On the multiplication of spherical functions of reductive spherical pairs of type A

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HOPE: Hypergeometric and Orthogonal Polynomials Event
Nijmegen, 1-3 May 2024

Let $K \subset G$ be connected reductive algebraic groups over $\mathbb{C}$.

Definition

G/K is called **spherical** *if $\mathbb{C}[G/K]$ is a multiplicity free G-module. Equivalently, (G, K) is called a* **reductive spherical pair***.*

EXAMPLES.

Symmetric varieties: $K = (G^{\vartheta})^{\circ}$ for some $\vartheta \in Inv(G)$, e.g.

$$SL_n/SO_n, \quad SL_{2n}/Sp_{2n}, \quad SO_n/SO_{n-1}$$

Non-symmetric exampes:

$$SL_{2n+1}/Sp_{2n}, \qquad SO_{2n+1}/GL_n, \qquad SO_8/G_2$$

We will assume that G is simple and simply connected.

The classification of the reductive spherical pairs with G simple goes back to Krämer (1979).

Let $X = G/K$ spherical. Consider the decomposition of G-modules

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda_X^+} E_X(\lambda)$$

with $E_X(\lambda) \simeq V(\lambda)$ irreducible of highest weight $\lambda \in \Lambda^+$

- Λ_X^+ is called the **weight monoid** of X.
- The **weight lattice** of X is the lattice Λ_X generated by Λ_X^+.

The weight monoid and weight lattice of X are well understood.

Problem

Given $\lambda, \mu \in \Lambda_X^+$, determine the decomposition of $E_X(\lambda) \cdot E_X(\mu)$: for which $\nu \in \Lambda_X^+$ does it hold $E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu)$?

Somehow, Λ_X^+ behaves as a monoid of dominant weights for a root datum (Φ_X, Λ_X) attached to X, which generalize the restricted root system of a symmetric variety.

We will be interested in the case where Φ_X is of type A.

The zonal spherical functions of a spherical pair

Let (G, K) be a reductive spherical pair, $X = G/K$. Then

$$\mathbb{C}[X] = \mathbb{C}[G]^K \simeq \bigoplus V(\lambda)^* \otimes V(\lambda)^K$$

Being multiplicity free amounts to the property

$$\dim V(\lambda)^K \leqslant 1 \quad \forall \lambda \in \Lambda^+$$

The weights in $\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda_X^+} E_X(\lambda)$ are given by

$$\Lambda_X^+ = \{\lambda \in \Lambda^+ \mid \dim V(\lambda)^K = 1\}$$

Fix nonzero elements $\varphi_\lambda \in E_X(\lambda)^K$, for all $\lambda \in \Lambda_X^+$: these are the **zonal spherical functions** of X. They form a basis of

$$\mathbb{C}[X]^K = \bigoplus_{\lambda \in \Lambda_X^+} E_X(\lambda)^K = \mathbb{C}[G]^{K \times K}$$

- Write $\varphi_\lambda \cdot \varphi_\mu = \sum a_{\lambda,\mu}^\nu \varphi_\nu$, then [Ruitenburg, 1989]

$$\boxed{E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff a_{\lambda,\mu}^\nu \neq 0}$$

Zonal spherical functions and Jacobi polynomials

Let $X = G/K$ symmetric, $K = G^{\vartheta}$. Fix in G a maximal *split torus* A, namely

$$\vartheta(a) = a^{-1} \quad \forall a \in A$$

Let $T \subset G$ be a maximal torus containing A, then T is ϑ-stable. Let $A_X = A/A \cap K$. Then $\Lambda_X^+ = \Lambda_X \cap \Lambda^+$ and

$$\Lambda_X = \{\chi - \vartheta(\chi) \ : \ \chi \in \Lambda\} \simeq \mathcal{X}(A_X).$$

- The **restricted root system** (possibly non-reduced) of X is

$$\widetilde{\Phi}_X = \{\alpha - \vartheta(\alpha) : \alpha \in \Phi, \alpha \neq \vartheta(\alpha)\}$$

  Its weight lattice is Λ_X, and we can choose a basis $\Delta_X \subset \widetilde{\Phi}_X$ so that Λ_X^+ identifies with the dominant weights of Φ_X.
- $\widetilde{\Phi}_X$ comes with a **multiplicity function**

$$m_X(\tilde{\alpha}) = |\{\alpha \in \Phi : \alpha - \vartheta(\alpha) = \tilde{\alpha}\}|$$

- Its Weyl group is the **little Weyl group**

$$W_X = N_K(A)/Z_K(A)$$

By a theorem of Richardson, restriction gives an isomorphism

$$\mathbb{C}[X]^K \xrightarrow{\sim} \mathbb{C}[A_X]^{W_X}$$

The image of the φ_λ is the **Jacobi polynomial** $P_\lambda^{(\underline{k})}$ associated to $\Phi_X \subset \mathcal{X}(A_X)$, evaluated at $\underline{k} = m_X/2$.

Problem. *How to decompose the product of two Jacobi polynomials?*

Assume that $\tilde{\Phi}_X$ of type A_n.

Conjecture (Stanley 1989)

Write $P_\lambda^{(1/k)} P_\mu^{(1/k)} = \sum b_{\lambda,\mu}^\nu(k) P_\nu^{(1/k)}$. Then $b_{\lambda,\mu}^\nu(k)$ is the ratio of two polynomials in k with non-negative integer coefficients.

Stanley's Pieri rule: the conjecture is true if $\widetilde{\Phi}_X$ is of type A_1.

Recall the decomposition $\varphi_\lambda \cdot \varphi_\mu = \sum a_{\lambda,\mu}^\nu \varphi_\nu$. Thus

$$E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \Longleftrightarrow a_{\lambda,\mu}^\nu \neq 0 \Longleftrightarrow b_{\lambda,\mu}^\nu(2/m_X) \neq 0$$

- Another important specialization of $P_\lambda^{(k)}$ is at $k = 1$:
  (slightly abusing, we identify partitions of length $\leqslant n$ and dominant weights for $\tilde{\Phi}_X$)
  up to scalars, $P_\lambda^{(1)}$ is (the image of) the Schur polynomial s_λ.

Write $s_\lambda \cdot s_\mu = \sum_\nu c_{\lambda,\mu}^\nu s_\nu$: then $c_{\lambda,\mu}^\nu \neq 0 \Longleftrightarrow b_{\lambda,\mu}^\nu(1) \neq 0$

Corollary

Suppose X is symmetric with $\widetilde{\Phi}_X$ of type A, and let $\lambda, \mu, \nu \in \Lambda_X^+$. Let G_X be the simple group with based root system $\Delta_X \subset \Phi_X$ and weight lattice Λ_X. If Stanley's conjecture is true, then

$$E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff V_X(\nu) \subset V_X(\lambda) \otimes V_X(\mu).$$

In particular, this is true if $\widetilde{\Phi}_X$ of type A_1.

In a slightly different form, considered by [Graham-Hunziker, 2009].

Table: Symmetric pairs with G simple and restricted root system of type A

G	K	$\widetilde{\Phi}_{G/K}$	$m_{G/K}$
SL_n, $n \geqslant 2$	SO_n	A_{n-1}	1
SL_{2n}, $n \geqslant 2$	Sp_{2n}	A_{n-1}	4
SO_n, $n \geqslant 5$	SO_{n-1}	A_1	$n-2$
E_6	F_4	A_2	8

The case of a spherical pair

Let now (G, K) be a reductive spherical pair, and set $X = G/K$. The **root monoid** of X is

$$\mathcal{M}_X = \langle \lambda + \mu - \nu \mid \lambda, \mu, \nu \in \Lambda_X^+, \;\; E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \rangle_{\mathbb{N}}$$

Theorem (Knop 1994 + Avdeev–Cupit-Foutou 2018)

$\mathcal{M}_X$ is a free monoid, and the set of free generators $\Delta_X \subset \mathcal{M}_X$ is the base of a (reduced) root system Φ_X.

In general, Λ_X is not contained in the weight lattice of Φ_X. However it is possible to define a map

$$\Phi_X \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda_X, \mathbb{Z}) \qquad \alpha \longmapsto \alpha^\vee$$

giving rise to a *based root datum*

$$\mathcal{R}_X = (\Lambda_X, \Delta_X, \Delta_X^\vee)$$

- In this way, the weight monoid Λ_X^+ gets identified with a submonoid of dominant weights for $\mathcal{R}_X$.
- If X is is symmetric, $\mathcal{R}_X$ is semisimple and Φ_X is the reduced root system associated to the restricted root system of X.

Let G_X be the reductive group defined by $\mathcal{R}_X$, then

$$\Lambda_X^+ \longrightarrow Irr(G_X) \qquad \lambda \longmapsto V_X(\lambda)$$

Recall the decomposition of G-modules

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda_X^+} E_X(\lambda), \qquad\qquad E_X(\lambda) \simeq V(\lambda)$$

Conjecture (Bravi-G.)

Recall that G is simple, and let $X = G/K$ with Φ_X of type A. If $\lambda, \mu, \nu \in \Lambda_X^+$, then

$$E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \Longleftrightarrow V_X(\nu) \subset V_X(\lambda) \otimes V_X(\mu)$$

Theorem (Bravi-G.)

- *If $X \neq \mathsf{F}_4/\mathsf{B}_4$, then the previous conjecture is a consequence of Stanley's conjecture.*
- *Suppose that Φ_X is direct sum of root systems of type A_1, and assume $X \neq \mathsf{F}_4/\mathsf{B}_4$. Then the previous conjecture is true.*

Table: The other reductive spherical pairs with G simple and Φ_X of type A

G	K	$\Phi_{G/K}$	Sym
SL_n, $n \geqslant 3$	SL_{n-1}	A_1	
SL_n, $n \geqslant 3$	GL_{n-1}	A_1	BC_1
SL_{2n+1}, $n \geqslant 2$	Sp_{2n}	$\mathsf{A}_n \times \mathsf{A}_{n-1}$	
SL_{2n+1}, $n \geqslant 2$	$\mathsf{GL}_1 \times \mathsf{Sp}_{2n}$	$\mathsf{A}_n \times \mathsf{A}_{n-1}$	
Sp_{2n}, $n \geqslant 2$	$\mathsf{GL}_1 \times \mathsf{Sp}_{2n-2}$	$\mathsf{A}_1 \times \mathsf{A}_1$	
Sp_{2n}, $n \geqslant 3$	$\mathsf{Sp}_2 \times \mathsf{Sp}_{2n-2}$	A_1	BC_1
Spin_7	G_2	A_1	
Spin_9	Spin_7	$\mathsf{A}_1 \times \mathsf{A}_1$	
Spin_8	G_2	$\mathsf{A}_1 \times \mathsf{A}_1 \times \mathsf{A}_1$	
F_4	Spin_9	A_1	BC_1
G_2	SL_3	A_1	

We do not have Jacobi polynomials in the non-symmetric setting. But we can reduce to the symmetric case!

- Several of the previous example correspond to exotic homogeneous actions on spheres:

$$G_2/SL_3 \simeq SO_7/SO_6$$

$$Spin_7/G_2 \simeq SO_8/SO_7$$

$$Spin_9/Spin_7 \simeq SO_{16}/SO_{15}$$

$$SL_n/SL_{n-1} \simeq SO_{2n}/SO_{2n-1}$$

- Other of the previous examples are also connected to exotic actions on symmetric varieties:

$$SL_{2n+1}/Sp_{2n} \simeq SL_{2n+2}/Sp_{2n+2}$$

$$Sp_{2n}/[GL_1 \times Sp_{2n-2}] \simeq SL_{2n}/GL_{2n-1}$$

$$SO_8/G_2 \simeq SO_8/SO_7 \times SO_8/Spin_7$$

- When the root systems of the two varieties are not isomorphic, we can find a symmetric subvariety which completes the picture.

For example:

$$\begin{array}{ccccc} GL_{2n}/Sp_{2n} & \hookrightarrow & SL_{2n+1}/Sp_{2n} & \simeq & SL_{2n+2}/Sp_{2n+2} \\ A_{n-1} & & A_n \times A_{n-1} & & A_n \end{array}$$

In the previous setting, the maps $Z \hookrightarrow X \xrightarrow{\sim} Y$ induce an isogeny

$$\mathcal{R}_X \longrightarrow \mathcal{R}_Y \oplus \mathcal{R}_Z \qquad \lambda \longmapsto (\widehat{\lambda}, \bar{\lambda})$$

compatible with the multiplication: if $\lambda, \mu, \nu \in \Lambda_X^+$, then

$$E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff \begin{array}{c} E_Y(\widehat{\nu}) \subset E_X(\widehat{\lambda}) \cdot E_X(\widehat{\mu}) \\ E_Z(\bar{\nu}) \subset E_Z(\bar{\lambda}) \cdot E_Z(\bar{\mu}) \end{array}$$

The isogeny is described as follows. If $\lambda \in \Lambda_X^+$, then

- $\widehat{\lambda} \in \Lambda_Y^+$ is the unique weight such that $E_X(\lambda) \subset E_Y(\widehat{\lambda})$
- $\bar{\lambda} \in \Lambda_Z^+$ is the weight such that $E_Z(\bar{\lambda})$ is the image of $E_X(\lambda)$

Thank you