Shift operators for a class of intermediate Jacobi polynomials of type BC2

HOPE

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May 3 2024
Introduction

Scalar-valued
A: Jacobi polynomials.
B: zsf rank one.
F: zsf rank > 1.
E: Heckman-Opdam.

Vector-valued
D: Koelink–de los Rios–Romàn.
C: Heckman-vP.
H: Shimeno, vP, . . .
Introduction

Goal

Find examples of classical systems \((\mathcal{W}, \mathcal{D}, (P_{\lambda})_{\lambda \in \Lambda})\) where

- \(\mathcal{W}\) is a matrix weight of size \(N \times N\),
- \(\mathcal{D}\) is a commutative algebra of differential operators acting on \(\mathbb{C}[x_1, \ldots, x_r] \otimes \mathbb{C}^N\)
- \((P_{\lambda})_{\lambda \in \Lambda}\) basis of \(\mathbb{C}[x_1, \ldots, x_r] \otimes \mathbb{C}^N\)

such that the \(P_{\lambda}\) are uniquely determined (up to scaling) as simultaneous eigenfunctions of \(\mathcal{D}\).

Moreover:

- The data \((\mathcal{W}, \mathcal{D}, (P_{\lambda})_{\lambda \in \Lambda})\) should depend on additional parameters.
- The families should be related by shift operators.

We also want the analogous \(q\)-cube.
Introduction

The intermediate Jacobi polynomials (vP 2023) give classical systems.
Rank one:
  ▶ Bouzeffour&Koornwinder, 2010
  ▶ Van Horssen&vP, 2024, Identification with spherical functions (discrete parameters)
    ▶ Non-symmetric shift operators, Harmonic Analysis.

The intermediate Macdonald polynomials (Schlösser 2023) give classical systems.
Rank one:
  ▶ Bouzeffour&Koornwinder, 2010
  ▶ Van Horssen&Schnösser, Non-symmetric shift in rank one.
Examples intermediate Jacobi polynomials in rank two:

- **Type $A_2$:** Classical pair of size $3 \times 3$ in two variables
  - no shifts yet.

- **Type $BC_2$:** Classical pair of size $2 \times 2$ in two variables
  - with shifts (jt. with van Horssen).
Intermediate Jacobi Polynomials

- $R \subset \alpha^*$ is a root system with Weyl group $W$, $R_+ \subset R$ choice of positive roots.
- $W' \subset W$ a subgroup generated by reflections.
- $k : R \to \mathbb{C}$ multiplicity function ($W$-invariant).
Intermediate Jacobi Polynomials

This gives

- $P$ is the weight lattice, $P^+$ dominant weights,
- $\mathbb{C}[P]$ is the group algebra of $P$,
- $\delta_k = \prod_{\alpha \in R}(1 - e^{\alpha})^{k_\alpha}$ is the usual weight function,
- $\mathfrak{h} = a_{\mathbb{C}}$ the complexification of the torus $a$.
- $H(R_+, k)$ graded Hecke algebra, isomorphic to $S(\mathfrak{h}) \otimes \mathbb{C}[W]$ as vector space.
Intermediate Jacobi Polynomials

- $W' \leq W$ a reflection subgroup
- $C[P]^W$ with inner product by weight $\delta(k)$.
- $(C[P]^W, \delta(k))$ is encoded by matrix-valued orthogonal polynomials.
- If $W' \leq W$ parabolic, then get classical system with parameter.
Intermediate Jacobi Polynomials

Let $R$ be of type $BC_2$.

- $W = \langle s_1 = s_{\epsilon_1 - \epsilon_2}, s_2 = s_{\epsilon_2} \rangle$
- $W' = \langle s_1, r = s_1 s_2 s_1 \rangle$ not parabolic!
- $W$ has 4 characters among which
  \[
  \sigma(s_1) = -1, \quad \sigma(s_2) = 1.
  \]
- $\sigma|_{W'} = \text{Id}$
- $\mathbb{C}[P]^{W'} = \mathbb{C}[P]^W \oplus \mathbb{C}[P]_{(\sigma)}^{(W)}$ has orthogonal basis
  \[
  \{ P(\lambda, k) \mid \lambda \in P^+ \} \cup \{ P^\sigma(\lambda + \epsilon_1, k) \mid \lambda \in P^+ \}
  \]
Intermediate Jacobi Polynomials

- \( P_{\epsilon_1}^{\sigma}(k) \in \mathbb{C}[P]^{(W)}_{(\sigma)} \)
- We have
  \[
  \frac{P_{\lambda}^{\sigma}(k)}{P_{\epsilon_1}^{\sigma}(k)} = P_{\lambda-\epsilon_1}(k')
  \]
- We found two operators \( C_1, C_2 \in \mathbf{H}(R_+, k) \):
  - \( C_i \) leaves \( \mathbb{C}[P]^{W'} \) invariant,
  - \( r \circ C_1 \circ r = C_2 \),
  - \( C_1 + C_2 \) is the order two Casimir operator.
  - \( C_1 - C_2 \) interchanges \( P(\lambda + \epsilon_1, k) \leftrightarrow P^{\sigma}(\lambda + \epsilon_1, k) \) for \( \lambda \in P^+ \).
Intermediate Jacobi Polynomials


$\delta(k) \leftrightarrow \begin{pmatrix} \delta(k) & 0 \\ 0 & \delta(k') \end{pmatrix}$

$C_1 - C_2 \leftrightarrow "internal"$ shift operator for $BC_2$.

**Result:** The shift operators

$$k \mapsto \tilde{k}$$

on the right (Koornwinder, Sprinkhuizen, Opdam) can be transferred to shift operators on

$$(\mathbb{C}[P]^W', \delta(k)) \rightarrow (\mathbb{C}[P]^W', \delta(\tilde{k}))$$

that symmetrize to the classical shift operators.
Intermediate Jacobi Polynomials

In this way we obtain a family

\[(\mathcal{W}(k), (P_\lambda(k))_{\lambda \in \Lambda})\]

of \(\text{End}(\mathbb{C}^2)\)-valued weight and its MVOPs with parameter \(k = (k_1, k_2, k_3)\) that are related by shift operators.

Thank you!