Shift operators for nonsymmetric Jacobi polynomials of type $BC_1$ and their norms

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Overview

- **Shift operators** for the Jacobi polynomials and their **norms**
- **Nonsymmetric** Jacobi polynomials as **matrix-valued orthogonal polynomials**
- **Construction** of (nonsymmetric) shift operators for nonsymmetric Jacobi polynomials
- Some **properties** and **applications** of nonsymmetric shift operators
DEFINITION: The monic Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$ are orthogonal polynomials, given by

1. $P_n^{(\alpha, \beta)}(x) = x^n + \text{lower order terms}$;
2. orthogonal with respect to $$(f, g)_{\alpha, \beta} = \int_{-1}^{1} f(x)g(x)w^{(\alpha, \beta)}(x)dx; \quad w^{(\alpha, \beta)}(x) = (1-x)^\alpha(1+x)^\beta.$$ 

REMARK: The $P_n^{(\alpha, \beta)}(x)$ are a special case of the symmetric Heckman-Opdam polynomials for the root system $R$ of type $BC_1$, i.e., $P_n^{(\alpha, \beta)}(x) = P_n(k)$ with $\alpha = k_1 + k_2 - \frac{1}{2}$ and $\beta = k_2 - \frac{1}{2}$.

Here $k = (k_1, k_2)$ is the multiplicity function on $R = \{\pm e_1, \pm 2e_1\}$, with $k_ie_1 = k_i$ for $i = 1, 2$. 
**Lemma:** The **forward** and **backward shift operators** are given by

\[
\partial_x P_n^{(\alpha, \beta)}(x) = n P_{n-1}^{(\alpha+1, \beta+1)}(x),
\]

\[
\partial^*_x P_n^{(\alpha, \beta)}(x) = (n + \alpha + \beta + 1) P_{n+1}^{(\alpha-1, \beta-1)}(x).
\]

**Key idea:**

\[
\| P_n^{(\alpha, \beta)} \|_{\alpha, \beta}^2 = \frac{1}{n+1} \left( \partial_x P_{n+1}^{(\alpha-1, \beta-1)}, P_n^{(\alpha, \beta)} \right)_{\alpha, \beta}
\]

\[
= \frac{1}{n+1} \left( P_{n+1}^{(\alpha-1, \beta-1)}, \partial^*_x P_n^{(\alpha, \beta)} \right)_{\alpha-1, \beta-1}
\]

\[
= \frac{n + \alpha + \beta + 1}{n+1} \| P_{n+1}^{(\alpha-1, \beta-1)} \|_{\alpha-1, \beta-1}^2.
\]

Applying this **repeatedly**, \( \| P_n^{(\alpha, \alpha)} \|_{\alpha, \alpha}^2 \) can be computed from \( \| \cdot \|_{0,0} \) for **integral** \( \alpha \).
**Remark:** The $P_n^{(\alpha, \beta)}$ are **eigenfunctions** of a second-order **differential operator**

$$L^{(\alpha, \beta)} = (x^2 - 1)\partial_x^2 + (\alpha - \beta + (\alpha + \beta + 2)x)\partial_x.$$

**Observation:** Writing $ML(k) = L^{(\alpha, \beta)} + ||\rho(k)||^2$, with $\rho(k) = \frac{1}{2}(k_1 + 2k_2)e_1$, we have

$$\partial_x ML(k) = ML(k + (0, 1))\partial_x,$$

$$\partial_x^* ML(k) = ML(k - (0, 1))\partial_x^*.$$

**Definition:** A **shift operator** $S$ with shift $l \in 2\mathbb{Z} \times \mathbb{Z}$ is an operator on $\mathbb{C}[x]$ satisfying the **transmutation property**:

$$S \circ ML(k) = ML(k + l) \circ S.$$
Transmutation property and fundamental shift operators

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Theorem (Heckman-Opdam): Each shift operator is can be constructed from the fundamental shift operators: $G_+ = \partial_x$, $G_- = \partial_x^*$, the $(2, -1)$-shift $E_+$, $E_- = E_+^*$, and $ML(k)$.
Remark: Writing \( x = (z + z^{-1})/2 \), the \( P_n(k) \) can be viewed as symmetric Laurent polynomials (invariant under the reflection \( r: z \leftrightarrow z^{-1} \)).

Definition: Consider the total order \( 1 < z < z^{-1} < z^2 < \ldots \). The monic nonsymmetric Jacobi polynomials are defined by

- \( E_n(k) = z^n + \text{lower order terms} \ (n \in \mathbb{Z}) \);
- orthogonal with respect to some inner product.
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- $E_n(k) = z^n + \text{lower order terms} \ (n \in \mathbb{Z})$;
- orthogonal with respect to some inner product.

Remark: The $E_n(k)$ are eigenfunctions of a differential-reflection operator

$$T(k) = z\partial_z + \sum_{\alpha \in R_+} \frac{(\alpha, e_1) k_\alpha}{1 - z^{-1}} (1 - r) - \rho(k)(e_1),$$

called the Cherednik-Dunkl operator.

**Definition:** A nonsymmetric shift operator $S$ with shift $l \in 2\mathbb{Z} \times \mathbb{Z}$ is a differential-reflection operator on $\mathbb{C}[z^{\pm}]$ satisfying the transmutation property:

$$S \circ T(k) = T(k + l) \circ S.$$
Nonsymmetric Jacobi polynomials as MVOPs

**Lemma:** Each \( f \in \mathbb{C}[z^\pm] \) can be written as \( f = f_1 + f_2z \), with \( f_i \in \mathbb{C}[x] \). Moreover, the map
\[
\mathbb{C}[z^\pm] \to \mathbb{C}[x] \otimes \mathbb{C}^2, \quad f \mapsto \underline{f} := (f_1, f_2)
\]
is a \( \mathbb{C}[x] \)-module isomorphism.

**Proposition** (van Pruijssen-vH): Let \( \mathbf{E}_n(k) = (\mathbf{E}_{-n}(k), \mathbf{E}_{n+1}(k)) \). Then \( \{\mathbf{E}_n(k)\}_{n \in \mathbb{N}} \) is a family of matrix-valued orthogonal polynomials (MVOPs), w.r.t.

\[
(F, G)_k = \frac{1}{2} \int_{-1}^{1} F(x)^* \mathcal{W}(x) G(x) w^{(\alpha, \beta)}(x) dx;
\]
\[
\mathcal{W}(x) = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}.
\]
**Lemma:** Each $f \in \mathbb{C}[z^\pm]$ can be written as $f = f_1 + f_2 z$, with $f_i \in \mathbb{C}[x]$. Moreover, the map $\mathbb{C}[z^\pm] \to \mathbb{C}[x] \otimes \mathbb{C}^2$, $f \mapsto \underline{f} := (f_1, f_2)$ is a $\mathbb{C}[x]$-module isomorphism.

**Proposition** (van Pruijssen-vH): Let $E_n(k) = (E_{-n}(k) \ E_{n+1}(k))$. Then $\{E_n(k)\}_{n \in \mathbb{N}}$ is a family of **matrix-valued orthogonal polynomials** (MVOPs), w.r.t.

$$(F, G)_k = \frac{1}{2} \int_{-1}^{1} F(x) W(x) G(x) w^{(\alpha, \beta)}(x) \, dx; \quad W(x) = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}. $$

Moreover, the matrix weight diagonalizes: $M^* W M = \text{diag}(w^{(\alpha+1, \beta)}, w^{(\alpha, \beta+1)})$, and

$$E_n(k) = M^* P_n(k) M C_n(k),$$

with $C_n(k)$ invertible, and $P_n(k) = \text{diag}(P_n^{(\alpha+1, \beta)}, P_n^{(\alpha, \beta+1)})$. 

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**Observation:** Notice that

\[
\partial_x P_n(k) = \begin{pmatrix}
\partial_x P_n^{(\alpha+1, \beta)} \\
0 \\
\partial_x P_n^{(\alpha, \beta+1)}
\end{pmatrix} = nP_n(k + (0, 1)),
\]

it follows that \( \partial_x E_n(k) = nE_n(k + (0, 1)) \).

**Remark:** \( T(k) \) can be pushed to an **differential operator** \( D(k) \) acting on \( \mathbb{C}[x] \otimes M_{2 \times 2}(\mathbb{C}) \), and

- The \( E_n(k) \) are **eigenfunctions** of \( D(k) \);
- Transmutation property: \( \partial_x \circ D(k) = D(k + (0, 1)) \circ \partial_x \).
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**Proposition:** The condition \( G_+ f = \partial_x f \) defines a differential-reflection operator satisfying:

- Transmutation property: \( G_+ \circ T(k) = T(k + (0, 1)) \circ G_+ \);
- \( G_+ E_{-n}(k) = n E_{-(n-1)}(k + (0, 1)) \) and \( G_+ E_{n+1}(k) = n E_n(k + (0, 1)) \) for all \( n \geq 0 \),

which is called the **forward nonsymmetric shift operator**.
There are **four fundamental** nonsymmetric shift operators: $G_\pm$ and $E_\pm$.

Each fundamental nonsymmetric shift operator can be **symmetrized** to the corresponding (symmetric) shift operator, e.g., $\text{Sym}(G_+)\big|_{\mathbb{C}[x]} = G_+$. 

**Theorem**: **Each** nonsymmetric shift operator can be **constructed** from the fundamental nonsymmetric shift operators and the Cherednik-Dunkl operator $T(k)$.

**Application I**: Computation of the **norms** $||E_n(k)||_2^2$ of the nonsymmetric Jacobi polynomials.

**Application II**: The **evaluation at the identity** $E_n(k; e)$ (viewed as functions on the torus).
Let $P_{n}^{AW}(a, b, c, d; q)$ and $E_{n}^{AW}(a, b, c, d; q)$ be the symmetric and nonsymmetric Askey-Wilson polynomials, respectively. Then “$\lim_{q \to 1} P_{n}^{AW} = P_{n}$” and “$\lim_{q \to 1} E_{n}^{AW} = E_{n}$”.

$P_{n}^{AW}(a, b, c, d)$ is symmetric in $a$, $b$, $c$, and $d$.

There are eight fundamental $q$-shift operators with shifts: $\pm \frac{1}{2}(1, 1, 1, 1)$, and permutations of $\frac{1}{2}(-1, -1, 1, 1)$.

The $E_{n}^{AW}(a, b, c, d)$ are symmetric in $a \leftrightarrow b$ and $c \leftrightarrow d$.

There are eight fundamental nonsymmetric $q$-shift operators with shifts: $\pm \frac{1}{2}(1, 1, 1, 1)$, permutations of $\frac{1}{2}(-1, -1, 1, 1)$, and permutations of $\frac{1}{2}(-1, 1, -1, 1)$.

$\pm \frac{1}{2}(1, 1, 1, 1) \stackrel{q \to 1}{\longrightarrow} \pm (0, 1)$, $\frac{1}{2}(-1, -1, 1, 1) \stackrel{q \to 1}{\longrightarrow} (0, 0)$ (constant), and $\frac{1}{2}(-1, 1, -1, 1) \stackrel{q \to 1}{\longrightarrow} (-2, 1)$. 