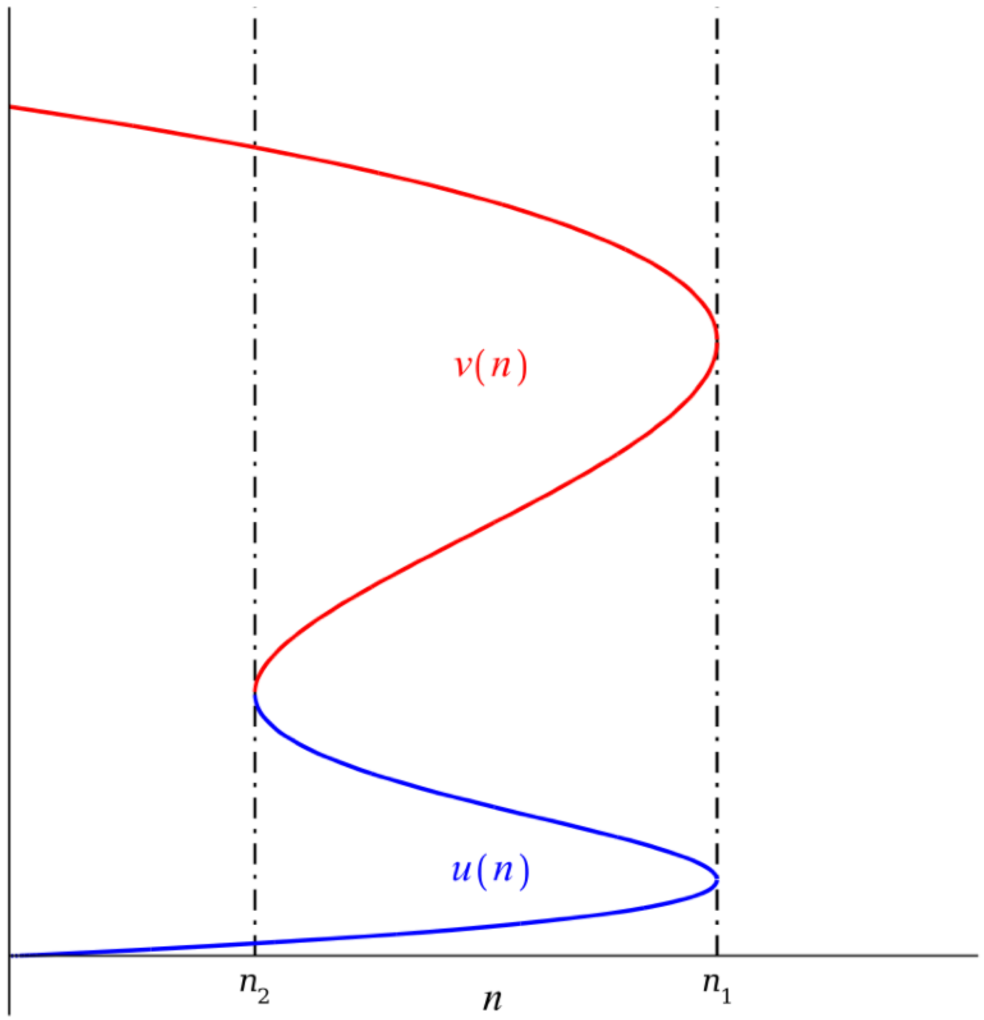
$$144\xi^3 - 72\tau\xi^2 + 4(2 + 3\kappa)\tau^2\xi - 2\kappa\tau^3 + 3n = 0$$
$$\eta = \left(\tfrac{1}{3}\xi^2 - \tfrac{2}{3}\tau\xi + \tfrac{1}{6}\kappa\tau^2\right)^{1/2}$$



$\kappa = \frac{1}{5}$ $\kappa = \frac{1}{6}$ $\kappa = \frac{1}{8}$

Case (iii) $0 < \kappa \leqslant \frac{1}{4} - \epsilon$ and $\tau = 20$ "two-branch case"

$$6u(u^2 + 6uv + 3v^2) - 4\tau u(u + 2v) + 2\kappa\tau^2 u = n$$
$$6v(3u^2 + 6uv + v^2) - 4\tau v(2u + v) + 2\kappa\tau^2 v = n$$



$$n_1 = \tfrac{4}{9}\left(\tfrac{1}{3} - \kappa\right)^{3/2}\tau^3$$

$$n_2 = \frac{\left[2(4 - 27\kappa) + (4 - 18\kappa)^{3/2}\right]\tau^3}{243}$$

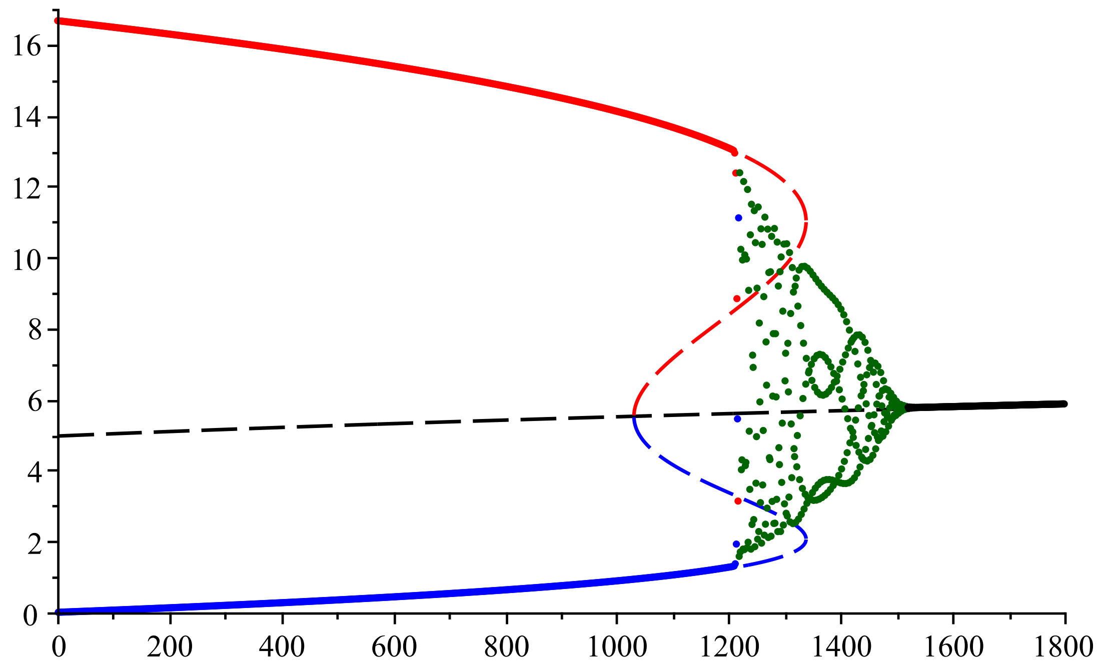
Note that

$$n_2 = 0 \quad \Rightarrow \quad \kappa = \tfrac{1}{6}$$

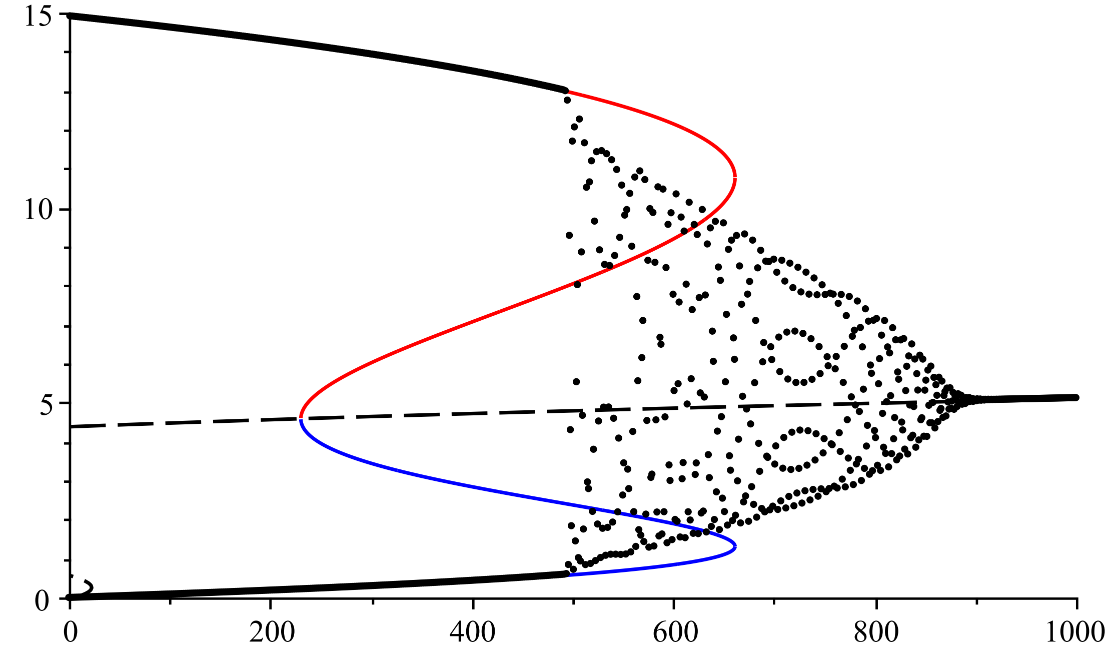
Also $n_1 = n_2$ when

$$\tfrac{4}{9}\left(\tfrac{1}{3} - \kappa\right)^{3/2} = \frac{\left[2(4 - 27\kappa) + (4 - 18\kappa)^{3/2}\right]}{243} \quad \Rightarrow \quad \kappa = -\tfrac{2}{3}$$

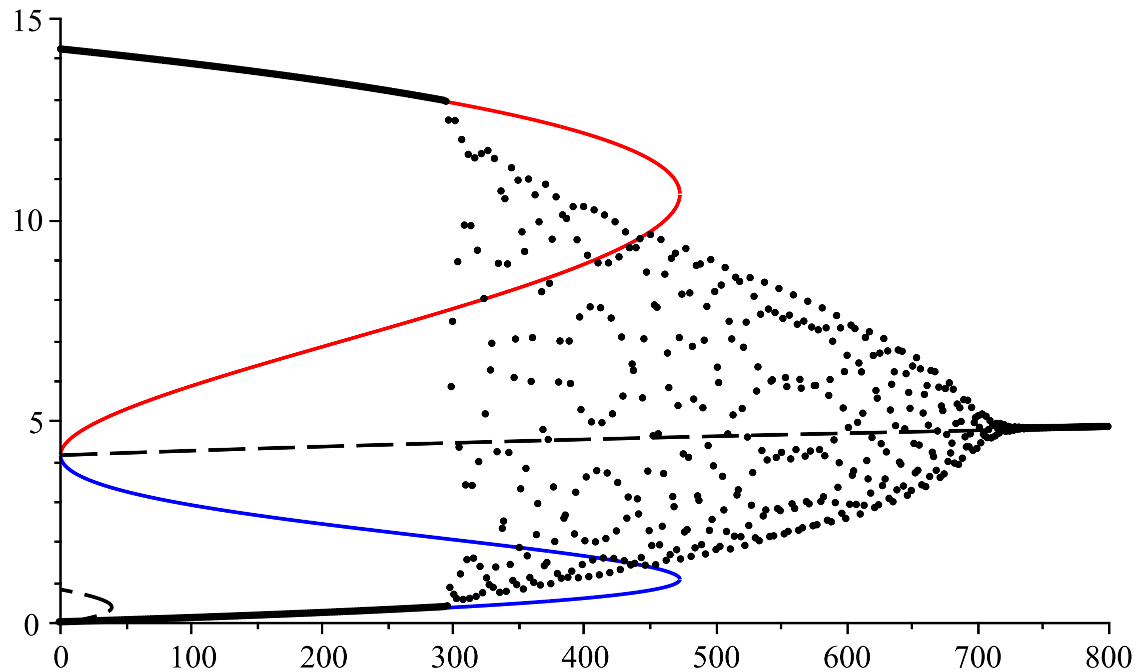
Case(iii): Evolution of $0 \le \kappa < \frac{1}{4} - \epsilon$ and $\tau = 25$



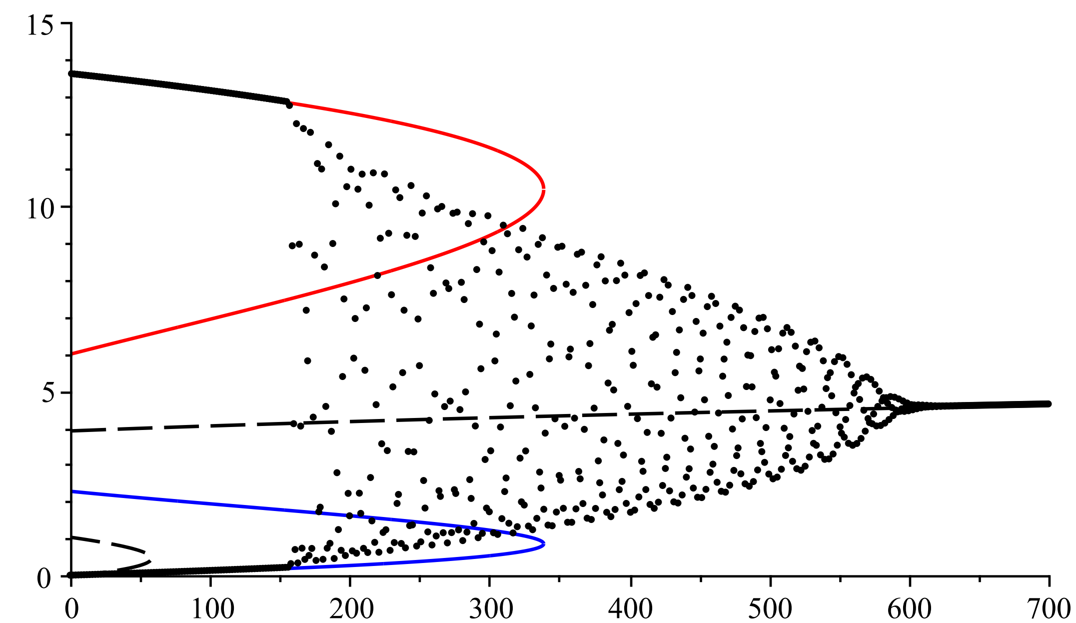
$\kappa = 0$



$\kappa = \frac{1}{8}$

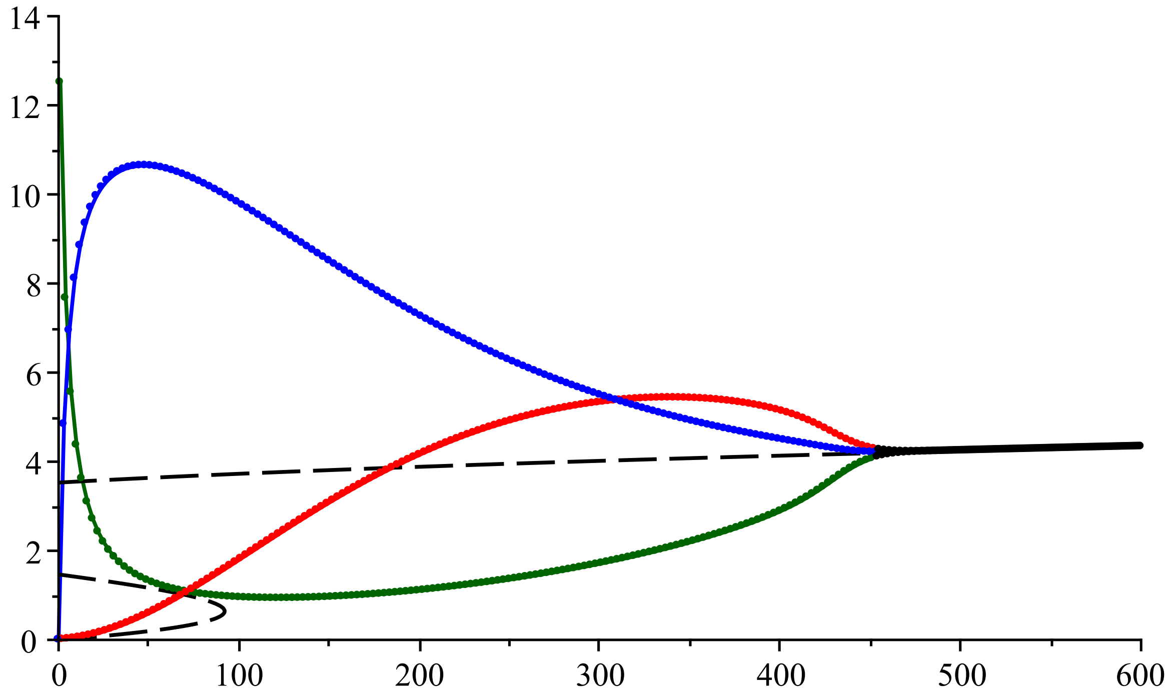


$\kappa = \frac{1}{6}$

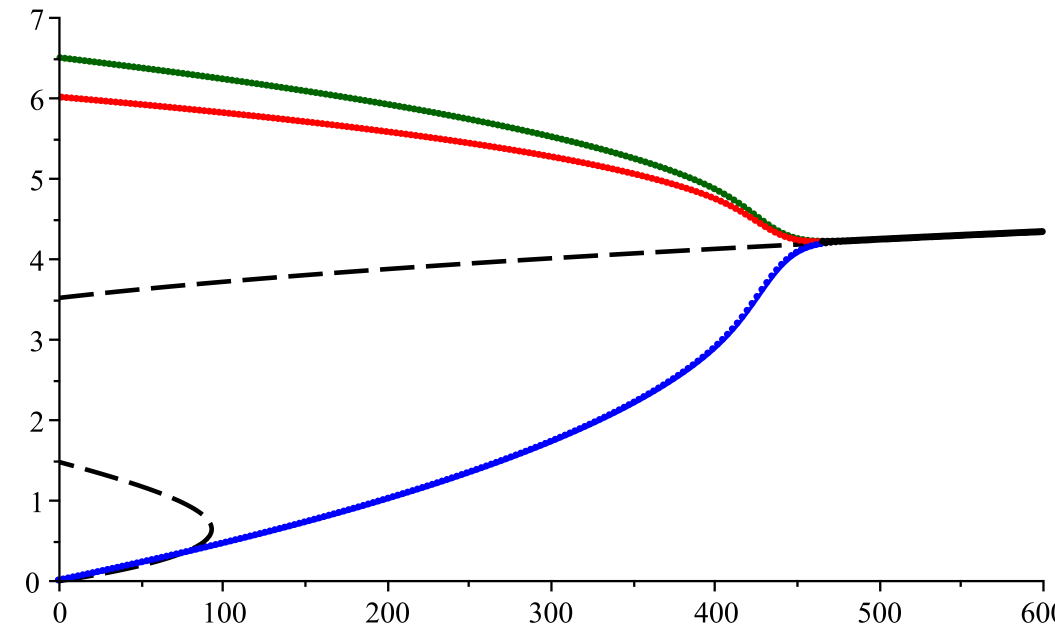


$\kappa = \frac{1}{5}$

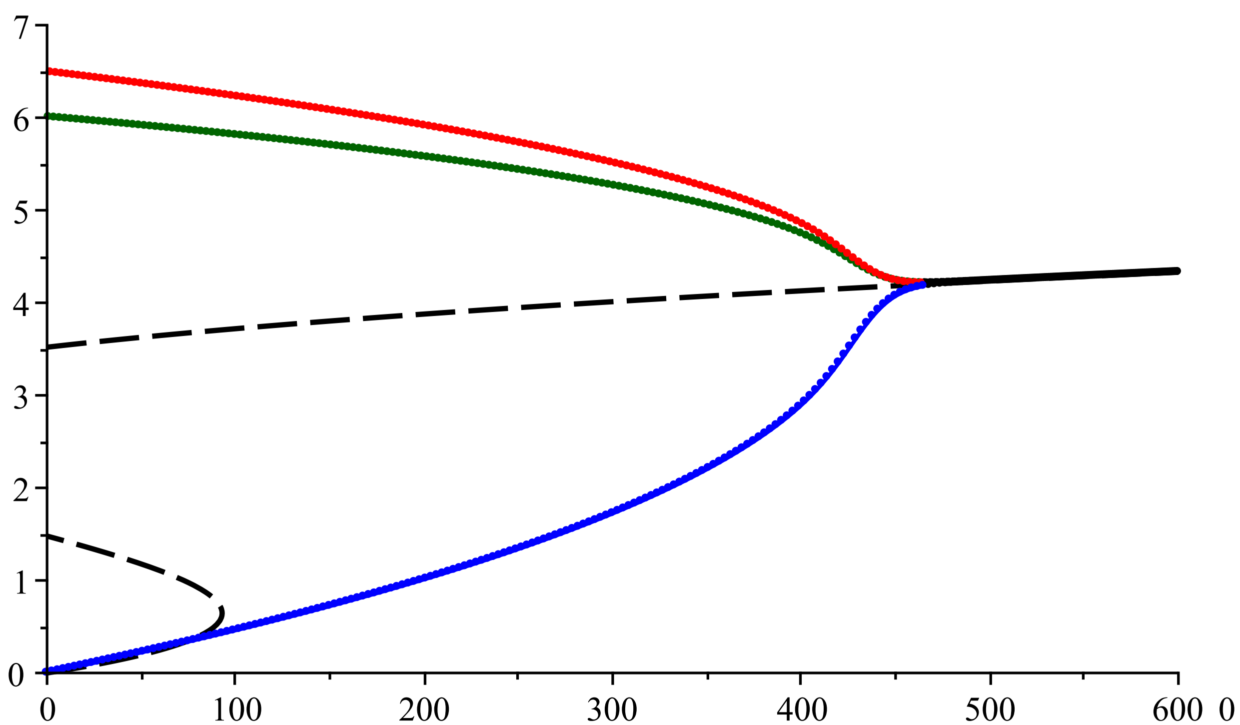
Evolution of $\left|\kappa-\frac{1}{4}\right| \leq \epsilon$ and $\tau = 25$



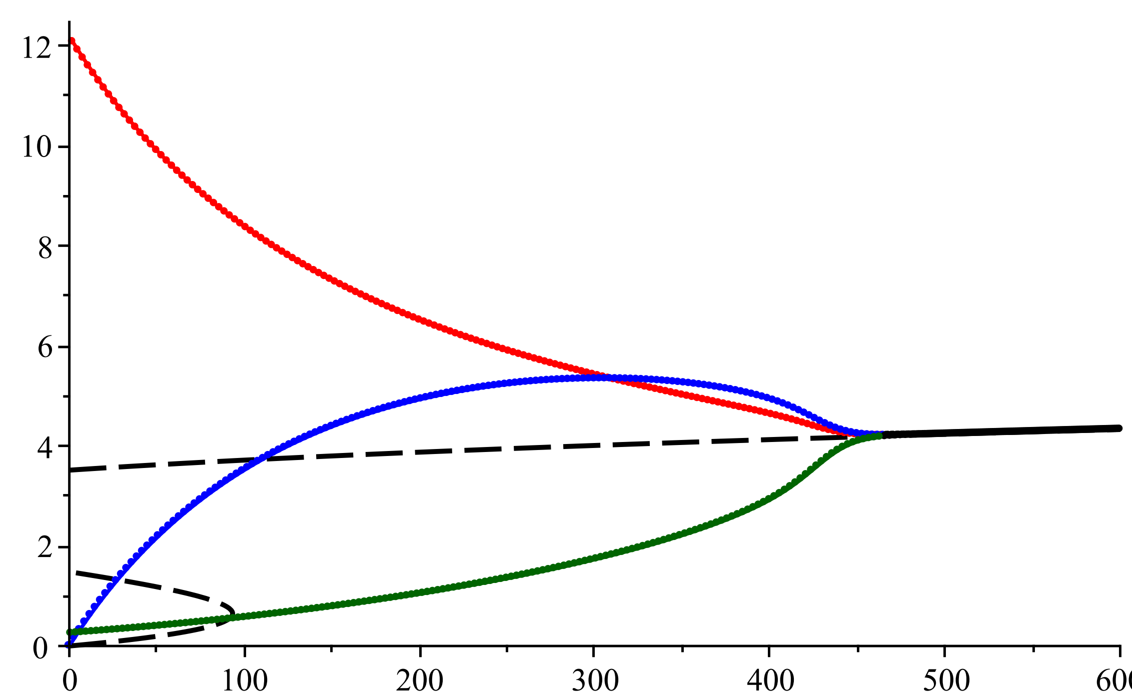
$\kappa = 0.249$



$\kappa = 0.24999$



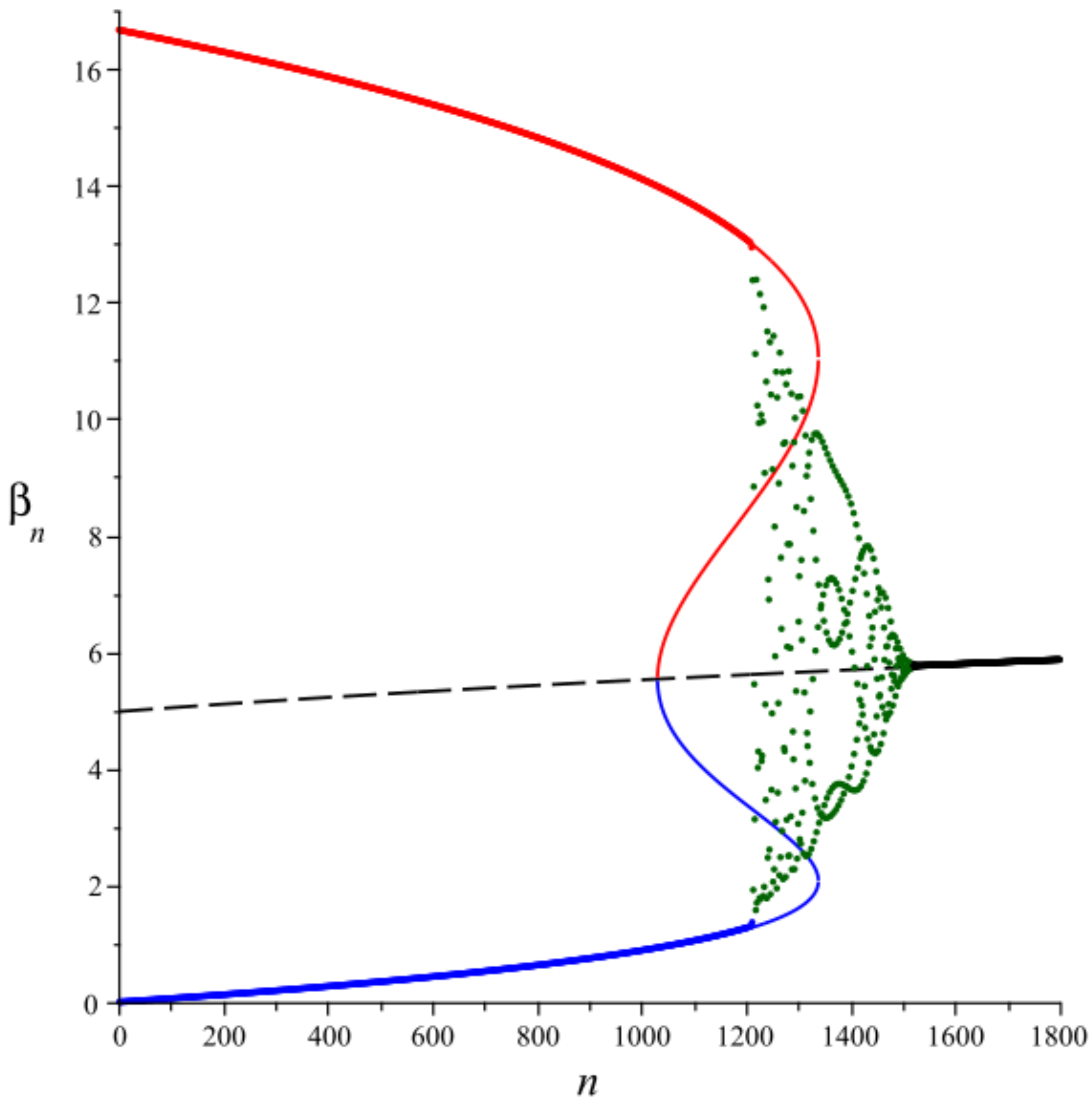
$\kappa = 0.25001$



$\kappa = 0.2505$

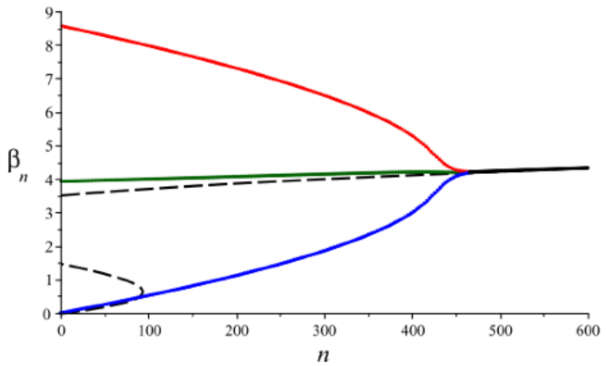
Evolution of $0 \leq \kappa \leq \frac{1}{4} + \epsilon$ and $\tau = 25$

$\kappa = 0$ and $\tau = 25$

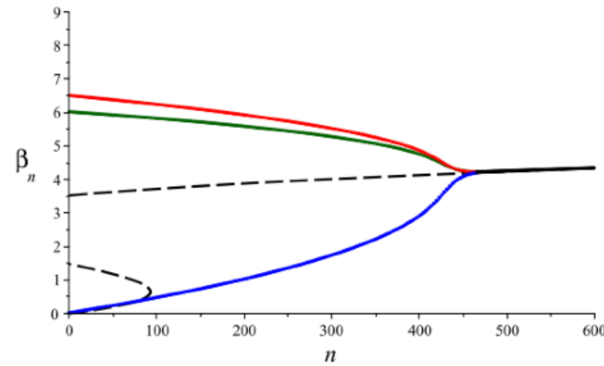


Case (i)-(iii): $\tau > 0$ and $\left|\kappa - \frac{1}{4}\right| < \epsilon$ "three-branch case"

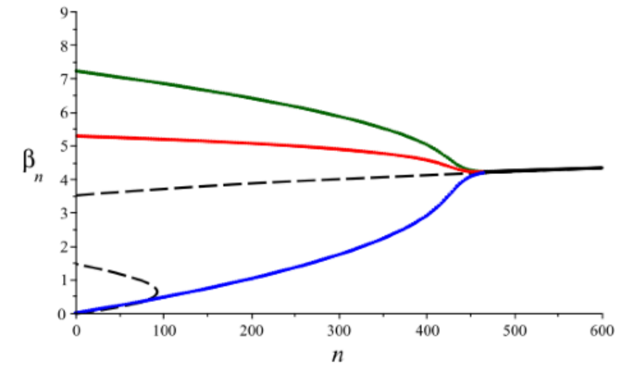
$$\tau = 25, \quad 0.2499 \le \kappa \le 0.2501, \quad \beta_{3n}, \ \beta_{3n+1}, \ \beta_{3n+2}$$



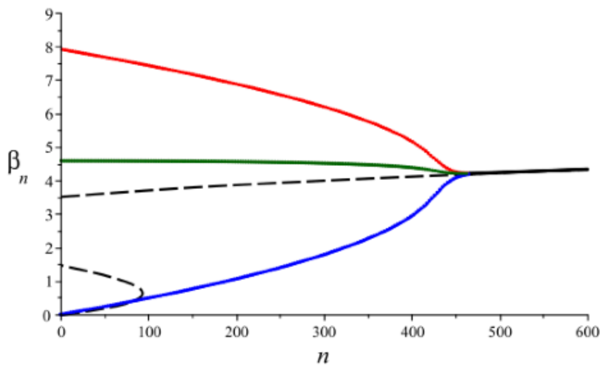
$\kappa = 0.2501$



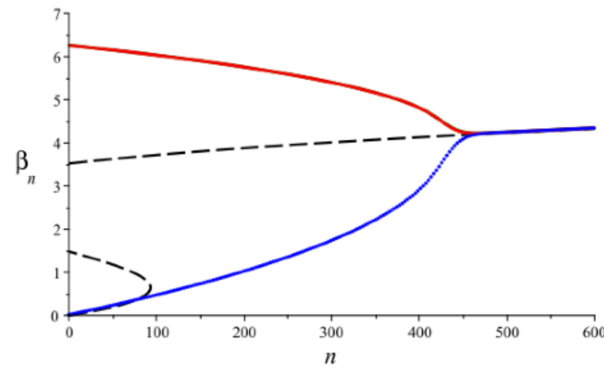
$\kappa = 0.25001$



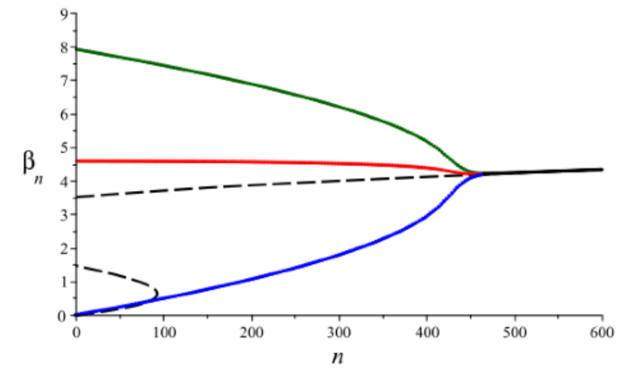
$\kappa = 0.24996$



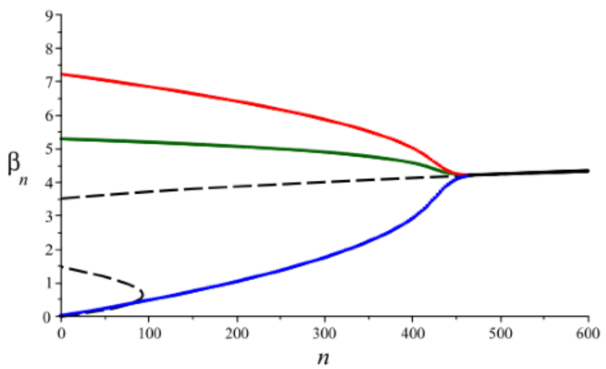
$\kappa = 0.25007$



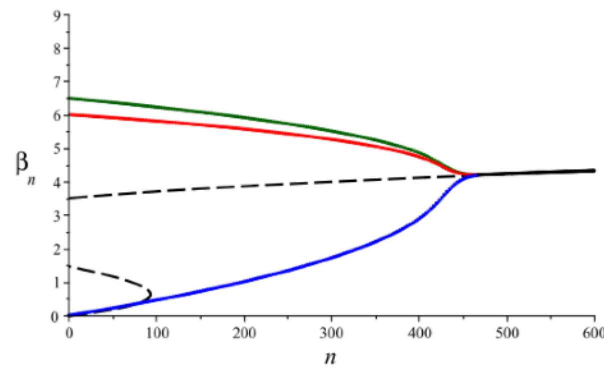
$\kappa = 0.25$



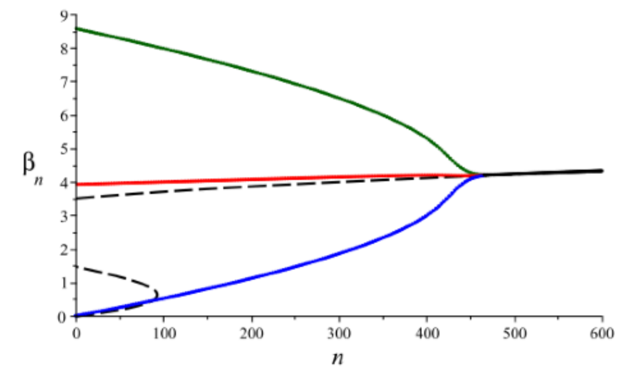
$\kappa = 0.24993$



$\kappa = 0.25004$

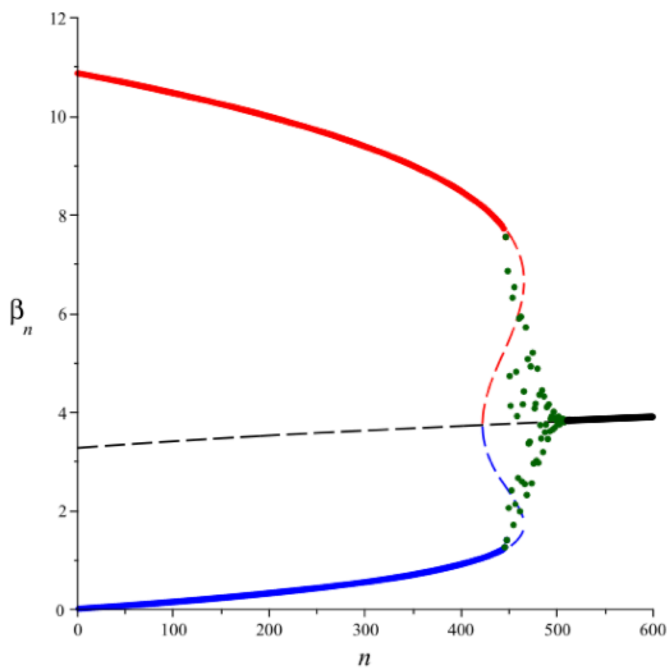


$\kappa = 0.24999$

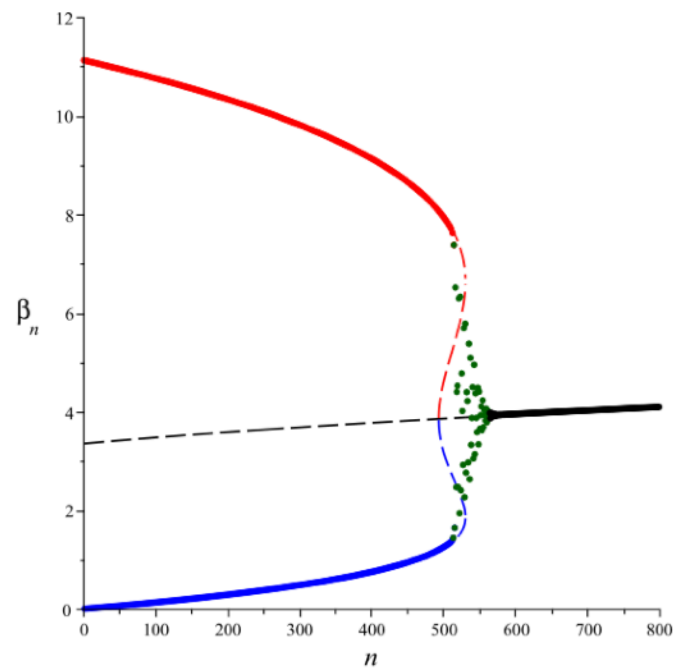


$\kappa = 0.2499$

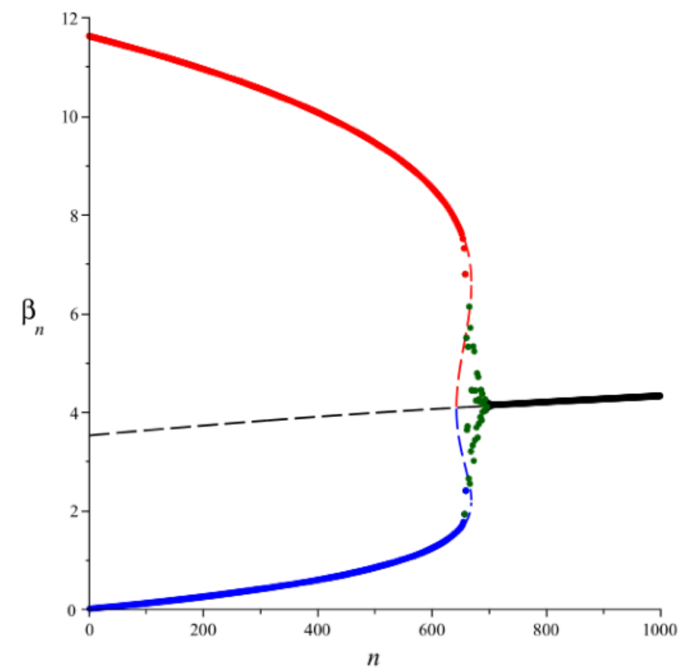
Case (v): $\kappa < 0$ and $\tau > 0$. Critical value $\kappa = -\frac{2}{3}$



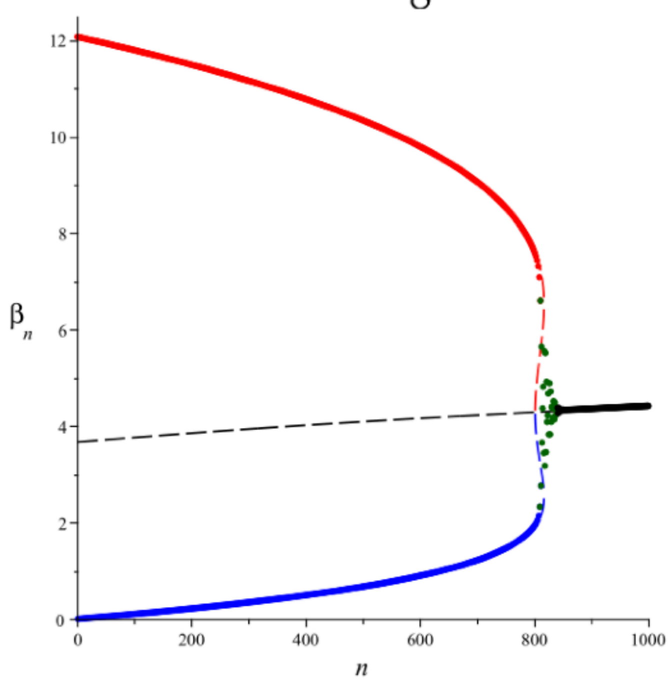
$\kappa = -\frac{1}{8}$



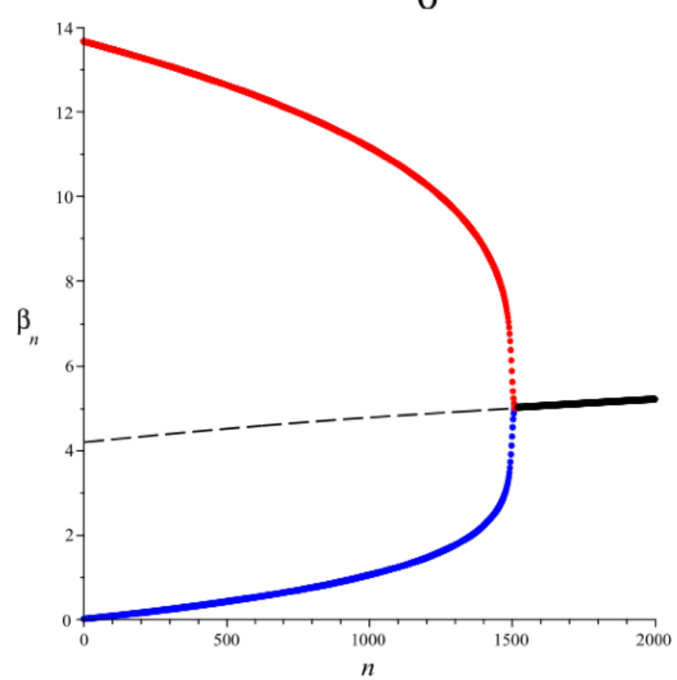
$\kappa = -\frac{1}{6}$



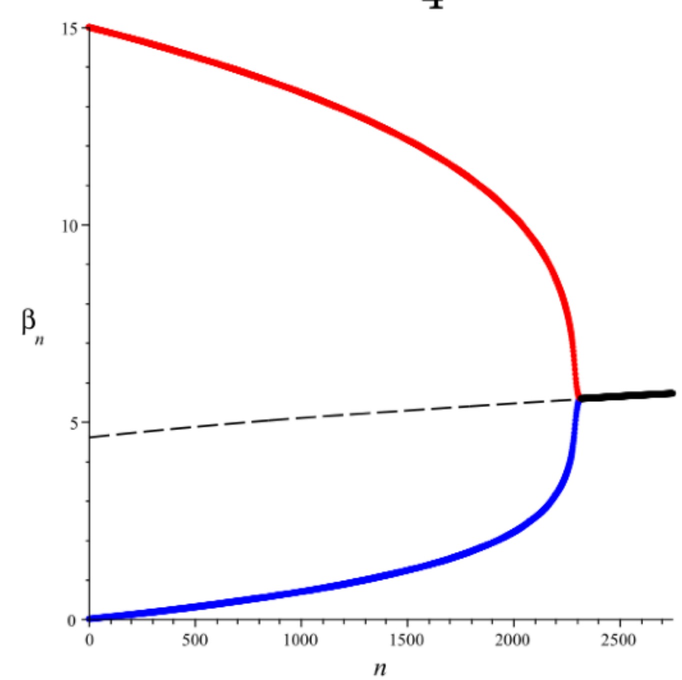
$\kappa = -\frac{1}{4}$



$\kappa = -\frac{1}{3}$



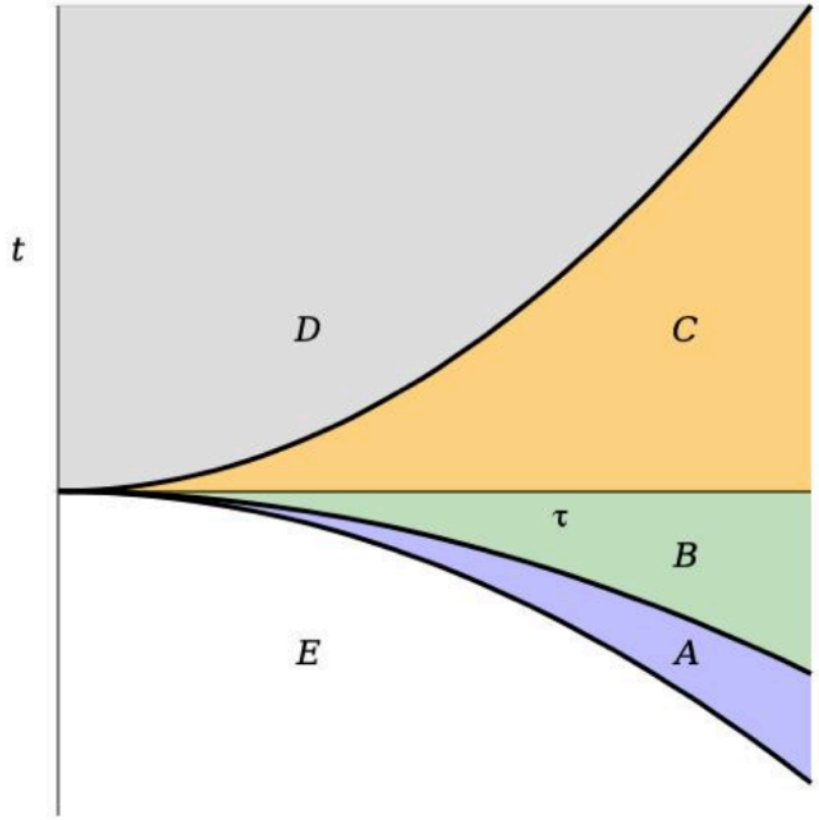
$\kappa = -\frac{2}{3}$



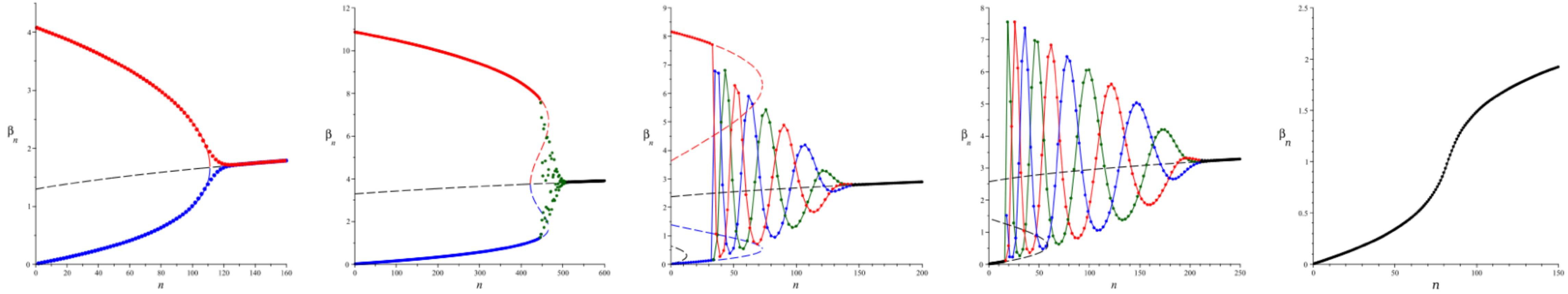
$\kappa = -1$

Summary

$$U(x) = x^6 - \tau x^4 - t x^2, \qquad \kappa = -t/\tau^2$$



$$\begin{aligned}
A:&\quad \tau > 0, \quad \tfrac{1}{4} < \kappa < \tfrac{2}{5}\\
B:&\quad \tau > 0, \quad 0 < \kappa < \tfrac{1}{4}\\
C:&\quad \tau > 0, \quad -\tfrac{2}{3} < \kappa < 0\\
D:&\quad \tau > 0, \quad \kappa < -\tfrac{2}{3}\\
E:&\quad \tau > 0, \quad \kappa > \tfrac{2}{5}
\end{aligned}$$



$k < -\frac{2}{3}$ $\quad -\frac{2}{3} < \kappa < 0$ $\quad 0 < \kappa < \frac{1}{4}$ $\quad \frac{1}{4} < \kappa < \frac{2}{5}$ $\quad \kappa > \frac{2}{5}$

Some references

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Thanks!