On the multiplication of spherical functions of reductive spherical pairs of type A

Jacopo Gandini
(Università di Bologna)

Joint work with Paolo Bravi
(Sapienza Università di Roma)

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Let $K \subset G$ be connected reductive algebraic groups over $\mathbb{C}$.

**Definition**

$G/K$ is called **spherical** if $\mathbb{C}[G/K]$ is a multiplicity free $G$-module. Equivalently, $(G, K)$ is called a **reductive spherical pair**.

**EXAMPLES.**

Symmetric varieties: $K = (G^\vartheta)^\circ$ for some $\vartheta \in Inv(G)$, e.g.

$$SL_n/SO_n, \quad SL_{2n}/Sp_{2n}, \quad SO_n/SO_{n-1}$$

Non-symmetric examples:

$$SL_{2n+1}/Sp_{2n}, \quad SO_{2n+1}/GL_n, \quad SO_8/G_2$$

We will assume that $G$ is simple and simply connected.

The classification of the reductive spherical pairs with $G$ simple goes back to Krämer (1979).
Let $X = G/K$ spherical. Consider the decomposition of $G$-modules

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda^+_X} E_X(\lambda)$$

with $E_X(\lambda) \cong V(\lambda)$ irreducible of highest weight $\lambda \in \Lambda^+$

- $\Lambda^+_X$ is called the weight monoid of $X$.
- The weight lattice of $X$ is the lattice $\Lambda_X$ generated by $\Lambda^+_X$.

The weight monoid and weight lattice of $X$ are well understood.

Problem

*Given $\lambda, \mu \in \Lambda^+_X$, determine the decomposition of $E_X(\lambda) \cdot E_X(\mu)$: for which $\nu \in \Lambda^+_X$ does it hold $E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu)$?*

Somehow, $\Lambda^+_X$ behaves as a monoid of dominant weights for a root datum $(\Phi_X, \Lambda_X)$ attached to $X$, which generalize the restricted root system of a symmetric variety.

We will be interested in the case where $\Phi_X$ is of type A.
Let \((G, K)\) be a reductive spherical pair, \(X = G/K\). Then
\[
\mathbb{C}[X] = \mathbb{C}[G]^K \simeq \bigoplus V(\lambda)^* \otimes V(\lambda)^K
\]
Being multiplicity free amounts to the property
\[
dim V(\lambda)^K \leq 1 \quad \forall \lambda \in \Lambda^+
\]
The weights in \(\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda^+_X} E_X(\lambda)\) are given by
\[
\Lambda^+_X = \{ \lambda \in \Lambda^+ \mid \dim V(\lambda)^K = 1 \}
\]
Fix nonzero elements \(\varphi_\lambda \in E_X(\lambda)^K\), for all \(\lambda \in \Lambda^+_X\): these are the zonal spherical functions of \(X\). They form a basis of
\[
\mathbb{C}[X]^K = \bigoplus_{\lambda \in \Lambda^+_X} E_X(\lambda)^K = \mathbb{C}[G]^{K \times K}
\]
• Write \(\varphi_\lambda \cdot \varphi_\mu = \sum a^K_{\lambda,\mu} \varphi_\nu\), then [Ruitenburg, 1989]
\[
E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff a^\nu_{\lambda,\mu} \neq 0
\]
Zonal spherical functions and Jacobi polynomials

Let \( X = G/K \) symmetric, \( K = G^\vartheta \). Fix in \( G \) a maximal split torus \( A \), namely

\[
\vartheta(a) = a^{-1} \quad \forall a \in A
\]

Let \( T \subset G \) be a maximal torus containing \( A \), then \( T \) is \( \vartheta \)-stable. Let \( A_X = A/A \cap K \). Then \( \Lambda^+_X = \Lambda_X \cap \Lambda^+ \) and

\[
\Lambda_X = \{ \chi - \vartheta(\chi) : \chi \in \Lambda \} \simeq \mathcal{X}(A_X).
\]

- The restricted root system (possibly non-reduced) of \( X \) is

\[
\tilde{\Phi}_X = \{ \alpha - \vartheta(\alpha) : \alpha \in \Phi, \alpha \neq \vartheta(\alpha) \}
\]

Its weight lattice is \( \Lambda_X \), and we can choose a basis \( \Delta_X \subset \tilde{\Phi}_X \) so that \( \Lambda^+_X \) identifies with the dominant weights of \( \Phi_X \).
- \( \tilde{\Phi}_X \) comes with a multiplicity function

\[
m_X(\tilde{\alpha}) = |\{ \alpha \in \Phi : \alpha - \vartheta(\alpha) = \tilde{\alpha} \}|
\]

- Its Weyl group is the little Weyl group

\[
W_X = N_K(A)/Z_K(A)
\]
By a theorem of Richardson, restriction gives an isomorphism
\[ \mathbb{C}[X]^K \cong \mathbb{C}[A_X]^{W_X} \]
The image of the \( \varphi_\lambda \) is the **Jacobi polynomial** \( P^{(k)}_\lambda \) associated to \( \Phi_X \subset \mathcal{X}(A_X) \), evaluated at \( k = m_X/2 \).

**Problem.** How to decompose the product of two Jacobi polynomials?

Assume that \( \tilde{\Phi}_X \) of type \( A_n \).

**Conjecture (Stanley 1989)**

Write \( P^{(1/k)}_\lambda P^{(1/k)}_\mu = \sum b^\nu_{\lambda, \mu}(k) P^{(1/k)}_\nu \). Then \( b^\nu_{\lambda, \mu}(k) \) is the ratio of two polynomials in \( k \) with non-negative integer coefficients.

**Stanley’s Pieri rule:** the conjecture is true if \( \tilde{\Phi}_X \) is of type \( A_1 \).

Recall the decomposition \( \varphi_\lambda \cdot \varphi_\mu = \sum a^\nu_{\lambda, \mu} \varphi_\nu \). Thus
\[ E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff a^\nu_{\lambda, \mu} \neq 0 \iff b^\nu_{\lambda, \mu}(2/m_X) \neq 0 \]

- Another important specialization of \( P^{(k)}_\lambda \) is at \( k = 1 \):
  (slightly abusing, we identify partitions of length \( \leq n \) and dominant weights for \( \tilde{\Phi}_X \))
  up to scalars, \( P^{(1)}_\lambda \) is (the image of) the Schur polynomial \( s_\lambda \).

Write \( s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda, \mu} s_\nu \): then \( c^\nu_{\lambda, \mu} \neq 0 \iff b^\nu_{\lambda, \mu}(1) \neq 0 \)
Corollary

Suppose \( X \) is symmetric with \( \tilde{\Phi}_X \) of type \( A \), and let \( \lambda, \mu, \nu \in \Lambda_X^+ \). Let \( G_X \) be the simple group with based root system \( \Delta_X \subset \Phi_X \) and weight lattice \( \Lambda_X \). If Stanley’s conjecture is true, then

\[
E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff V_X(\nu) \subset V_X(\lambda) \otimes V_X(\mu).
\]

In particular, this is true if \( \tilde{\Phi}_X \) of type \( A_1 \).

In a slightly different form, considered by [Graham-Hunziker, 2009].

Table: Symmetric pairs with \( G \) simple and restricted root system of type \( A \)

<table>
<thead>
<tr>
<th>( G )</th>
<th>( K )</th>
<th>( \tilde{\Phi}_{G/K} )</th>
<th>( m_{G/K} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{SL}_n, \ n \geq 2 )</td>
<td>( \text{SO}_n )</td>
<td>( A_{n-1} )</td>
<td>1</td>
</tr>
<tr>
<td>( \text{SL}_{2n}, \ n \geq 2 )</td>
<td>( \text{Sp}_{2n} )</td>
<td>( A_{n-1} )</td>
<td>4</td>
</tr>
<tr>
<td>( \text{SO}_n, \ n \geq 5 )</td>
<td>( \text{SO}_{n-1} )</td>
<td>( A_1 )</td>
<td>( n - 2 )</td>
</tr>
<tr>
<td>( \text{E}_6 )</td>
<td>( \text{F}_4 )</td>
<td>( A_2 )</td>
<td>8</td>
</tr>
</tbody>
</table>
The case of a spherical pair

Let now \((G, K)\) be a reductive spherical pair, and set \(X = G/K\).

The root monoid of \(X\) is

\[
\mathcal{M}_X = \langle \lambda + \mu - \nu \mid \lambda, \mu, \nu \in \Lambda_X^+, \ E_X(\nu) \subseteq E_X(\lambda) \cdot E_X(\mu) \rangle_{\mathbb{N}}
\]

**Theorem (Knop 1994 + Avdeev–Cupit-Foutou 2018)**

\(\mathcal{M}_X\) is a free monoid, and the set of free generators \(\Delta_X \subset \mathcal{M}_X\) is the base of a (reduced) root system \(\Phi_X\).

In general, \(\Lambda_X\) is not contained in the weight lattice of \(\Phi_X\).

However it is possible to define a map

\[
\Phi_X \longrightarrow \text{Hom}_\mathbb{Z}(\Lambda_X, \mathbb{Z}) \quad \alpha \longmapsto \alpha^\vee
\]

giving rise to a based root datum

\[
\mathcal{R}_X = (\Lambda_X, \Delta_X, \Delta_X^\vee)
\]

- In this way, the weight monoid \(\Lambda_X^+\) gets identified with a submonoid of dominant weights for \(\mathcal{R}_X\).

- If \(X\) is is symmetric, \(\mathcal{R}_X\) is semisimple and \(\Phi_X\) is the reduced root system associated to the restricted root system of \(X\).
Let $G_X$ be the reductive group defined by $\mathcal{R}_X$, then

$$\Lambda_X^+ \rightarrow \text{Irr}(G_X) \quad \lambda \mapsto V_X(\lambda)$$

Recall the decomposition of $G$-modules

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda_X^+} E_X(\lambda), \quad E_X(\lambda) \cong V(\lambda)$$

**Conjecture (Bravi-G.)**

*Recall that $G$ is simple, and let $X = G/K$ with $\Phi_X$ of type $A$. If $\lambda, \mu, \nu \in \Lambda_X^+$, then*

$$E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff V_X(\nu) \subset V_X(\lambda) \otimes V_X(\mu)$$

**Theorem (Bravi-G.)**

- *If $X \neq F_4/B_4$, then the previous conjecture is a consequence of Stanley’s conjecture.*

- *Suppose that $\Phi_X$ is direct sum of root systems of type $A_1$, and assume $X \neq F_4/B_4$. Then the previous conjecture is true.*
Table: The other reductive spherical pairs with $G$ simple and $\Phi_X$ of type $A$

<table>
<thead>
<tr>
<th>$G$</th>
<th>$K$</th>
<th>$\Phi_{G/K}$</th>
<th>Sym</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SL_n$, $n \geq 3$</td>
<td>$SL_{n-1}$</td>
<td>$A_1$</td>
<td></td>
</tr>
<tr>
<td>$SL_n$, $n \geq 3$</td>
<td>$GL_{n-1}$</td>
<td>$A_1$</td>
<td>$BC_1$</td>
</tr>
<tr>
<td>$SL_{2n+1}$, $n \geq 2$</td>
<td>$Sp_{2n}$</td>
<td>$A_n \times A_{n-1}$</td>
<td></td>
</tr>
<tr>
<td>$SL_{2n+1}$, $n \geq 2$</td>
<td>$GL_1 \times Sp_{2n}$</td>
<td>$A_n \times A_{n-1}$</td>
<td></td>
</tr>
<tr>
<td>$Sp_{2n}$, $n \geq 2$</td>
<td>$GL_1 \times Sp_{2n-2}$</td>
<td>$A_1 \times A_1$</td>
<td></td>
</tr>
<tr>
<td>$Sp_{2n}$, $n \geq 3$</td>
<td>$Sp_2 \times Sp_{2n-2}$</td>
<td>$A_1$</td>
<td>$BC_1$</td>
</tr>
<tr>
<td>$Spin_7$</td>
<td>$G_2$</td>
<td>$A_1$</td>
<td></td>
</tr>
<tr>
<td>$Spin_9$</td>
<td>$Spin_7$</td>
<td>$A_1 \times A_1$</td>
<td></td>
</tr>
<tr>
<td>$Spin_8$</td>
<td>$G_2$</td>
<td>$A_1 \times A_1 \times A_1$</td>
<td></td>
</tr>
<tr>
<td>$F_4$</td>
<td>$Spin_9$</td>
<td>$A_1$</td>
<td>$BC_1$</td>
</tr>
<tr>
<td>$G_2$</td>
<td>$SL_3$</td>
<td>$A_1$</td>
<td></td>
</tr>
</tbody>
</table>
We do not have Jacobi polynomials in the non-symmetric setting. But we can reduce to the symmetric case!

- Several of the previous example correspond to exotic homogeneous actions on spheres:
  \[ G_2/SL_3 \simeq SO_7/SO_6 \]
  \[ Spin_7/G_2 \simeq SO_8/SO_7 \]
  \[ Spin_9/Spin_7 \simeq SO_{16}/SO_{15} \]
  \[ SL_n/SL_{n-1} \simeq SO_{2n}/SO_{2n-1} \]

- Other of the previous examples are also connected to exotic actions on symmetric varieties:
  \[ SL_{2n+1}/Sp_{2n} \simeq SL_{2n+2}/Sp_{2n+2} \]
  \[ Sp_{2n}/[GL_1 \times Sp_{2n-2}] \simeq SL_{2n}/GL_{2n-1} \]
  \[ SO_8/G_2 \simeq SO_8/SO_7 \times SO_8/Spin_7 \]
When the root systems of the two varieties are not isomorphic, we can find a symmetric subvariety which completes the picture.

For example:

\[
\begin{align*}
GL_{2n}/Sp_{2n} & \hookrightarrow SL_{2n+1}/Sp_{2n} \cong SL_{2n+2}/Sp_{2n+2} \\
A_{n-1} & \quad A_n \times A_{n-1} \quad A_n
\end{align*}
\]

In the previous setting, the maps \(Z \hookrightarrow X \sim \rightarrow Y\) induce an isogeny

\[
\mathcal{R}_X \longrightarrow \mathcal{R}_Y \oplus \mathcal{R}_Z \quad \lambda \longmapsto (\hat{\lambda}, \bar{\lambda})
\]

compatible with the multiplication: if \(\lambda, \mu, \nu \in \Lambda^+_X\), then

\[
E_X(\nu) \subset E_X(\lambda) \cdot E_X(\mu) \iff E_Y(\hat{\nu}) \subset E_X(\hat{\lambda}) \cdot E_X(\hat{\mu})
\]

\[
E_Z(\bar{\nu}) \subset E_Z(\bar{\lambda}) \cdot E_Z(\bar{\mu})
\]

The isogeny is described as follows. If \(\lambda \in \Lambda^+_X\), then

- \(\hat{\lambda} \in \Lambda^+_Y\) is the unique weight such that \(E_X(\lambda) \subset E_Y(\hat{\lambda})\)
- \(\bar{\lambda} \in \Lambda^+_Z\) is the weight such that \(E_Z(\bar{\lambda})\) is the image of \(E_X(\lambda)\)
Thank you