Humbert functions and sums of squares

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Hypergeometric and Orthogonal Polynomials Event

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Introduction: Sums of k squares

Let $k \in \mathbb{N}$ and consider the zeta function attached to the sum of k squares,

$$\zeta_k(s) = \sum_{n_1,\ldots,n_k \in \mathbb{Z}^k \setminus \{\mathbf{0}\}} \frac{1}{\left(n_1^2 + \ldots + n_k^2\right)^s}, \quad \text{Re}(s) > \frac{k}{2}. \qquad (1)$$

When $k = 1$ we are reduced to $\zeta_1(s) = 2\zeta(2s)$. Therefore, the ideas that led Riemann to his study of the functional equation for $\zeta(s)$ can be used to study $\zeta_k(s)$.

Riemann's second proof of the functional equation

Riemann considered two drastically different proofs of the functional equation in his paper. For our purposes, we just give an overview of his second proof. Riemann started with Jacobi's theta functions

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-n^2\pi x}, \qquad \psi(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$$

so that $\theta(x) = 2\psi(x) + 1$. He related $\eta(s)$ with the **Mellin transform** of the Jacobi theta function

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(x)\,dx = \sum_{n=1}^{\infty}\int_0^\infty x^{\frac{s}{2}-1}e^{-n^2\pi x}dx =$$

$$= \sum_{n=1}^{\infty}\frac{1}{(n^2\pi)^{\frac{s}{2}}}\int_0^\infty x^{\frac{s}{2}-1}e^{-x}dx = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) := \eta(s).$$

Next, to connect the left-hand side to what we expect to be $\eta(1-s)$ we need to understand the symmetries of Jacobi's function. Riemann then cites Jacobi's 1829's *Fundamenta Nova Theoriae Functionum Ellipticarum* to present the reflection formula:

Reflection formula for Jacobi's theta function:

The identity

$$\theta(x) = \frac{1}{\sqrt{x}}\theta\left(\frac{1}{x}\right)$$

is valid for $\mathrm{Re}(x) > 0$. An equivalent form is

$$x^{1/2}\left(1 + 2\psi(x)\right) = 1 + 2\psi\left(\frac{1}{x}\right). \quad (2)$$

with $\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}$.

Using Riemann's main ideas

Looking at Riemann's main ideas, one may study the zeta function $\zeta_k(s)$ as a single Dirichlet series.

In this presentation, $r_k(n)$ will always denote the number of representations of a positive integer n as the sum of k squares, counting different signs and different orders of the summands. This allows us to write

$$\zeta_k(s) = \sum_{n_1,\ldots,n_k \neq 0} \frac{1}{\left(n_1^2 + n_2^2 + \ldots + n_k^2\right)^s} := \sum_{n=1}^{\infty} \frac{r_k(n)}{n^s}, \quad \mathrm{Re}(s) > \frac{k}{2}.$$

Note that, when $k = 1$, $r_1(n) = 0$ unless n is a perfect square, in which case $r_1(n) = 2$. Since the functional equation of Riemann's function was attached to Jacobi's theta function $\theta(x)$,....

$$\theta^k(x) = \left(\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}\right)^k = \sum_{n_1\in\mathbb{Z}} \sum_{n_2\in\mathbb{Z}} \ldots \sum_{n_k\in\mathbb{Z}} e^{-\pi\left(n_1^2+n_2^2+\ldots+n_k^2\right)x}$$

$$= 1 + \sum_{n_1,\ldots,n_k \neq 0} e^{-\pi\left(n_1^2+n_2^2+\ldots+n_k^2\right)x} := 1 + \sum_{n=1}^{\infty} r_k(n)\, e^{-\pi\, nx}$$

Thus, for $\mathrm{Re}(s) > \frac{k}{2}$,

$$\int_0^\infty x^{s-1}\left(\theta^k(x) - 1\right)\, dx = \pi^{-s}\Gamma(s)\,\zeta_k(s) := \eta_k(s).$$

Hence, the symmetries of the completed zeta function $\eta_k(s)$ are now revealed if we use

$$\theta^k(x) = x^{-\frac{k}{2}}\theta^k\left(\frac{1}{x}\right),$$

or, in an equivalent form,

$$\sum_{n=0}^{\infty} r_k(n)\, e^{-\pi n x} = x^{-\frac{k}{2}} \sum_{n=0}^{\infty} r_k(n)\, e^{-\pi n/x}, \quad \mathrm{Re}(x) > 0.$$

Hence, $\zeta_k(s)$ can be described as:

- An analytic function everywhere except at the point $s = \frac{k}{2}$;
- Having a functional equation of the form

$$\eta_k(s) := \pi^{-s}\Gamma(s)\,\zeta_k(s) = \pi^{-\left(\frac{k}{2}-s\right)}\Gamma\left(\frac{k}{2} - s\right)\zeta_k\left(\frac{k}{2} - s\right) = \eta_k(k/2-s),$$

Other simple formulas involving $r_k(n)$

Formulas involving $r_k(n)$ (most of them in the case $k = 2$) were relatively standard in the times of Ramanujan and Hardy[1].
For example, for $\mathrm{Re}(x) > 0$, Hardy[2] proved the identity

$$\sum_{n=1}^{\infty} r_2(n)\, e^{-\sqrt{n}x} = \frac{2\pi}{x^2} - 1 + 2\pi x \sum_{n=1}^{\infty} \frac{r_2(n)}{(4\pi^2 n + x^2)^{\frac{3}{2}}},$$

which was employed by him to derive a lower bound for the error term in the circle problem. This formula has a beautiful companion due to Ramanujan,

$$\sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+a}} e^{-2\pi\sqrt{b(n+a)}} = \sum_{n=0}^{\infty} \frac{r_2(n)}{\sqrt{n+b}} e^{-2\pi\sqrt{a(n+b)}}, \quad a, b > 0.$$

[1] B. C. Berndt, A. Dixit, S. Kim, A. Zaharescu, Sums of squares and products of Bessel functions, *Advances in Mathematics*, **338** (2018), 305–338.

[2] G.H. Hardy, On the expression of a number as the sum of two squares, Quart. J. Pure Appl. Math. 46 (1915) 263–283.

A formula of Berndt, Dixit, Kim and Zaharescu[3]

If x, y are two positive numbers such that $x > y$ and $\mathrm{Re}(\nu) > 0$, then the following formula holds

$$\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu} + \sum_{n=1}^{\infty} r_k(n)\, I_{\nu}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}x\right)$$

$$= \frac{2\Gamma\left(\frac{k}{2}+\nu\right)}{\pi^{k/2}\Gamma(\nu+1)} \sum_{n=0}^{\infty} r_k(n) \frac{\left(\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}\right)^{k-2}}{\left(n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2\right)^{\frac{k-1}{2}}}$$

$$\times \left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{\nu}$$

$$\times\, {}_2F_1\left(1-\frac{k}{2}+\nu,\, 1-\frac{k}{2};\, \nu+1;\, \left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{2}\right).$$

[3] B. C. Berndt, A. Dixit, S. Kim, A. Zaharescu, Sums of squares and products of Bessel functions, *Advances in Mathematics*, **338** (2018), 305–338.

Some interesting corollaries

1. Take $y \to 0^+$ on both sides of the previous formula and use the limiting relation

$$\lim_{y\to 0} y^{-\nu} I_\nu(y) = \frac{2^{-\nu}}{\Gamma(\nu+1)}.$$

Then the following identity of Popov[4] holds

$$\frac{x^\nu\,\Gamma\left(\nu+\frac{k}{2}\right)}{2\pi^{\nu+\frac{k}{2}}}\sum_{n=0}^{\infty}\frac{r_k(n)}{(n+x^2)^{\nu+\frac{k}{2}}}=\frac{\Gamma(\nu)}{2\pi^\nu x^\nu}+\sum_{n=1}^{\infty}r_k(n)\,n^{\frac{\nu}{2}}K_\nu\left(2\pi\sqrt{n}\,x\right), \quad (3)$$

valid for $\mathrm{Re}(\nu)>0$ and $x>0$. This can be also seen as a consequence of the Chowla-Selberg formula and Berndt's generalized Bessel expansion of Hecke Dirichlet series[5].

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[4] A.I. Popov, Uber die zylindrische Funktionen enthaltenden Reihen, C. R. Acad. Sci. URSS 2 (1935) 96–99 (in Russian).

[5] B. C. Berndt, Generalized Dirichlet series and Hecke's functional equation, *Proc. Edinburgh Math. Soc.*, **15** (1967), 309–313.

2. For $\nu = \frac{1}{2}$, let us use the particular cases for $I_\nu(x)$ and $K_\nu(x)$

$$I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sinh(x), \quad K_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}}\,e^{-x}.$$

One is able to obtain the identity

$$\begin{aligned} 2\pi y + \sum_{n=1}^{\infty} \frac{r_k(n)}{\sqrt{n}} e^{-2\pi\sqrt{n}x} \sinh\left(2\pi\sqrt{n}y\right) \\ = \frac{\Gamma\left(\frac{k-1}{2}\right)}{2\pi^{\frac{k-1}{2}}} \left\{\frac{1}{(x-y)^{k-1}} - \frac{1}{(x+y)^{k-1}}\right\} \\ + \frac{\Gamma\left(\frac{k-1}{2}\right)}{2\pi^{\frac{k-1}{2}}} \sum_{n=1}^{\infty} \left\{\frac{r_k(n)}{\left(n+(x-y)^2\right)^{\frac{k-1}{2}}} - \frac{r_k(n)}{\left(n+(x+y)^2\right)^{\frac{k-1}{2}}}\right\}, \end{aligned}$$

also due to Popov (when $k = 2$)[6].

[6] B. C. Berndt, A. Dixit, S. Kim, A. Zaharescu, Sums of squares and products of Bessel functions, *Advances in Mathematics*, **338** (2018), 305–338.

A common Genesis? The transformation formula for the $\theta-$function

The proof of the formula stated three slides ago uses three main ingredients:

1. An analogue of Voronoï's summation formula due to Guinand[7];
2. The evaluation of a Hankel transform due to Koshliakov[8];
3. Several transformation formula concerning the hypergeometric function ${}_2F_1(a, b; c; z)$.

However, the corollaries previously stated can be proved via the transformation formula for the $\theta-$function,

$$\sum_{n=0}^{\infty} r_k(n)\, e^{-\pi n x} = x^{-\frac{k}{2}} \sum_{n=0}^{\infty} r_k(n)\, e^{-\pi n/x}.$$

[7] A.P. Guinand, Summation formulae and self-reciprocal functions (II), Quart. J. Math. 1 (1939) 104–118.

[8] N.S. Koshliakov, On a certain definite integral connected with the cylindric function $J_\mu(x)$, C. R. Acad. Sci. URSS 2 (1934) 145–147.

A Common Genesis: Why are the indices of the Bessel functions the same?

"The summands in (1.31) [formula above] contain a product of Bessel functions $I_\nu(X)$ and $K_\nu(x)$. Dixon and Ferrar obtained integral representations for the product $I_\mu(X)\,K_\nu(x)$, where the orders μ and ν are not necessarily equal. Therefore, perhaps exists a more general transformation than our formula."[a]

[a] B. C. Berndt, A. Dixit, S. Kim, A. Zaharescu, Sums of squares and products of Bessel functions, *Advances in Mathematics*, **338** (2018), 305–338.

Why we care about $\theta-$functions: Zeros shifted Combinations

Inspired by some remarks written by Dixit, Kumar, Maji and Zaharescu [9], our first approach to these kind of summation formulas was the study the number of zeros of a function of the form

$$F_{z,k}(s) = \sum_{j\neq 0} c_j \, \pi^{-(s+i\lambda_j)} \Gamma\left(s+i\lambda_j\right) \, \zeta_k(s+i\lambda_j) \, \mathrm{Re}\left\{ {}_1F_1\left(\frac{k}{2}-s-i\lambda_j;\, \frac{k}{2};\, \frac{z^2}{4}\right)\right\}$$

where we take the convention that $\lambda_{-j} := -\lambda_j$ and where ${}_1F_1(a;c;x)$ denotes the confluent hypergeometric function

$${}_1F_1(a;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!}.$$

[9] A. Dixit, R. Kumar, B. Maji and A. Zaharescu, Zeros of combinations of the Riemann Ξ-function and the confluent hypergeometric function on bounded vertical shifts, *J. Math. Anal. Appl.* **466** (2018), 307–323.

Here comes Hankel!

The main idea, essentially dating back to Hardy[10], is to study an integral involving $F_{k,z}(s)$ with an analogue of the theta function.
This is done once we invoke Hankel's formula[11]

$$\int_0^\infty t^{s-1} e^{-p^2 t^2} J_\nu(at)\, dt = \frac{\Gamma\left(\frac{s+\nu}{2}\right) a^\nu}{2^{\nu+1} p^{s+\nu} \Gamma(\nu+1)} \, {}_1F_1\left(\frac{s+\nu}{2};\, \nu+1;\, -\frac{a^2}{4p^2}\right), \tag{4}$$

valid for $\mathrm{Re}(s) > -\mathrm{Re}(\nu)$, $\mathrm{Re}(p) > 0$.

[10] 4G. H. Hardy, Sur les zeros de la fonction $\zeta(s)$ de Riemann, Comptes rendus, 158 (1914), 1012–1014.

[11] P. Ribeiro, S. Yakubovich, Certain extensions of results of Siegel, Wilton and Hardy, *Adv. Appl. Math.*, in press.

Theorem (R., Yakubovich) (2023):

The following $\theta-$transformation formula takes place

$$\sqrt{x}\,e^{z^2/8}\sum_{n=1}^{\infty} r_k(n)\,n^{\frac{1}{2}-\frac{k}{4}}\,e^{-\pi n x}\,J_{\frac{k}{2}-1}(\sqrt{\pi\,n\,x}\,z) - \left(\sqrt{\frac{\pi}{x}}\,\frac{z}{2}\right)^{\frac{k}{2}-1}\frac{e^{-z^2/8}}{\Gamma\left(\frac{k}{2}\right)\sqrt{x}}$$
$$= \frac{e^{-z^2/8}}{\sqrt{x}}\sum_{n=1}^{\infty} r_k(n)\,n^{\frac{1}{2}-\frac{k}{4}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{k}{2}-1}\left(\sqrt{\frac{\pi n}{x}}\,z\right) - \left(\sqrt{\pi x}\,\frac{z}{2}\right)^{\frac{k}{2}-1}\frac{\sqrt{x}\,e^{z^2/8}}{\Gamma\left(\frac{k}{2}\right)}. \tag{5}$$

Moreover, both sides of the previous formula are equal to the integral:

$$\left(\frac{\sqrt{\pi}z}{2}\right)^{\frac{k}{2}-1}\frac{e^{z^2/8}}{2\pi\,\Gamma\left(\frac{k}{2}\right)}$$
$$\times\int_{-\infty}^{\infty}\pi^{-\frac{k}{4}-it}\Gamma\left(\frac{k}{4}+it\right)\zeta_k\left(\frac{k}{4}+it\right)\,{}_1F_1\left(\frac{k}{4}+it;\,\frac{k}{2};\,-\frac{z^2}{4}\right)x^{-it}\,dt.$$

The idea of the proof

Using Hankel's integral formula, we may start with the formula

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{1}{2}-\frac{k}{4}}\, e^{-\pi n x}\, J_{\frac{k}{2}-1}\left(\sqrt{\pi n}\, y\right) = \frac{(y/2)^{\frac{k}{2}-1}}{2\pi i\, x^{\frac{k}{4}-\frac{1}{2}}\Gamma\left(\frac{k}{2}\right)}$$

$$\int_{\sigma-i\infty}^{\sigma+i\infty} \pi^{-s}\,\Gamma\left(s+\frac{k}{4}-\frac{1}{2}\right)\zeta_k\left(s+\frac{k}{4}-\frac{1}{2}\right){}_1F_1\left(s+\frac{k}{4}-\frac{1}{2};\,\frac{k}{2};\,-\frac{y^2}{4x}\right)x^{-s}ds, \tag{6}$$

where σ is some large real number which assures that $\zeta_k(\sigma+\frac{\nu}{2})$ converges absolutely!

The proof of the generalized transformation formula comes from a combination of three steps:

1. Shift the line of integration to a region where $\zeta_k(s)$ "is" on the critical line $\mathrm{Re}(s) = \frac{k}{4}$;
2. Use the functional equation for $\zeta_k(s)$,

$$\pi^{-s}\Gamma(s)\zeta_k(s) = \pi^{-\left(\frac{k}{2}-s\right)}\Gamma\left(\frac{k}{2}-s\right)\zeta_k\left(\frac{k}{2}-s\right).$$

3. Use Kummer's transformation for ${}_1F_1$,

$$ {}_1F_1\left(a;\, c;\, x\right) = e^{x}\,{}_1F_1\left(c-a;\, c;\, -x\right), \tag{7}$$

we get

$$ {}_1F_1\left(s+\frac{k}{4}-\frac{1}{2};\, \frac{k}{2};\, -\frac{y^2}{4x}\right) = e^{-y^2/4x}\,{}_1F_1\left(\frac{k}{4}-s+\frac{1}{2};\, \frac{k}{2};\, \frac{y^2}{4x}\right)$$

4. Change the variables as $s \leftrightarrow \frac{k}{2}-s$, return to the region of absolute convergence of $\zeta_k(s)$ and reverse the process leading to Hankel's formula. Since the argument $-y^2$ was converted into y^2, the $J-$Bessel function will be replaced by an $I-$Bessel function!

There are two main consequences of this approach:

1. From the integral representation, one can devise an argument to count the number of zeros of any shifted combination of the form[12]

$$\sum_{j\neq 0} c_j\,\pi^{-(s+i\lambda_j)}\Gamma\left(s+i\lambda_j\right)\,\zeta_k\left(s+i\lambda_j\right)\,\mathrm{Re}\left\{{}_1F_1\left(\frac{k}{2}-s-i\lambda_j;\,\frac{k}{2};\,\frac{z^2}{4}\right)\right\};$$

2. The transformation formula without the integral is[13]

$$\sum_{n=1}^{\infty} r_k(n)\,n^{\frac{1}{2}-\frac{k}{4}}\,e^{-\pi n x}\,J_{\frac{k}{2}-1}(\sqrt{\pi n}y) = -\frac{y^{\frac{k}{2}-1}\pi^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)}+\frac{y^{\frac{k}{2}-1}\pi^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{2}-1}x^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)}\,e^{-\frac{y^2}{4x}}$$
$$+\frac{e^{-\frac{y^2}{4x}}}{x}\sum_{n=1}^{\infty} r_k(n)n^{\frac{1}{2}-\frac{k}{4}}\,e^{-\frac{\pi n}{x}}\,I_{\frac{k}{2}-1}\left(\frac{\sqrt{\pi n}\,y}{x}\right), \qquad (8)$$

valid for $\mathrm{Re}(x)>0$ and any $y\in\mathbb{C}$.

[12] P. Ribeiro, Estimates for the number of zeros of shifted combinations of completed Dirichlet series , arXiv:2401.02813

[13] A. Popov, On some summation formulas (in Russian), Bull. Acad. Sci. L'URSS 7 (1934), 801–802.

If we let $y \to 0$ in the previous formula and use the limiting relations

$$\lim_{y\to 0} y^{-\nu} J_\nu(y) = \lim_{y\to 0} y^{-\nu} I_\nu(y) = \frac{2^{-\nu}}{\Gamma(\nu+1)},$$

then the previous formula gives

$$\sum_{n=0}^{\infty} r_k(n)\, e^{-\pi n x} = x^{-\frac{k}{2}} \sum_{n=0}^{\infty} r_k(n)\, e^{-\pi n/x}, \quad \mathrm{Re}(x) > 0. \tag{9}$$

But we have emphasized that (9) implies the Bessel expansion

$$\frac{x^{\nu}\,\Gamma\left(\nu+\frac{k}{2}\right)}{2\pi^{\nu+\frac{k}{2}}} \sum_{n=0}^{\infty} \frac{r_k(n)}{(n+x^2)^{\nu+\frac{k}{2}}} = \frac{\Gamma(\nu)}{2\pi^{\nu}x^{\nu}} + \sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu}{2}} K_\nu\left(2\pi\sqrt{n}\,x\right). \tag{10}$$

Question:

What is the generalization of (10) in this setting?

Theorem (R., Yakubovich) (2023)

For any $\mathrm{Re}(\nu) > 0$ and $x > y > 0$, the following transformation formula holds

$$\frac{2\,\Gamma\left(\frac{k}{2}\right)\,\pi^{\nu+1}x^{-\nu}y^{1-\frac{k}{2}}}{\Gamma\left(\nu+\frac{k}{2}\right)}\sum_{n=1}^{\infty} r_k(n)\,n^{\frac{\nu+1}{2}-\frac{k}{4}}\,I_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right)$$

$$=\frac{1}{(x^2-y^2)^{\nu+\frac{k}{2}}}-\frac{\pi^{\frac{k}{2}}\,\Gamma(\nu)\,x^{-2\nu}}{\Gamma(s)}$$

$$+\sum_{n=1}^{\infty}\frac{r_k(n)}{(n+x^2-y^2)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4},\,\frac{\nu+1}{2}+\frac{k}{4};\,\frac{k}{2};\,-\frac{4n\,y^2}{(n+x^2-y^2)^2}\right)$$

Déjà vu?

Under the same conditions, we have seen previously the formula

$$\frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu}+\sum_{n=1}^{\infty} r_k(n)\, I_{\nu}\left(2\pi\sqrt{n}y\right)\, K_{\nu}\left(2\pi\sqrt{n}x\right)$$

$$=\frac{2\Gamma\left(\frac{k}{2}+\nu\right)}{\pi^{k/2}\Gamma(\nu+1)}\sum_{n=0}^{\infty} r_k(n)\,\frac{\left(\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}\right)^{k-2}}{\left(n^2+2n\left(x^2+y^2\right)+\left(x^2-y^2\right)^2\right)^{\frac{k-1}{2}}}$$

$$\times\left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{\nu}$$

$$\times\,{}_2F_1\left(1-\frac{k}{2}+\nu,\ 1-\frac{k}{2};\ \nu+1;\ \left(\frac{\sqrt{n+(x+y)^2}-\sqrt{n+(x-y)^2}}{\sqrt{n+(x+y)^2}+\sqrt{n+(x-y)^2}}\right)^{2}\right).$$

What are the differences?

Some new particular cases

Taking the limit $\nu \to 0^+$ in our formula and using an argument of analytic continuation, it is possible to get a transformation formula for the infinite series

$$\sum_{n=1}^{\infty} r_k(n)\, I_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\, K_0\left(2\pi\sqrt{n}\,x\right),$$

which seem to be "nearly impossible" (due to elaborated integral transforms!) to establish via other methods. For example, when $k=1$, such a formula gives, for $x>y>0$,

$$2\sum_{n=1}^{\infty}\left\{\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{n^2+x^2-y^2+\sqrt{n^4+2n^2\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}}}{\sqrt{n^4+2n^2\left(x^2+y^2\right)+\left(x^2-y^2\right)^2}}-\frac{1}{n}\right\}$$

$$=-\frac{1}{\sqrt{x^2-y^2}}+2\log\left(\frac{2}{x}\right)-2\gamma+4\sum_{n=1}^{\infty}\cosh\left(2\pi ny\right)K_0\left(2\pi nx\right).$$

Other identity obtained by the same principle is the following. Let $\tau(n)$ denote Ramanujan's $\tau-$function, defined by the generating function

$$\sum_{n=1}^{\infty}\tau(n)\,q^{n}:=q\,\prod_{n=1}^{\infty}(1-q^{n})^{24},\quad q:=\exp(2\pi iz),\ \ \mathrm{Im}(z)>0.$$

If $x,y>0$ are such that $x>y$, then the the following formula takes place

$$\sum_{n=1}^{\infty}\frac{\tau(n)\left(n^{2}+2n\left(x^{2}+y^{2}\right)+\left(x^{2}-y^{2}\right)^{2}\right)^{-\frac{1}{2}}}{\left(n+x^{2}-y^{2}+\sqrt{n^{2}+2n\,(x^{2}+y^{2})+(x^{2}-y^{2})^{2}}\right)^{11}}$$

$$=\frac{4\pi}{(2y)^{11}}\sum_{n=1}^{\infty}\tau(n)\,n^{-\frac{11}{2}}\,I_{11}\left(4\pi\sqrt{n}y\right)\,K_{0}\left(4\pi\sqrt{n}\,x\right).$$

Let us note that we have proved a summation formula for

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu+1}{2}-\frac{k}{4}}\, I_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\, K_{\nu}\left(2\pi\sqrt{n}\,x\right)$$

motivated by a transformation formula that starts with an infinite sum of the form

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{1}{2}-\frac{k}{4}}\, e^{-\pi n x}\, J_{\frac{k}{2}-1}(\sqrt{\pi n}y)$$

and ends up with an infinite sum of the form

$$\sum_{n=1}^{\infty} r_k(n) n^{\frac{1}{2}-\frac{k}{4}}\, e^{-\frac{\pi n}{x}}\, I_{\frac{k}{2}-1}\left(\frac{\sqrt{\pi n}\,y}{x}\right).$$

It is somewhat clear that the index of J and I Bessel functions are connected in all these formulas.

Thus, if we want to prove a transformation formula involving $r_k(n)$ and the more general product

$$I_\mu(2\pi\sqrt{n}y)\, K_\nu\left(2\pi\sqrt{n}\,x\right),$$

we need to look at a summation formula for

$$\sum_{n=1}^{\infty} r_k(n)\, n^{-\frac{\mu}{2}}\, e^{-\pi x n}\, J_\mu(\sqrt{\pi n}\, y),$$

where μ is no longer dependent on k!

Bad news for the "symmetrists":

The formula that we seek does not end up in

$$\sum_{n=1}^{\infty} r_k(n)\, n^{-\frac{\mu}{2}}\, e^{-\pi n/x}\, I_\mu(\sqrt{\pi n}\, y/x)!$$

Theorem (R.) (2023):

For any $\mathrm{Re}(\nu) > -1$ and $\mathrm{Re}(x) > 0$, $y \in \mathbb{C}$, the following summation formula holds

$$\pi^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} r_k(n)\, n^{-\frac{\nu}{2}}\, e^{-\pi x n}\, J_{\nu}(\sqrt{\pi n}\, y)$$

$$= -\frac{y^{\nu}}{2^{\nu}\Gamma(\nu+1)} + \frac{\pi^{\frac{k}{2}} y^{\nu} x^{-\frac{k}{2}}}{2^{\nu}\Gamma(\nu+1)}\, {}_1F_1\left(\frac{k}{2};\, \nu+1;\, -\frac{y^2}{4x}\right)$$

$$+\frac{y^{\nu} x^{-\frac{k}{2}}}{2^{\nu}\Gamma(\nu+1)} e^{-\frac{y^2}{4x}} \sum_{n=1}^{\infty} r_k(n) e^{-\frac{\pi n}{x}}\, \Phi_3\left(1-\frac{k}{2}+\nu; \nu+1; \frac{y^2}{4x}, \frac{\pi y^2 n}{4x^2}\right),$$

where $\Phi_3(b; c; w, z)$ is the usual Humbert function,

$$\Phi_3(b;\, c;\; w, z) = \sum_{k,m=0}^{\infty} \frac{(b)_k}{(c)_{k+m}} \frac{w^k z^m}{k!\; m!},$$

which converges absolutely for any $w, z \in \mathbb{C}$.

Note that the series on the right-hand side converges absolutely because

$$|\Phi_3(b;\, c;\; w, z)| \leq e^{2\sqrt{|z|}+2|w|+\frac{|bw|}{|c|}}.$$

Connection formulas for Humbert functions

Let $\Psi_2(a; b, c; w, z)$ denote the Humbert function

$$\Psi_2(a; b, c; w, z) := \sum_{k,m=0}^{\infty} \frac{(a)_{k+m}}{(b)_k (c)_m} \frac{w^k z^m}{k!\, m!}. \qquad (11)$$

If $b, c \notin \mathbb{N}_0^-$, then Ψ_2 and Φ_3 are connected via the transformation formula[a]

$$\Psi_2\left(b; b, c; w, z\right) = e^{w+z} \Phi_3\left(c - b; c; -z, wz\right).$$

[a] V. V. Manako, A connection formula between double hypergeometric series Ψ_2 and Φ_3, *Int. Transf. Spec. Funct.*, 23, 503-508.

Thus, our previous formula can be rewritten as

$$\pi^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} r_k(n)\, n^{-\frac{\nu}{2}}\, e^{-\pi n x}\, J_\nu(\sqrt{\pi n} y) = -\frac{y^\nu}{2^\nu \Gamma(\nu+1)} + \frac{y^\nu x^{-\frac{k}{2}} \pi^{\frac{k}{2}}}{2^\nu \Gamma(\nu+1)}\, {}_1F_1\left(\frac{k}{2};\, \nu+1;\, -\frac{y^2}{4x}\right) + \frac{y^\nu x^{-\frac{k}{2}}}{2^\nu \Gamma(\nu+1)} \sum_{n=1}^{\infty} r_k(n)\, \Psi_2\left(\frac{k}{2};\, \frac{k}{2}, \nu+1; -\frac{\pi n}{x},\, -\frac{y^2}{4x}\right),$$

$$\pi^{-\frac{\nu}{2}} \sum_{n=1}^{\infty} r_k(n)\, n^{-\frac{\nu}{2}}\, e^{-\pi n x}\, I_\nu(\sqrt{\pi n} y) = -\frac{y^\nu}{2^\nu \Gamma(\nu+1)} + \frac{y^\nu x^{-\frac{k}{2}} \pi^{\frac{k}{2}}}{2^\nu \Gamma(\nu+1)}\, {}_1F_1\left(\frac{k}{2};\, \nu+1;\, \frac{y^2}{4x}\right) + \frac{y^\nu x^{-\frac{k}{2}}}{2^\nu \Gamma(\nu+1)} \sum_{n=1}^{\infty} r_k(n)\, \Psi_2\left(\frac{k}{2};\, \frac{k}{2}, \nu+1; -\frac{\pi n}{x},\, \frac{y^2}{4x}\right),$$

In this new general framework, Humbert functions play the role of the Bessel functions in the theta transformation formula....

The Humbert functions can be seen as two variable generalizations of the Bessel functions...

What function will replace the Gauss' hypergeometric function in our previous formulas?

The Appell F4-function

One of the extensions of the hypergeometric function ${}_2F_1\left(a, b; c; x\right)$ to two variables is the Appell F_4 function, defined as the double infinite series

$$F_4\left(\alpha, \beta; \gamma, \gamma'; x, y\right) := \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{m!\, n!\, (\gamma)_m(\gamma')_n} x^m y^n, \quad \sqrt{|x|} + \sqrt{|y|} < 1.$$

Theorem (R.) (2023):

Assume that $\mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0$ and x, y are two positive real numbers such that $x > y$. Then the following identity holds

$$\pi^{\frac{\nu-\mu}{2}} \sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu-\mu}{2}}\, I_{\mu}\left(2\sqrt{\pi n y}\right) K_{\nu}\left(2\sqrt{\pi n}\, x\right)$$

$$= -\frac{y^{\nu}\Gamma(\nu)}{2x^{\nu}\Gamma(\mu+1)} + \frac{x^{-\nu-k}y^{\nu}}{2}\frac{\Gamma(\nu+1)}{\Gamma(\mu+1)}\,{}_2F_1\left(\frac{k}{2}, \nu+1;\ \mu+1;\ \frac{y^2}{x^2}\right)$$

$$+\frac{y^{\mu}\Gamma\left(\nu+\frac{k}{2}\right)}{2x^{\nu+k}\Gamma(\mu+1)} \sum_{n=1}^{\infty} r_k(n)\, F_4\left(\frac{k}{2}, \nu+\frac{k}{2};\ \frac{k}{2}, \mu+1;\ -\frac{\pi n}{x^2}, \frac{y^2}{x^2}\right). \quad (12)$$

The convergence of the series containing the Appell F_4 terms is assured by the transformation formula,

$$
\begin{aligned}
& F_4\left(\nu+\frac{k}{2}; \frac{k}{2}; \mu+1,\, \frac{k}{2}; \frac{y^2}{x^2}, -\frac{\pi n}{x^2}\right) \\
= \left(\frac{\pi n}{x^2}\right)^{-\nu-\frac{k}{2}} & F_4\left(\nu+\frac{k}{2}; \nu+1; \mu+1; \nu+1; -\frac{y^2}{\pi n}, -\frac{x^2}{\pi n}\right) \\
& = \frac{x^{2\nu+k}}{\pi^{\nu+\frac{k}{2}}}\, n^{-\nu-\frac{k}{2}} \left\{1+O\left(\frac{1}{n}\right)\right\}.
\end{aligned}
$$

Thus, the series is assured if $\mathrm{Re}(\nu) > 0$.

The idea of the proof

The proof uses the generalization of the theta transformation formula involving Humbert functions. Let us look at the left-hand side of it: we can write it in the form

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{\nu-\mu}{2}}\; I_{\mu}\left(2\sqrt{\pi n y}\right)\; K_{\nu}\left(2\sqrt{\pi n}\, x\right)$$

$$= \frac{x^{\nu}}{2\pi^{\frac{\mu}{2}}} \int_{0}^{\infty} t^{\nu-1} e^{-x^2 t} \sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{\mu}{2}}}\, e^{-\frac{\pi n}{t}} I_{\mu}\left(2\sqrt{\pi n}\, y\right)\, dt,$$

where the essential ingredient here is essentially absolute convergence and the well-known integral representation for the Macdonald function

$$\int_{0}^{\infty} t^{\nu-1} e^{-\beta t}\, e^{-\frac{\gamma}{t}}\, dt = 2\left(\frac{\gamma}{\beta}\right)^{\nu/2} K_{\nu}\left(2\sqrt{\beta\,\gamma}\right), \quad \mathrm{Re}(\beta),\, \mathrm{Re}(\gamma) > 0.$$

The formula follows if we use the previous transformation

$$\sum_{n=1}^{\infty} r_k(n)\, n^{-\frac{\mu}{2}}\, e^{-\pi n x}\, I_\mu(2\sqrt{\pi n} y) = -\frac{y^\mu \pi^{\frac{\nu}{2}}}{\Gamma(\mu+1)}$$

$$+\frac{y^\mu x^{-\frac{k}{2}} \pi^{\frac{\nu+k}{2}}}{\Gamma(\mu+1)}\, {}_1F_1\left(\frac{k}{2};\, \mu+1;\, \frac{y^2}{4x}\right)$$

$$+\frac{\pi^{\frac{\nu}{2}} y^\nu x^{-\frac{k}{2}}}{\Gamma(\mu+1)} \sum_{n=1}^{\infty} r_k(n)\, \Psi_2\left(\frac{k}{2};\, \frac{k}{2}, \mu+1; -\frac{\pi n}{x},\, \frac{y^2}{4x}\right),$$

together with the Mellin transform

$$\int_0^\infty x^{s-1} e^{-px} \Psi_2\left(a; c, c'; wx, zx\right)\, dx = \frac{\Gamma(s)}{p^s} F_4\left(s, a; c, c'; \frac{w}{p}, \frac{z}{p}\right),$$

which is valid for $\mathrm{Re}(p) > \max\left\{|\mathrm{Re}(w)|, |\mathrm{Re}(z)|\right\}$ and $\mathrm{Re}(s) > 0$.

Reduction formulas

What is the most famous reduction formula for F_4?
Most people say that it is Bailey's reduction formula[14]

$$\begin{aligned} & F_4\left(\alpha, \beta; \gamma, \alpha+\beta-\gamma+1; w(1-z), z(1-w)\right) \\ = & {}_2F_1\left(\alpha, \beta; \gamma; w\right) {}_2F_1\left(\alpha, \beta; \alpha+\beta-\gamma+1; z\right) \end{aligned} \tag{13}$$

and indeed we can use (13) to deduce new formulas about concerning the product of Hypergeometric functions and sums of squares....

but making an excuse à la Fermat...

[14] On the Reducibility of Appell's function F_4, Quart. Journ. Math. 5 (1934), 291-292.

Using the reduction formula

$$\begin{aligned} F_4\left(\alpha,\beta;1+\alpha-\beta,\beta;-\frac{w}{(1-w)(1-z)},-\frac{z}{(1-w)(1-z)}\right) \\ =(1-z)^{\alpha}\,{}_2F_1\left(\alpha,\beta;1+\alpha-\beta;-\frac{w(1-z)}{1-w}\right) \\ =(1-w)^{\alpha}\ {}_2F_1\left(\alpha,\beta;1+\alpha-\beta;-\frac{z(1-w)}{1-z}\right), \end{aligned}$$

it is also a matter of simple (but tedious!) computations and transformations to rederive, from our general formula, the identity of Berndt, Dixit, Kim and Zaharescu....

Using yet another standard reduction formula

$$F_4\left(\alpha, \beta; \beta, \beta; -\frac{w}{(1-w)(1-z)}, -\frac{z}{(1-w)(1-z)}\right) = (1-w)^{\alpha}(1-z)^{\alpha}\,{}_2F_1\left(\alpha, 1+\alpha-\beta; \beta; wz\right),$$

it is a matter of simple computations to check that, when $\mu = \frac{k}{2} - 1$, our general theorem gives the formula

$$\frac{2\,\Gamma\left(\frac{k}{2}\right)\,\pi^{\nu+1}x^{-\nu}y^{1-\frac{k}{2}}}{\Gamma\left(\nu+\frac{k}{2}\right)}\sum_{n=1}^{\infty} r_k(n)\,n^{\frac{\nu+1}{2}-\frac{k}{4}}\,I_{\frac{k}{2}-1}(2\pi\sqrt{n}y)\,K_{\nu}\left(2\pi\sqrt{n}\,x\right) = \frac{1}{(x^2-y^2)^{\nu+\frac{k}{2}}} - \frac{\pi^{\frac{k}{2}}\,\Gamma(\nu)\,x^{-2\nu}}{\Gamma(s)}$$

$$+\sum_{n=1}^{\infty}\frac{r_k(n)}{(n+x^2-y^2)^{\nu+\frac{k}{2}}}\,{}_2F_1\left(\frac{\nu}{2}+\frac{k}{4}, \frac{\nu+1}{2}+\frac{k}{4}; \frac{k}{2}; -\frac{4n\,y^2}{(n+x^2-y^2)^2}\right)$$

which is nothing more the formula that motivated this whole work!

There are at least four nontrivial reduction formulas for Appell's $F4$ function.

However, there are also other similar formulas that can be deduced under a similar setting. If x, y are two positive real numbers such that $x > y > 0$, we have the transformation formula

$$\sum_{n=1}^{\infty} \frac{r_k(n)}{n^{\frac{k}{4}-\frac{1}{2}}} e^{-\pi n x} I_{\frac{k}{4}-\frac{1}{2}}(\pi n y) = \frac{1}{\Gamma\left(\frac{k}{4}+\frac{1}{2}\right)} \left(\frac{\pi y}{2}\right)^{\frac{k}{4}-\frac{1}{2}} \left\{ \frac{1}{(x^2-y^2)^{\frac{k}{4}}} - 1 \right\} + \frac{1}{\sqrt{x^2-y^2}} \sum_{n=1}^{\infty} r_k(n) n^{\frac{1}{2}-\frac{k}{4}} e^{-\frac{\pi n x}{x^2-y^2}} I_{\frac{k}{4}-\frac{1}{2}}\left(\frac{\pi n y}{x^2-y^2}\right),$$

where $x, y > 0$.

Note that this is not a consequence or reformulation of the previous identity

$$\sum_{n=1}^{\infty} r_k(n)\, n^{\frac{1}{2}-\frac{k}{4}}\, e^{-\pi n x}\, J_{\frac{k}{2}-1}(\sqrt{\pi n} y) = -\frac{y^{\frac{k}{2}-1} \pi^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{2}-1}\Gamma\left(\frac{k}{2}\right)} + \frac{y^{\frac{k}{2}-1} \pi^{\frac{k}{4}-\frac{1}{2}}}{2^{\frac{k}{2}-1} x^{\frac{k}{2}} \Gamma\left(\frac{k}{2}\right)} e^{-\frac{y^2}{4x}} + \frac{e^{-\frac{y^2}{4x}}}{x} \sum_{n=1}^{\infty} r_k(n) n^{\frac{1}{2}-\frac{k}{4}} e^{-\frac{\pi n}{x}} I_{\frac{k}{2}-1}\left(\frac{\sqrt{\pi n}\, y}{x}\right)!$$

For example, let us note that, for $k = 4$,

$$2\pi y + \sum_{n=1}^{\infty} \frac{r_4(n)}{n} \left\{ e^{-\pi n(x-y)} - e^{-\pi n(x+y)} \right\}$$
$$= \frac{2\pi y}{x^2 - y^2} + \sum_{n=1}^{\infty} \frac{r_4(n)}{n} \left\{ e^{-\frac{\pi n}{x+y}} - e^{-\frac{\pi n}{x-y}} \right\},$$

valid whenever $x > y > 0$.
Another curious consequence is, for $x > y > 0$,

$$\sqrt{x^2 - y^2} \left(1 + \sum_{n=1}^{\infty} r_2(n)\, e^{-\pi n x} I_0 (\pi n y) \right)$$
$$= 1 + \sum_{n=1}^{\infty} r_2(n)\, e^{-\frac{\pi n x}{x^2 - y^2}} I_0 \left(\frac{\pi n y}{x^2 - y^2} \right).$$

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Dankjewel!