zeta Mahler functions

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Zeta Mahler functions

For a Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}] \setminus \{0\}$, define the *zeta Mahler function* (ZMF) as

$$Z(P;s) := \frac{1}{(2\pi i)^r} \int_{\mathbb{T}^r} |P(x_1, \ldots, x_r)|^s \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r}, \tag{1}$$

where s is a complex parameter and
$\mathbb{T}^r = \{(x_1, \ldots, x_r) \in \mathbb{C}^r : |x_1| = \cdots = |x_r| = 1\}$.

$Z(P;s)$ is the "arithmetic average" of $|P(x_1, \ldots, x_r)|^s$ over the r-dimensional torus.

$Z(P;s)$ converges absolutely in a certain half-plane $\mathrm{Re}(s) > s_0$ for some $s_0 < 0$.
Introduced in 2009 by Hirotaka Akatsuka under the name *zeta Mahler measure*.

Relation to the Mahler Measure

The *logarithmic Mahler measure* is defined as

$$\mathrm{m}(P) := \frac{1}{(2\pi i)^r} \int_{\mathbb{T}^r} \log |P(x_1, \dots, x_r)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_r}{x_r}. \tag{2}$$

We have the following relation

$$\mathrm{m}(P) = \frac{\mathrm{d}Z(P; s)}{\mathrm{d}s}\Big|_{s=0}. \tag{3}$$

If $P = a(x_1 - \alpha_1) \cdots (x_1 - \alpha_d) \in \mathbb{C}[x_1]$, this value coincides with

$$\log |a| + \sum_{j=1}^{d} \log \max\{1, |\alpha_j|\}.$$

Conjecture (Lehmer)

For all $P \in \mathbb{Z}[x_1]$ monic with $\mathrm{m}(P) > 0$, there exists an (absolute) constant $\epsilon > 0$ such that $\mathrm{m}(P) > \epsilon$.

Relation to the Mahler Measure

Examples (Smyth, 1981):

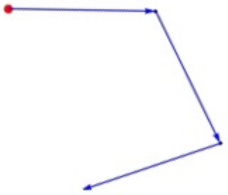
$$\mathrm{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) \quad \text{and} \quad \mathrm{m}(1+x+y+z) = \frac{7\zeta(3)}{2\pi^2},$$

(where $\chi_{-3} = \left(\frac{-3}{n}\right)$ is the quadratic character modulo 3).

Probabilistic interpretation of $Z(P;s)$

$Z(P;s)$ is the s-th moment of the random variable $|P(X_1,\ldots,X_r)|$, where $X_1,\ldots,X_r$ are independent and uniformly distributed random variables on the complex unit circle $\{z \in \mathbb{C}\colon |z| = 1\}$.

Example: $P(x,y) = 1 + x + y$.



What is the expected distance to the origin: $Z(P;1)$?
What is its variance: $Z(P;2) - Z(P;1)^2$?

The family W_r

We consider the Zeta Mahler functions for the Laurent polynomials $k + (x_1 + x_1^{-1}) \cdots (x_r + x_r^{-1})$ for real k. We denote this function by $W_r(k; s)$,

$$W_r(k; s) = \frac{1}{(2\pi i)^r} \int_{\mathbb{T}^r} \left| k + (x_1 + x_1^{-1}) \cdots (x_r + x_r^{-1}) \right|^s \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_r}{x_r}.$$

Note that $W_r(|k|; s) = W_r(k; s)$, so we will assume $k \geq 0$.

Define $p_r(k; -)$ to be the probability density function of the random variable $|k + (X_1 + X_1^{-1}) \cdots (X_r + X_r^{-1})|$.

We can relate p_r to W_r via

$$W_r(k; s) = \int_0^\infty x^s p_r(k; x) \, \mathrm{d}x.$$

W_1 for $k \geq 2$

In this case, the probability density is given by

$$p_1(k;x) = \frac{1}{2\pi} \frac{1}{\sqrt{1 - \frac{(x-k)^2}{4}}},$$

with the support on $(k-2, k+2)$.

Therefore,

$$W_1(k;s) = \int_{k-2}^{k+2} x^s p_1(k;x)\mathrm{d}x = \frac{1}{2\pi} \int_{k-2}^{k+2} \frac{x^s \mathrm{d}x}{\sqrt{1 - \frac{(x-k)^2}{4}}}.$$

W_1 for $k \geq 2$

This expression simplifies to

$$W_1(k;s) = k^s \cdot {}_2F_1\left(\frac{-s}{2}, \frac{1-s}{2}; 1; \frac{4}{k^2}\right).$$

When $k = 2$, this reduces further to

$$W_1(k;s) = \frac{2^s \Gamma(\frac{1}{2}+s)}{\Gamma(1+\frac{s}{2})\Gamma(\frac{1+s}{2})}.$$

W_1 for $k < 2$

In this case, the probability density is given by

$$p_1(k;x) = \begin{cases} \frac{1}{2\pi}\frac{1}{\sqrt{1-\frac{(x-k)^2}{4}}} & \text{for } 2-k \leq x < 2+k, \\ \frac{1}{2\pi}\frac{1}{\sqrt{1-\frac{(x-k)^2}{4}}} + \frac{1}{2\pi}\frac{1}{\sqrt{1-\frac{(x+k)^2}{4}}} & \text{for } 0 \leq x < 2-k, \\ 0 & \text{for } x < 0 \text{ or } x \geq 2+k. \end{cases}$$

A similar computation shows

$$\begin{aligned} W_1(k;s) &= \int_0^{k+2} x^s p_1(k;x)\,\mathrm{d}x \\ &= \frac{4^s\Gamma(\frac{1+s}{2})^2}{\pi\Gamma(1+s)} \cdot {}_2F_1\left(\frac{-s}{2}, \frac{-s}{2}; \frac{1}{2}; \frac{k^2}{4}\right). \end{aligned}$$

Functional equations for W_1

Using Euler's transformation formula

$$ {}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} {}_2F_1(c - a, c - b; c; z), $$

we find:

Theorem

1. *For $k > 2$ and $s \in \mathbb{C}$, we have*

$$ W_1(k; -s - 1) = (k^2 - 4)^{-s-\frac{1}{2}} W_1(k; s). $$

2. *For $0 \le k < 2$ and $-1 < \mathrm{Re}(s) < 0$, we have*

$$ W_1(k; -s - 1) = \cot\left(-\frac{\pi s}{2}\right) (4 - k^2)^{-s-\frac{1}{2}} W_1(k; s). $$

Zeros of W_1

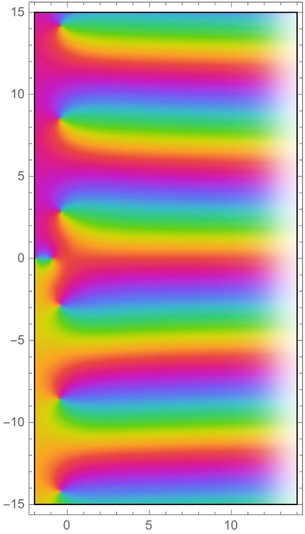


Figure: Here is a picture of $W_1(1; s)$

RH-type result for W_1

Theorem (R.,2020)

For any real k, all the non-trivial zeros of $W_1(k;s)$ lie on the critical line $\mathrm{Re}(s) = -\frac{1}{2}$.

Proof of the Theorem

For $\lambda \in \mathbb{C}$ and $(\alpha, \beta) \in \mathbb{C}^2$, we consider the *Jacobi function*

$$\phi_\lambda^{(\alpha,\beta)}(t) := (\cosh t)^{-\alpha-\beta-1-i\lambda} \times {}_2F_1\left(\frac{1}{2}(\alpha+\beta+1+i\lambda), \frac{1}{2}(\alpha-\beta+1+i\lambda); \alpha+1; \tanh^2 t\right),$$

for $t \in \mathbb{R}$.

Further, let

$$\Delta_{\alpha,\beta}(t) := (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}.$$

For brevity we write ϕ_λ for $\phi_\lambda^{(\alpha,\beta)}$, similarly for Δ.

ϕ_λ satisfies the differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + \frac{\Delta'(t)}{\Delta(t)}\frac{\mathrm{d}f}{\mathrm{d}t} + \left(\lambda^2 + (\alpha+\beta+1)^2\right) f = 0.$$

Proof of the Theorem

Lemma

For any $x > 0$, $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \pm\mu$ and $\alpha, \beta \in \mathbb{R}$,

$$\int_0^x \phi_\lambda(t)\phi_\mu(t)\Delta(t)\,\mathrm{d}t = (\mu^2 - \lambda^2)^{-1}\Delta(x)\left(\phi'_\lambda(x)\phi_\mu(x) - \phi_\lambda(x)\phi'_\mu(x)\right).$$

Lemma

For any $x > 0$, if $\phi_\lambda(x) = 0$ then $\lambda \in \mathbb{R} \cup i\mathbb{R}$.

Apply these lemmas to $(\alpha, \beta) = (-\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. The Theorem is an immediate consequence.

W_r for $k \geq 2^r$

Theorem (R.,2020)

Let $r \geq 1$.

(i) *For $k > 2^r$ and all $s \in \mathbb{C}$,*

$$W_r(k;s) = k^s \cdot {}_{r+1}F_r\left(\frac{-s}{2}, \frac{1-s}{2}, \frac{1}{2}, \ldots, \frac{1}{2}; 1, \ldots, 1; \frac{4^r}{k^2}\right).$$

(ii) *For $k = 2^r$ and $\mathrm{Re}(s) > -\frac{r}{2}$,*

$$W_r(k;s) = 2^{rs} \cdot {}_{r+1}F_r\left(\frac{-s}{2}, \frac{1-s}{2}, \frac{1}{2}, \ldots, \frac{1}{2}; 1, \ldots, 1; 1\right).$$

Proof

$$\begin{aligned} W_r(k;s) &= \frac{1}{(2\pi i)^r}\int_{\mathbb{T}^r}\left(k+(x_1+x_1^{-1})\cdots(x_r+x_r^{-1})\right)^s\frac{\mathrm{d}x_1}{x_1}\ldots\frac{\mathrm{d}x_r}{x_r}\\ &= \frac{k^s}{(2\pi i)^r}\int_{\mathbb{T}^r}\sum_{n\geq 0}\frac{1}{k^n}\binom{s}{n}(x_1+x_1^{-1})^n\cdots(x_r+x_r^{-1})^n\frac{\mathrm{d}x_1}{x_1}\ldots\frac{\mathrm{d}x_r}{x_r}. \end{aligned}$$

We have,

$$\frac{1}{(2\pi i)^r}\int_{\mathbb{T}^r}(x_1+x_1^{-1})^n\cdots(x_r+x_r^{-1})^n=\begin{cases}\binom{n}{n/2}^r & \text{if } n \text{ is even,}\\ 0 & \text{otherwise.}\end{cases}$$

Proof

Thus,

$$W_r(k;s) = k^s \sum_{n\geq 0} \binom{s}{2n}\binom{2n}{n}^r \frac{1}{k^{2n}}$$
$$= k^s \cdot {}_{r+1}F_r\left(\frac{-s}{2}, \frac{1-s}{2}, \frac{1}{2}, \ldots, \frac{1}{2}; 1, \ldots, 1; \frac{4^r}{k^2}\right).$$

W_2 for $0 \le k < 4$

Theorem (R.,2020)

For $0 \le k < 4$ *and* $\mathrm{Re}(s) > -1$ *and* s *not an odd integer, we have*

$$W_2(k;s) = \frac{1}{2\pi}\frac{\tan(\frac{\pi s}{2})}{s+1}k^{1+s} \cdot {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1+\frac{s}{2},\frac{3}{2}+\frac{s}{2};\frac{k^2}{16}\right)$$
$$+\frac{\Gamma(s+1)^2}{\Gamma(\frac{s}{2}+1)^4} \cdot {}_3F_2\left(\frac{-s}{2},\frac{-s}{2},\frac{-s}{2};\frac{1-s}{2},\frac{1}{2};\frac{k^2}{16}\right).$$

Theorem (R.,2020)

For odd positive integers n and $0 \leq k < 4$,

$$W_2(k;n) = (-1)^{\frac{n+1}{2}} \frac{2^n n!}{\pi^3} \cdot G_{3,3}^{2,3}\left(1+\frac{n}{2}, 1+\frac{n}{2}, 1+\frac{n}{2}; 0, \frac{n+1}{2}, \frac{1}{2}; \frac{k^2}{16}\right).$$

Where

$$G_{p,q}^{m,n}(a_1, \ldots, a_p; b_1, \ldots, b_q; z)$$

is the *Meijer G-function*.

Mahler measure of $k + (x_1 + x_1^{-1})(x_2 + x_2^{-1})$

We compute,

$$\left.\frac{\mathrm{d}W_2(k;s)}{\mathrm{d}s}\right|_{s=0} = \frac{k}{4}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};\frac{k^2}{16}\right).$$

Corollary

For $0 \le k < 4$,

$$\mathrm{m}(k + (x_1 + x_1^{-1})(x_2 + x_2^{-1})) = \frac{k}{4}\,{}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};\frac{k^2}{16}\right).$$

W_3 for $0 \le k < 8$

Theorem (R.,2020)

For $0 \le k < 8$ and $\mathrm{Re}(s) > -1$, s not an odd positive integer, we have

$$
\begin{aligned}
W_3(k;s) =& \frac{\Gamma(1+s)^3}{\Gamma(1+\frac{s}{2})^6} \cdot {}_4F_3\left(\frac{-s}{2}, \frac{-s}{2}, \frac{-s}{2}, \frac{-s}{2}; \frac{1-s}{2}, \frac{1-s}{2}, \frac{1}{2}; \frac{k^2}{64}\right) \\
&- \frac{\tan(\frac{\pi s}{2})^2}{4\pi(1+s)} k^{1+s} \cdot {}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, 1+\frac{s}{2}, \frac{3+s}{2}; \frac{k^2}{64}\right) \\
&+ \frac{4^s \tan(\frac{\pi s}{2}) \Gamma(s+1)}{\pi^{7/2}} \\
&\times G^{2,4}_{4,4}\left(\frac{2+s}{2}, \frac{2+s}{2}, \frac{2+s}{2}, \frac{2+s}{2}; \frac{1+s}{2}, \frac{1+s}{2}, 0, \frac{1}{2}; \frac{k^2}{64}\right).
\end{aligned}
$$

Mahler measure of $k + (x_1 + x_1^{-1})(x_2 + x_2^{-1})(x_3 + x_3^{-1})$

Corollary

For $0 \leq k < 8$,

$$\mathrm{m}(k + (x_1 + x_1^{-1})(x_2 + x_2^{-1})(x_3 + x_3^{-1})) = \frac{1}{2\pi^{5/2}} G_{4,4}^{2,4}\left(1, 1, 1, 1; \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}; \frac{k^2}{64}\right).$$

We may also write this Mahler measure as a triple integral for $0 < k < 8$,

$$\mathrm{m}(k + (x_1 + x_1^{-1})(x_2 + x_2^{-1})(x_3 + x_3^{-1})) = \frac{k}{16\pi^2} \int_{[0,1]^3} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\sqrt{xyz(1-x)(1-y)(1-x+\frac{k^2}{64}xyz)}}.$$

Concluding remarks / HOPEs

- What can be said about the location of the zeros of $W_r(k;s)$ for general r?
- The zeros of $W_2(X;n)$ for integers n ($k>4$) seem to be purely imaginary.
- What can be said about the arithmetic of $W_r(k;n)$ when n is a positive odd integer (at integer values $k<2^r$)?

  For example: $W_1(1;n)\in\mathbb{Q}+\frac{\sqrt{3}}{\pi}\mathbb{Q}$ for odd integers n.

  What can be said about the values $W_2(k;n)$

$$=(-1)^{\frac{n+1}{2}}\frac{2^n n!}{\pi^3}\cdot G_{3,3}^{2,3}\left(1+\frac{n}{2},1+\frac{n}{2},1+\frac{n}{2};0,\frac{n+1}{2},\frac{1}{2};\frac{k^2}{16}\right)$$

  for fixed odd integers n and (integer) values of k?