Matrix valued orthogonal polynomials, Fourier algebras and non-abelian Toda-type equations

HOPE, Nijmegen 2024

Pablo Román

May 1, 2024

Universidad Nacional de Córdoba - Argentina

joint with A. Deaño and L. Morey.

Matrix valued orthogonal polynomials

Matrix valued polynomials:

$$P(x) = A_n x^n + A_{n-1} x^{n-1} + \ldots + A_0, \qquad A_j \in M_N(\mathbb{C}) = \mathbb{C}^{N\times N}.$$

Weight matrix: Let $W : [a, b] \to M_N(\mathbb{C})$ such that

- $W(x)$ is positive definite for $x \in (a, b)$.
- W has finite moments:

$$\int_a^b x^n W(x) dx < \infty, \qquad \forall n \in \mathbb{N} \cup \{0\}.$$

Matrix valued inner product

For polynomials P, Q we have:

$$\langle P, Q\rangle = \int_a^b P(x)W(x)Q(x)^* dx \in M_N(\mathbb{C}).$$

$\exists$ a unique sequence of monic orthogonal polynomials $(P_n)_n$, (MVOPs).

Matrix valued differential and difference operators

Matrix differential operators:

$$\mathcal{D} = \sum_{k=0}^{\ell} \frac{d^k}{dx^k} F_k(x), \qquad F_k \text{ are rational functions on } M_N(\mathbb{C}).$$

Right action of $\mathcal{D}$ on matrix polynomials:

$$Q(x) \cdot \mathcal{D} = \sum_{k=0}^{\ell} Q^{(k)}(x) F_k(x), \qquad Q \in M_N(\mathbb{C}).$$

Matrix difference operators:

$$M = \sum_{k=-r}^{s} G_k(n) \delta^k, \qquad \delta_k(P_n(x)) = P_{n+k}(x).$$

Left action of M on matrix polynomials:

$$M \cdot P_n(x) = \sum_{k=-r}^{s} G_k(n) P_{n+k}(x).$$

The Fourier algebras

[Casper and Yakimov, 2022, American Journal of Math.]

The right Fourier algebra:

$$\mathcal{F}_R(P) = \{\mathcal{D} \colon \exists M,\, M \cdot P_n = P_n \cdot \mathcal{D}\}.$$

The left Fourier algebra:

$$\mathcal{F}_L(P) = \{M \colon \exists \mathcal{D},\, M \cdot P_n = P_n \cdot \mathcal{D}\}.$$

The Fourier algebras are isomorphic:

The following map is an isomorphism:

$$\begin{aligned} \psi : \quad \mathcal{F}_L(P) &\to \quad \mathcal{F}_R(P), \\ M &\mapsto \qquad \mathcal{D}, \end{aligned}$$

where $M \cdot P_n = P_n \cdot \mathcal{D}$.

Example:

Every sequence of matrix valued orthogonal polynomials satisfies a three term-recurrence relation:

$$xP_n(x) = P_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x).$$

This can be seen as:

$$\mathcal{L} \cdot P_n(x) = P_n(x) \cdot \mathcal{D},$$

where

$$\mathcal{L} = \delta^1 + B_n\delta^0 + C_n\delta^{-1}, \qquad \mathcal{D} = x.$$

In this case:

$$\mathcal{L} \in \mathcal{F}_L(P), \qquad \mathcal{D} \in \mathcal{F}_R(P).$$

Characterization of Fourier algebras

Theorem: For every $M \in \mathcal{F}_L(P)$, there exists $M^\dagger \in \mathcal{F}_L(P)$ such that:

$$\langle M \cdot P_n, P_m \rangle = \langle P_n, M^\dagger \cdot P_m \rangle.$$

Theorem: The left Fourier algebra is characterized by

$$\mathcal{F}_L(P) = \left\{ M \,:\, \mathrm{Ad}_{\mathscr{L}}^{k+1}(M) = 0 \text{ for some } k \geq 0 \right\}, \quad \mathrm{Ad}_S(T) = ST - TS.$$

Theorem: The right Fourier algebra is characterized by

$$\mathcal{F}_R(P) = \left\{ D \,:\, D \text{ is has an adjoint } D^\dagger \right\}.$$

This is:

$$\langle P \cdot D, Q \rangle = \langle P, Q \cdot D^\dagger \rangle, \qquad P, Q \in M_N(\mathbb{C}).$$

Characterization of Fourier algebras

Corollary 1: The Fourier algebras $\mathcal{F}_L(P)$ and $\mathcal{F}_R(P)$ are closed under the adjoint operation $\dagger$. Moreover, the Fourier map ψ satisfies $\psi(M^\dagger) = \psi(M)^\dagger$.

$$D(W) = \{\mathcal{D} : \; P_n \cdot \mathcal{D} = \Gamma_n \cdot P_n \quad \text{for some } \Gamma_n \in M_N(\mathbb{C})\} \subset \mathcal{F}_R(P).$$

Corollary 2: $D(W)$ is closed under $\dagger$.

Corollary 3:

If D is symmetric:

$$\langle P \cdot D, Q \rangle = \langle P, Q \cdot D \rangle, \qquad P, Q \in M_N(\mathbb{C}),$$

then $D \in \mathcal{F}_R(P)$.

[Casper and Yakimov, 2022, American Journal of Math.]

Application: The matrix valued Bochner problem

[Durán, 1998] The Matrix valued Bochner problem

Find all orthogonal polynomial solutions to the eigenvalue problem:

$$P_n(x) \cdot \mathcal{D} = \Gamma_n \cdot P_n(x), \qquad \Gamma_n \in M_N(\mathbb{C}),$$

where $\mathcal{D}$ is a second order differential operator.

$N = 1$: **Bochner problem:** Hermite, Laguerre and Jacobi.

For $N > 1$, the problem is **much harder**.

- The first examples are related to the harmonic analysis on the compact symmetric pairs. [Grünbaum, Pacharoni, Tirao, Koelink, van Pruijssen, R., Heckman].
- Other examples: Durán, Grünbaum, de la Iglesia, Castro, ...
- Solution: [Casper and Yakimov, 2022, American Journal of Math.]

Matrix valued deformations of the weight

The classical Toda deformation for W is:

$$W(x;t) = e^{-tx}W(x), \qquad t \in \mathbb{R},$$

Three-term recurrence relation for the monic MVOP:

$$xP_n(x;t) = P_{n+1}(x;t) + B_n(t)P_n(x;t) + C_n(t)P_{n-1}(x;t).$$

The recurrence coefficients $B_n(t)$ and $C_n(t)$ satisfy the following differential equations

$$\begin{aligned} \dot{B}_n(t) &= C_n(t) - C_{n+1}(t), \\ \dot{C}_n(t) &= C_n(t)B_{n-1}(t) - B_n(t)C_n(t). \end{aligned}$$

[Ismail, Koelink, R., 2019]

These equations appear in [Gekhtman] (who attributes the equations to Polyakov).

Matrix valued deformations of the weight

Question: Is it possible to have matrix valued deformations?

A first approach was given in [Ismail, Koelink, R., 2019]. We consider a constant matrix Λ such that

$$\Lambda W(x) = W(x)\Lambda^*.$$

The deformation is now given by:

$$W(x;t) = e^{-t\Lambda x} W(x), \qquad t \in \mathbb{R},$$

The Toda equations are:

$$\begin{aligned} \dot{B}_n(t) &= \Lambda\big(C_n(t) - C_{n+1}(t)\big), \\ \dot{C}_n(t) &= \Lambda\big(C_n(t)B_{n-1}(t) - B_n(t)C_n(t)\big). \end{aligned}$$

Problem: The condition $\Lambda W(x) = W(x)\Lambda^*$ implies that the weight matrix is **reducible**.

Matrix valued deformations of the weight

The classic deformation

$$W(x;t) = e^{-tx}W(x),$$

involves a symmetric operator: x.

Idea: Replace x by more general symmetric operators.

Symmetric operators

We denote by $S(W)$ the space of all symmetric operators of order zero:

$$\mathcal{S}(W) := \{\Lambda(x) \,:\, \Lambda(x)W(x) = W(x)\Lambda(x)^*, \quad \forall x \in [a, b]\}.$$

These operators have the following properties:

- $\langle P \cdot \Lambda(x), Q\rangle = \langle P, Q \cdot \Lambda(x)\rangle$, for all $P, Q \in M_N(\mathbb{C})[x]$.
- By the characterization of the Fourier algebras, we have that $\Lambda \in \mathcal{F}_R(P)$ for all $\Lambda \in S(W)$.
- Then the following holds true:

$$P_n(x) \cdot \Lambda(x) = \psi^{-1}(\Lambda(x)) \cdot P_n(x), \qquad \forall n \in \mathbb{N}.$$

- $\Lambda(x)$ is a matrix valued polynomial:

$$\Lambda(x) = \Lambda_k x^k + \Lambda_{k-1}x^{k-1} + \cdots + \Lambda_0,$$

The deformed weight matrix

Let $\Lambda \in S(W)$. For $t \in \mathbb{R}$ we define:

$$W(x;t) = e^{-t\Lambda(x)} W(x) = e^{-\frac{t}{2}\Lambda(x)} W(x) e^{-\frac{t}{2}\Lambda(x)^*}.$$

Now we have:

- Matrix inner product:

$$\langle P, Q \rangle_t = \int_{\mathbb{R}} P(x) W(x;t) Q(x)^* d\mu(x)$$

- Deformed monic matrix valued polynomials: $(P_n(x;t))_{n \geq 0}$

$$\langle P_n(x;t), P_m(x;t) \rangle_t = \mathcal{H}_n(t) \delta_{n,m},$$

- Deformed Fourier algebras $\mathcal{F}_L(P;t)$ and $\mathcal{F}_R(P;t)$.

Deformed difference operator:

The relation:

$$\Lambda(x)W(x;t) = \Lambda(x)e^{-t\Lambda(x)}W(x) = e^{-t\Lambda(x)}W(x)\Lambda(x)^* = W(x;t)\Lambda(x)^*,$$

implies $\Lambda(x) \in \mathcal{F}_R(P;t)$ for all $t \in \mathbb{R}$.

Therefore we have that:

$$P_n(x;t) \cdot \Lambda(x) = M(t) \cdot P_n(x;t), \qquad n \in \mathbb{N}_0.$$

Remark: $\Lambda(x) = \Lambda(x)^\dagger \quad \Longrightarrow \quad M(t) = M(t)^\dagger$.

The operator $M(t)$ has the form

$$M(t) = \sum_{j=-k}^{k} G_j(n;t)\delta^j.$$

Deformed difference operator:

The operator $M(t)$ has the form

$$M(t) = \sum_{j=-k}^{k} G_j(n;t)\delta^j.$$

Remark: If $\Lambda(x) = x$, then

$$M(t) = \delta^1 + B_n(t)\delta^0 + C_n(t)\delta^{-1},$$

where $B_n(t), C_n(t)$ are the deformed recurrence coefficients.

Goal: Describe the time evolution of the coefficients $G_j(t)$.

Time evolution of the coefficients $G_j(t)$

We recall that:

$$M(t) = G_{-k}(n;t)\delta^{-k} + \cdots + G_k(n;t)\delta^k.$$

The coefficients $G_j(t)$ have the following properties:

- The coefficient G_k of $M(t)$ is independent of t and n.
- The following relation hold:

$$G_\ell(n;t)\mathcal{H}_{n+\ell}(t) = \mathcal{H}_n(t)G_{-\ell}(n+\ell;t)^*, \qquad \ell = -k,\dots,k.$$

- $\dot{\mathcal{H}}_n(t) = -G_0(n;t)\mathcal{H}_n(t) = -\mathcal{H}_n(t)G_0(n;t)^*.$
- $G_m(n;t)\mathcal{H}_{n+m}(t) = \langle P_n(x;t), P_{n+m}(x,t)\cdot\Lambda(x)\rangle_t$

Time evolution of the coefficients $G_j(t)$

Theorem:

The coefficients $G_m(n;t)$ satisfy the following time evolution equations:

$$\dot{G}_m(n;t) = \sum_{j=-k}^{m} G_j(n;t)G_{m-j}(n+j;t) - \sum_{j=0}^{k+m} G_j(n;t)G_{m-j}(n+j;t),$$

if $m = -k, \ldots, -1$, and

$$\dot{G}_m(n;t) = \sum_{j=-k+m}^{-1} G_j(n;t)G_{m-j}(n+j;t) - \sum_{j=m+1}^{k} G_j(n;t)G_{m-j}(n+j;t),$$

if $m = 0, \ldots, k$.

[Deaño, Morey, R., PAMS 2024].

Toda-type lattice equations

Example 1: $k = 1$:

In this case, $\Lambda(x)$ is a polynomial of degree 1.

$$\begin{aligned}
\dot{G}_0(n;t) &= G_{-1}(n;t)G_1(n-1;t) - G_1(n;t)G_{-1}(n+1;t),\\
\dot{G}_{-1}(n;t) &= G_{-1}(n;t)G_0(n-1;t) - G_0(n;t)G_{-1}(n;t),\\
\dot{G}_1(n;t) &= 0.
\end{aligned}$$

Example 1: $k = 1$, $\Lambda(x) = x$: Nonabelian Toda

$$\begin{aligned}
\dot{B}(n;t) &= C(n;t) - C(n+1;t),\\
\dot{C}(n;t) &= C(n;t)B(n-1;t) - B(n;t)C(n;t).
\end{aligned}$$

Toda-type lattice equations

$k = 2$, (M of order five) gives the equations

In this case, $\Lambda(x)$ is a polynomial of degree 2. Therefore:

$$M(t) = G_2(n;t)\delta^2 + G_1(n;t)\delta + G_0(n;t) + G_{-1}(n;t)\delta^{-1} + G_{-2}(n;t)\delta^{-1}.$$

The equations are:

$$\begin{aligned}
\dot{G}_0(n;t) &= G_{-2}(n;t)G_2(n-2;t) + G_{-1}(n;t)G_1(n-1;t) \\
&\quad - G_1(n;t)G_{-1}(n+1;t) - G_2(n;t)G_{-2}(n+2;t), \\
\dot{G}_{-1}(n;t) &= G_{-2}(n;t)G_1(n-2;t) - G_0(n;t)G_{-1}(n;t) \\
&\quad - G_1(n;t)G_{-2}(n+1;t) + G_{-1}(n;t)G_0(n-1;t), \\
\dot{G}_1(n;t) &= G_{-1}(n;t)G_2(n-1;t) - G_2(n;t)G_{-1}(n+2;t), \\
\dot{G}_{-2}(n;t) &= G_{-2}(n;t)G_0(n-2;t) - G_0(n;t)G_{-2}(n;t), \\
\dot{G}_2(n;t) &= 0.
\end{aligned}$$

Lax pair

The time evolution equations can also be written in the form of a Lax pair. Let us consider the block infinite matrices L, L^+, where $i, j \in \mathbb{N}_0$,

$$L_{i,j} = \begin{cases} 0 & |i-j| > k \\ G_{j-i}(i;t) & |i-j| \leq k \end{cases} \qquad L^+_{i,j} = \begin{cases} L_{i,j} & i \leq j \\ 0 & i > j \end{cases}$$

Theorem

The following relation holds

$$\dot{L} = [L, L^+].$$

where $[L, L^+] = LL^+ - L^+L$.

A Hermite–type example

We consider the $N \times N$ Hermite-type weight matrix W supported on $\mathbb{R}$:

$$W^{(a)}(x) = e^{-x^2} e^{xA} e^{xA^*}, \qquad A_{j,k} = \begin{cases} a_k & k = j-1, \\ 0 & \text{otherwise.} \end{cases}$$

The right Fourier algebra $\mathcal{F}_R(P^{(a)})$ contains four relevant differential operators: x, and

$$D^{(a)} = \frac{d}{dx} + A, \qquad (D^{(a)})^\dagger = -D^{(a)} + 2x,$$

$$\mathcal{D}^{(a)} = \frac{d^2}{dx^2} + \frac{d}{dx}(2A - 2x) + A^2 - 2J, \quad J_{ii} = i, \quad J_{ij} = 0,\ i \neq j.$$

Remark: $D^{(a)}$ and $(D^{(a)})^\dagger$ are mutually adjoint and $\mathcal{D}^{(a)}$ is symmetric.

The operators $D^{(a)}$, $(D^{(a)})^\dagger$, $\mathcal{D}^{(a)}$, x and the identity matrix generate a four dimensional Lie algebra (isomorphic to the harmonic oscillator alg.).

A Hermite–type example

The Casimir operator for this Lie algebra is the polynomial of degree one which commutes with $D^{(a)}$, $(D^{(a)})^{\dagger}$, $\mathcal{D}^{(a)}$ and x:

$$\mathcal{C}^{(a)}(x) = \mathcal{D}^{(a)} - \frac{1}{2}(D^{(a)})^{\dagger} D^{(a)} = J - xA,$$

[Deaño, Eijsvoogel, R., Stud. Appl. Math. 2021]

Remark: $\mathcal{C}^{(a)}(x)$ is a symmetric operator and, hence $\mathcal{C}^{(a)}(x) \in \mathcal{S}(W)$.

The deformed weight with respect to $\mathcal{C}^{(a)}(x)$ is :

$$W(x;t) = e^{-t\mathcal{C}^{(a)}(x)} W^{(a)}(x) = e^{-x^2} e^{xA} e^{-tJ} e^{xA^*}.$$

A Hermite–type example

$\mathcal{C}^{(a)}$ acts on $P_n(x,t)$ by a three-term difference operator:

$$P_n(x;t)\cdot\mathcal{C}^{(a)}(x) = G_1(n;t)P_{n+1}(x;t) + G_0(n;t)P_n(x;t) + G_{-1}(n;t)P_{n-1}(x;t).$$

The coefficients can be written in terms of the recurrence coefficients:

$$G_1(n;t) = -A,$$

$$G_0(n;t) = n + J - 2C(n;t) - AB(n;t),$$

$$G_{-1}(n;t) = C(n;t)A - 2C(n;t)B(n-1;t).$$

A Hermite–type example

If we let $N = 2$:

$$W(x;t) = e^{-t\mathcal{C}^{(a)}(x)} = e^{-x^2-t}\begin{pmatrix} 1 & xa \\ xa & x^2a^2 + e^{-t} \end{pmatrix}$$

The monic orthogonal polynomials $P_n(x;t)$ can be written as a matrix linear combination of **scalar Hermite polynomials** as

$$\begin{aligned} 2^n P_n(x;t) = H_n(x) - na \begin{pmatrix} 0 & \frac{2}{a^2+2e^t} \\ 1 & 0 \end{pmatrix} H_{n-1}(x) \\ + n(n-1)\begin{pmatrix} \frac{2a^2}{a^2+2e^t} & 0 \\ 0 & 0 \end{pmatrix} H_{n-2}(x). \end{aligned}$$

The coefficients are: $G_1(n;t) = -A$ and

$$G_0(n;t) = \begin{pmatrix} 2\frac{na^2+e^t}{na^2+2e^t} & 0 \\ 0 & \frac{(n+1)a^2+4e^t}{(n+1)a^2+2e^t} \end{pmatrix}, \qquad G_{-1}(n;t) = \begin{pmatrix} 0 & -\frac{2nae^t}{(na^2+2e^t)^2} \\ 0 & 0 \end{pmatrix}.$$

A Hermite–type example

There are 4 linearly independent symmetric operators of order zero and degree ≤ 1:

$$I, \qquad , x \qquad , \qquad \mathcal{C}^{(a)}, \qquad E = x\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Further questions and remarks:

- The deformation with respect to the operator E is much more complicated
- Is there a characterizations of symmetric operators or order zero?
- Asymptotics of these polynomials as $n \to \infty$.
- Deformation with respect to higher order symmetric operators.