Elliptic hypergeometric functions and the Ruijsenaars model

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Overview

1. q-Difference operators
2. Integral operators
3. Commutativity

Theta functions and elliptic functions

Realize complex torus multiplicatively as

$$\mathbb{C}_{\neq 0}/\{x = px\}, \qquad |p| < 1.$$

An elliptic function is meromorphic for $x \neq 0$ and satisfies $f(px) = f(x)$. Equivalently,

$$f(x) = C\frac{\theta(a_1x;p)\cdots\theta(a_nx;p)}{\theta(b_1x;p)\cdots\theta(b_nx;p)},$$

where

$$\theta(x;p) = \prod_{j=0}^{\infty}(1-p^jx)(1-p^{j+1}/x),$$

$$a_1\cdots a_n = b_1\cdots b_n.$$

Throughout we use compact notation like

$$\theta(a_1x,\dots,a_nx) = \theta(a_1x;p)\cdots\theta(a_nx;p).$$

Ruijsenaars operators

Ruijsenaars operators

$$(D^{(k)}f)(x_1,\dots,x_n) = \sum_{\substack{I\subseteq\{1,\dots,n\},\\ |I|=k}} \prod_{i\in I,\, j\in I^c} \frac{\theta(tx_i/x_j;p)}{\theta(x_i/x_j;p)} \cdot f(x_1,\cdots,\underbrace{qx_i}_{i\in I},\dots,x_n).$$

The case $p=0$, $k=1$ is the A-type Macdonald operator.

Commutativity (Ruijsenaars 1987)

$$[D^{(k)},D^{(l)}]=0, \qquad k,l=0,1,\dots,n.$$

Defines integrable system of relativistic quantum particles.
Generalizes various Calogero–Moser–Sutherland-type models.

Noumi–Sano operators

Notation

$$(a;q,p)_k = \theta(a;p)\theta(aq;p)\cdots\theta(aq^{k-1};p).$$

Sano (2008) and Noumi–Sano (2021) introduced the q-difference operators

$$\begin{aligned}&(H^{(k)}f)(x_1,\dots,x_n)\\&=\sum_{\substack{\mu_1,\dots,\mu_n\geq 0\\ \mu_1+\dots+\mu_n=k}}\prod_{1\leq i<j\leq n}\frac{q^{\mu_j}\theta(q^{\mu_i-\mu_j}x_i/x_j;p)}{\theta(x_i/x_j;p)}\prod_{i,j=1}^{n}\frac{(tx_i/x_j;q,p)_{\mu_i}}{(qx_i/x_j;q,p)_{\mu_i}}\\&\qquad\times f(q^{\mu_1}x_1,\dots,q^{\mu_n}x_n).\end{aligned}$$

The coefficients are typical for elliptic hypergeometric series on the (affine) root system A_{n-1}.

Ruijsenaars and Noumi–Sano

The Noumi–Sano operators are polynomials in the Ruijsenaars operators and vice versa.

Analogous to relation between elementary and complete homogeneous symmetric functions.

Follows that

$$[H^{(k)}, H^{(l)}] = [H^{(k)}, D^{(l)}] = 0.$$

LSW transformation

Hallnäs–Langmann–Noumi–R. (2022) observed that $[H^{(k)}, H^{(l)}] = 0$ is equivalent to the Langer–Schlosser–Warnaar (2009) elliptic hypergeometric transformation

$$\sum_{\substack{k_1,\ldots,k_n\geq 0,\\ k_1+\cdots+k_n=N}} \prod_{i,j=1}^{n} \frac{(qx_i/x_j)_{k_i-k_j}(tx_i/x_j, x_iy_j)_{k_i}}{(tx_i/x_j)_{k_i-k_j}(qx_i/x_j, qx_iy_j/t)_{k_i}}$$

$$= \sum_{\substack{k_1,\ldots,k_n\geq 0,\\ k_1+\cdots+k_n=N}} \prod_{i,j=1}^{n} \frac{(qy_i/y_j)_{k_i-k_j}(ty_i/y_j, y_ix_j)_{k_i}}{(ty_i/y_j)_{k_i-k_j}(qy_i/y_j, qy_ix_j/t)_{k_i}},$$

where

$$(a_1,\ldots,a_m)_k = (a_1;q,p)_k \cdots (a_m;q,p)_k.$$

Kernel functions

$K = K(\mathbf{x};\mathbf{y})$ is a kernel function for the operator A if

$$A_{\mathbf{x}}K = A_{\mathbf{y}}K.$$

In a nice setting, this means that

$$K(\mathbf{x};\mathbf{y}) = \sum_k a_k\, e_k(\mathbf{x}) e_k(\mathbf{y}),$$

where a_k are arbitrary scalars and e_k eigenfunctions of A.

Kernel functions for the Ruijsenaars operators

It is hard to find a complete system of eigenfunctions for the Ruijsenaars operators. However, they have explicit kernel functions (Ruijsenaars, 2006):

$$K_u(\mathbf{x};\mathbf{y}) = \prod_{i,j=1}^{n} \frac{\Gamma(ux_iy_j;p,q)}{\Gamma(tux_iy_j;p,q)},$$

where

$$\Gamma(x;p,q) = \prod_{j,k=0}^{\infty} \frac{1-p^{j+1}q^{k+1}/x}{1-p^jq^kx}.$$

Note that

$$(a;q,p)_k = \frac{\Gamma(aq^k;p,q)}{\Gamma(a;p,q)}.$$

Integral operators

If A is symmetric with respect to $d\mu$ and

$$(If)(\mathbf{x}) = \int f(\mathbf{y})K(\mathbf{x};\mathbf{y})\,d\mu(\mathbf{y}),$$

then at least formally

$$[A, I] = 0.$$

Starting with Ruijsenaars (2005), many people have used versions of such operators to study Calogero–Moser–Sutherland-type systems.

Following Belousov–Derkachov–Kharchev–Khoroshkin (many recent preprints) we call them Q-operators.

Q-operators for Ruijsenaars model

Ruijsenaars difference operators are formally symmetric with respect to the pairing

$$\int f(x_1,\ldots,x_n)g(x_1^{-1},\ldots,x_n^{-1}) \prod_{1\leq i\neq j\leq n} \frac{\Gamma(tx_i/x_j;p,q)}{\Gamma(x_i/x_j;p,q)} \frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

For this reason it is natural to look at integrals

$$\int f(\mathbf{x})K_u(\mathbf{x}^{-1};\mathbf{y}) \prod_{1\leq i\neq j\leq n} \frac{\Gamma(tx_i/x_j)}{\Gamma(x_i/x_j)} \frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.$$

Q-operators

Our version of Q-operators is

$$(Q_u f)(y_1,\dots,y_n) = \int_{\mathbf{x}\in\mathbb{T}_{\mathbf{y}}^{n-1}} f(x_1,\dots,x_n) \prod_{1\leq i\neq j\leq n} \frac{\Gamma(tx_i/x_j)}{\Gamma(x_i/x_j)} \prod_{i,j=1}^{n} \frac{\Gamma(uy_j/x_i)}{\Gamma(tuy_j/x_i)}\, |d\mathbf{x}|.$$

Here,

$$\mathbb{T}_{\mathbf{y}}^{n-1} = \{x\in\mathbb{C}^n;\, |x_1| = |x_2| = \cdots = |x_n|,\, x_1\cdots x_n = y_1\cdots y_n\},$$

$$|d\mathbf{x}| = \frac{dx_1}{2\pi \mathrm{i} x_1}\cdots\frac{dx_{n-1}}{2\pi \mathrm{i} x_{n-1}}.$$

We will not go into details about parameter conditions etc.

From Q-operators to Noumi–Sano operators

$$(Q_u f)(\mathbf{y}) = \int_{x\in\mathbb{T}^{n-1}_{\mathbf{y}}} f(\mathbf{x}) \prod_{1\leq i\neq j\leq n} \frac{\Gamma(tx_i/x_j)}{\Gamma(x_i/x_j)} \prod_{i,j=1}^{n} \frac{\Gamma(uy_j/x_i)}{\Gamma(tuy_j/x_i)} \, |d\mathbf{x}|.$$

The integrand has poles at $x_j = uq^{k_j} y_j$ where $k_j \in \mathbb{Z}_{\geq 0}$ (and lots of other poles).

Suppose this holds for all j. Since $x_1 \cdots x_n = y_1 \cdots y_n$, we must have $u = q^{-N/n}$, where $N = \sum_j k_j \in \mathbb{Z}_{\geq 0}$. The residue sum over these poles is essentially[1] the Noumi–Sano operator $H^{(N)}$.

[1] Up to a "change of gauge".

Commutativity

Natural to believe that all the operators D_k, H_k and Q_u and mutually commute.

The relation

$$[Q_u, Q_v] = 0$$

is not obvious!

Belousov, Derkachov, Kharchev and Khoroshkin prove a hyperbolic version in a recent preprint (2023).

We will consider $[Q_u, Q_v] = 0$ as an identity for formal integral operators, that is, the integral kernel of the commutator vanishes.

Two proofs

Together with Rains we found (I HOPE) two proofs.

- Clean hands approach. Uses formal eigenfunctions of Langmann, Noumi, Shiraishi (2022).
- Dirty hands approach. Proves conjecture of Gadde, Rastelli, Razamat and Yan (2010).

First proof: Clean hands

Work in the space of formal power series

$$V = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]^{\mathrm{S}_n}[[p]] \qquad \mathrm{mod} \quad x_1 \cdots x_n = C.$$

Note that any Laurent polynomial is a polynomial modulo $x_1 \cdots x_n = C$.

Expansion of integral kernel

We have

$$\Gamma(x;p,q)=\prod_{j,k=0}^{\infty}\frac{1-p^{j+1}q^{k+1}/x}{1-p^jq^kx}=\frac{1}{(x;q)_\infty}\sum_{k=0}^{\infty}\phi_k(x)p^k,$$

where $\phi_k\in\mathbb{C}[x^\pm]$.
Integral kernel of Q_u is

$$\prod_{1\leq i\neq j\leq n}\frac{\Gamma(tx_i/x_j)}{\Gamma(x_i/x_j)}\prod_{i,j=1}^{n}\frac{\Gamma(uy_j/x_i)}{\Gamma(tuy_j/x_i)}$$

$$=\underbrace{\prod_{1\leq i\neq j\leq n}\frac{(x_i/x_j;q)_\infty}{(tx_i/x_j;q)_\infty}}_{\text{Macdonald weight function}}\underbrace{\prod_{i,j=1}^{n}\frac{(tuy_j/x_i;q)_\infty}{(uy_j/x_i;q)_\infty}}_{\text{Macdonald Cauchy kernel}}\times\underbrace{(\Phi_0+p\Phi_1+\cdots+)}_{\Phi_k\text{ are Laurent polynomials}}.$$

This implies that Q_u acts on V.

Elliptic Macdonald polynomials

Langmann, Noumi and Shiraishi (2022) showed that the Ruijsenaars operators have formal eigenfunctions

$$P_\lambda(\mathbf{x};p) = \sum_{k=0}^{\infty} P_\lambda^{(k)}(\mathbf{x}) p^k,$$

where $P_\lambda^{(k)} \in \mathbb{C}[x_1^\pm, \dots, x_n^\pm]^{\mathrm{S}_n}$.
The trigonometric case $P_\lambda^{(0)}$ are Macdonald polynomials.

Any element of V can be written

$$\sum_{\lambda_1 \geq \cdots \geq \lambda_n = 0} A_\lambda(p) P_\lambda(\mathbf{x};p),$$

with convergence as formal power series in p.

Conclusion of first proof

We can expand

$$Q_u P_\lambda(\mathbf{x};p) = \sum_\mu A_{\lambda\mu}(p) P_\mu(\mathbf{x};p).$$

Using that $[Q_u, D_k] = 0$ for all k we can conclude that $P_\lambda(\mathbf{x};p)$ are formal eigenfunctions of Q_u.

It follows that $[Q_u, Q_v] = 0$ on V. This is enough to conclude that $[Q_u, Q_v] = 0$ as formal integral operators.

Second proof: Dirty hands

Compute the commutator explicitly. We see that $[Q_u, Q_v] = 0$ is equivalent to (after a change of parameters)

$$\int_{x\in\mathbb{T}^{n-1}} \frac{\prod_{i=1}^{n}\prod_{j=1}^{2n}\Gamma(ay_jx_i)\Gamma(b/y_jx_i)}{\prod_{1\leq i\neq j\leq n}\Gamma(x_i/x_j, abx_i/x_j)}\,|d\mathbf{x}| = \int_{x\in\mathbb{T}^{n-1}} \frac{\prod_{i=1}^{n}\prod_{j=1}^{2n}\Gamma(by_jx_i)\Gamma(a/y_jx_i)}{\prod_{1\leq i\neq j\leq n}\Gamma(x_i/x_j, abx_i/x_j)}\,|d\mathbf{x}|,$$

where

$$x_1\cdots x_n = y_1\cdots y_{2n} = 1.$$

Conjectured by Gadde, Rastelli, Razamat and Yan (2010). Appeared from quantum field theory.

Known cases

$$\int_{x\in\mathbb{T}^{n-1}} \frac{\prod_{i=1}^{n}\prod_{j=1}^{2n}\Gamma(ay_jx_i)\Gamma(b/y_jx_i)}{\prod_{1\le i\neq j\le n}\Gamma(x_i/x_j, abx_i/x_j)}\,|d\mathbf{x}| = \int_{x\in\mathbb{T}^{n-1}} \frac{\prod_{i=1}^{n}\prod_{j=1}^{2n}\Gamma(by_jx_i)\Gamma(a/y_jx_i)}{\prod_{1\le i\neq j\le n}\Gamma(x_i/x_j, abx_i/x_j)}\,|d\mathbf{x}|,$$

$$x_1\cdots x_n = y_1\cdots y_{2n} = 1.$$

One-dimensional case ($n=2$) due to Van de Bult (2011).

Hyperbolic version proved by Belousov, Derkachov, Kharchev, Khoroshkin (arXiv 2023).

Sketch of proof of GRRY identity

Write the identity as

$$J(y_1,\dots,y_{2n};a,b)=J(y_1,\dots,y_{2n};b,a).$$

Prove that

$$\int_{\mathbf{T}^{2n-1}} J(\mathbf{y};a,b)\phi(\mathbf{y})\,|d\mathbf{y}| = \int_{\mathbf{T}^{2n-1}} J(\mathbf{y};b,a)\phi(\mathbf{y})|d\mathbf{y}|$$

for “enough” test functions ϕ.

Using an integral transformation of Rains (2010), we can prove this for

$$\phi(\mathbf{y})=\frac{\prod_{i=1}^{2n}\prod_{j=1}^{n}\Gamma(cz_j^{\pm}y_i,dw_j^{\pm}/y_i)}{\prod_{1\leq i\neq j\leq n}\Gamma(y_i/y_j)},$$

where $z_1,\dots,z_n,w_1,\dots,w_n$ are free parameters and $abc^2d^2=pq$.

Rains' integral transformation

We need the case $m = 2n$ of

$$\kappa_n \int_{\mathbb{T}^{n-1}} \frac{\prod_{i=1}^{n} \prod_{j=1}^{n+m} \Gamma(a_j x_i, b_j/x_i)}{\prod_{1\leq i\neq j\leq n} \Gamma(x_i/x_j)} |d\mathbf{x}|,$$

$$= \prod_{i,j=1}^{n+m} \Gamma(a_i b_j) \cdot \kappa_m \int_{\mathbb{T}^{m-1}} \frac{\prod_{i=1}^{m} \prod_{j=1}^{n+m} \Gamma(\nu x_i/a_j, \nu/x_i b_j)}{\prod_{1\leq i\neq j\leq m} \Gamma(x_i/x_j)} |d\mathbf{x}|,$$

where

$$\kappa_n = \frac{(p;p)_\infty^{n-1}(q;q)_\infty^{n-1}}{n!},$$

$$\nu^m = a_1 \cdots a_{m+n} = (pq)^m / b_1 \cdots b_{n+m}.$$

Do we have enough test functions?

To go from

$$\int_{\mathbf{T}^{2n-1}} J(\mathbf{y};a,b)\phi(\mathbf{y})\,|d\mathbf{y}| = \int_{\mathbf{T}^{2n-1}} J(\mathbf{y};b,a)\phi(\mathbf{y})|d\mathbf{y}|$$

to

$$J(\mathbf{y};a,b) = J(\mathbf{y};b,a)$$

we look at the limit case $p, q \to 0$.

More precisely, we write $(a,b,c,d,\mathbf{z},\mathbf{w},p,q) \mapsto (r^2a, r^2b, rc\mathbf{z}, r^{-1}d^{-1}\mathbf{w}, r^4p, r^4q)$ and expand everything as formal power series in r.

We have enough test functions!

If

$$J(\mathbf{y}; a, b) - J(\mathbf{y}; b, a) = E(\mathbf{y}) r^N + \mathcal{O}(r^{N+1})$$

then E is a symmetric Laurent polynomial such that

$$\int_{\mathbb{T}^{2n-1}} E(\mathbf{y}) \frac{\prod_{1 \leq i < j \leq 2n} (y_i - y_j)(y_i^{-1} - y_j^{-1})}{\prod_{i=1}^{2n} \prod_{j=1}^{n} (1 - y_i/z_j)(1 - w_j/y_i)} |d\mathbf{y}| = 0,$$

where $|z_j| > 1$ and $|w_j| < 1$.

Using standard facts on Schur polynomials we can deduce that

$$E(\mathbf{y}) = 0.$$

Fokko van de Bult, Eric Rains, Ole Warnaar and Erik Koelink, Leiden, 2013.