Shift operators for a class of intermediate Jacobi polynomials of type BC2
HOPE

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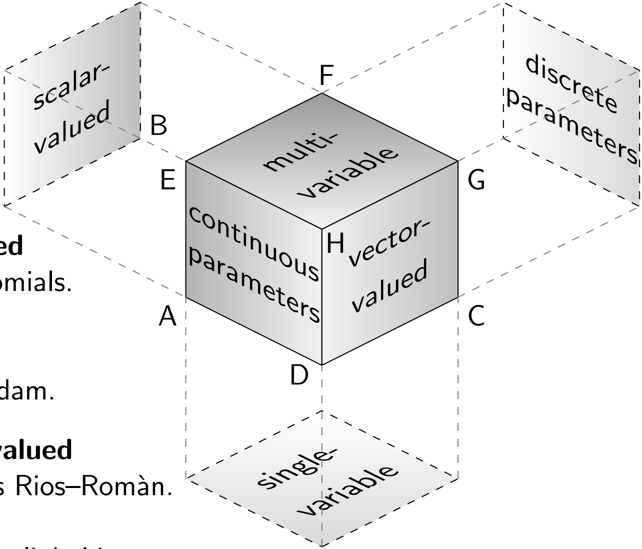
Introduction



Scalar-valued

A: Jacobi polynomials.
B: zsf rank one.
F: zsf rank > 1.
E: Heckman-Opdam.

Vector-valued

D: Koelink–de los Rios–Romàn.
C: Heckman-vP.
G: Pezzini–vP, Koelink–Liu.
H: Shimeno, vP, ...

Introduction

Goal

Find examples of classical systems $(\mathcal{W}, \mathbb{D}, (P_\lambda)_{\lambda\in\Lambda})$ where

- $\mathcal{W}$ *is a matrix weight of size* $N \times N$*,*
- $\mathbb{D}$ *is a commutative algebra of differential operators acting on*

$$\mathbb{C}[x_1, \dots, x_r] \otimes \mathbb{C}^N$$

- $(P_\lambda)_{\lambda\in\Lambda}$ *basis of* $\mathbb{C}[x_1, \dots, x_r] \otimes \mathbb{C}^N$

such that the P_λ *are uniquely determined (up to scaling) as simultaneous eigenfunctions of* $\mathbb{D}$*.*

Moreover:

- The data $(\mathcal{W}, \mathbb{D}, (P_\lambda)_{\lambda\in\Lambda})$ should depend on additional parameters.
- The families should be related by shift operators.

We also want the analogous q-cube.

Introduction

The intermediate Jacobi polynomials (vP 2023) give classical systems.
Rank one:

- Bouzeffour&Koornwinder, 2010
- Van Horssen&vP, 2024, Identification with spherical functions (discrete parameters)
  - Non-symmetric shift operators, Harmonic Analysis.

The intermediate Macdonald polynomials (Schlösser 2023) give classical systems.
Rank one:

- Bouzeffour&Koornwinder, 2010
- Van Horssen&Schlösser, Non-symmetric shift in rank one.

Introduction

Examples intermediate Jacobi polynomials in rank two:

- Type A_2: Classical pair of size 3×3 in two variables **no shifts** yet.
- Type BC_2: Classical pair of size 2×2 in two variables **with shifts** (jt. with van Horssen).

Intermediate Jacobi Polynomials

- $R \subset \mathfrak{a}^*$ is a root system with Weyl group W, $R_+ \subset R$ choice of positive roots.
- $W' \subset W$ a subgroup generated by reflections.
- $k : R \to \mathbb{C}$ multiplicity function (W-invariant).

Intermediate Jacobi Polynomials

This gives

- P is the weight lattice, P^+ dominant weights,
- $\mathbb{C}[P]$ is the group algebra of P,
- $\delta_k = \prod_{\alpha \in R}(1 - e^{\alpha})^{k_\alpha}$ is the usual weight function,
- $\mathfrak{h} = \mathfrak{a}_{\mathbb{C}}$ the complexification of the torus $\mathfrak{a}$.
- $\mathbf{H}(R_+, k)$ graded Hecke algebra, isomorphic to $S(\mathfrak{h}) \otimes \mathbb{C}[W]$ as vector space.

Intermediate Jacobi Polynomials

- $W' \leq W$ a reflection subgroup
- $\mathbb{C}[P]^{W'}$ with inner product by weight $\delta(k)$.
- $(\mathbb{C}[P]^{W'}, \delta(k))$ is encoded by matrix-valued orthogonal polynomials.
- If $W' \leq W$ parabolic, then get classical system with parameter.

Intermediate Jacobi Polynomials

Let R be of type BC_2.

- $W = \langle s_1 = s_{\epsilon_1 - \epsilon_2}, s_2 = s_{\epsilon_2} \rangle$
- $W' = \langle s_1, r = s_1 s_2 s_1 \rangle$ **not parabolic!**
- W has 4 characters among which

$$\sigma(s_1) = -1, \quad \sigma(s_2) = 1.$$

- $\sigma|_{W'} = \mathrm{Id}$
- $\mathbb{C}[P]^{W'} = \mathbb{C}[P]^W \oplus \mathbb{C}[P]^{(W)}_{(\sigma)}$ has orthogonal basis

$$\{P(\lambda, k) \mid \lambda \in P^+\} \cup \{P^\sigma(\lambda + \epsilon_1, k) \mid \lambda \in P^+\}$$

Intermediate Jacobi Polynomials

- $P^{\sigma}_{\epsilon_1}(k) \in \mathbb{C}[P]^{(W)}_{(\sigma)}$
- We have
$$\frac{P^{\sigma}_{\lambda}(k)}{P^{\sigma}_{\epsilon_1}(k)} = P_{\lambda-\epsilon_1}(k')$$
- We found two operators $\mathcal{C}_1, \mathcal{C}_2 \in \mathbf{H}(R_+, k)$:
  - $\mathcal{C}_i$ leaves $\mathbb{C}[P]^{W'}$ invariant,
  - $r \circ \mathcal{C}_1 \circ r = \mathcal{C}_2$,
  - $\mathcal{C}_1 + \mathcal{C}_2$ is the order two Casimir operator.
  - $\mathcal{C}_1 - \mathcal{C}_2$ interchanges $P(\lambda+\epsilon_1, k) \leftrightarrow P^{\sigma}(\lambda+\epsilon_1, k)$ for $\lambda \in P^+$.

Intermediate Jacobi Polynomials

Under the identification $\mathbb{C}[P]^{W'} \cong \mathbb{C}[P]^{W} \oplus \mathbb{C}[P]^{W}$

- $\delta(k) \leftrightarrow \begin{pmatrix} \delta(k) & 0 \\ 0 & \delta(k') \end{pmatrix}$
- $\mathcal{C}_1 - \mathcal{C}_2 \leftrightarrow$ "internal" shift operator for BC_2.

Result: The shift operators

$$k \mapsto \widetilde{k}$$

on the right (Koornwinder, Sprinkhuizen, Opdam) can be transferred to shift operators on

$$(\mathbb{C}[P]^{W'}, \delta(k)) \to (\mathbb{C}[P]^{W'}, \delta(\widetilde{k}))$$

that symmetrize to the classical shift operators.

Intermediate Jacobi Polynomials

In this way we obtain a family

$$(\mathcal{W}(k), (P_\lambda(k))_{\lambda \in \Lambda})$$

of $\mathrm{End}(\mathbb{C}^2)$-valued weight and its MVOPs with parameter $k = (k_1, k_2, k_3)$ that are related by shift operators.

Thank you!