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Multiple orthogonal polynomials with hypergeometric moment generating functions

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Overview

MOP with hypergeometric moment generating functions:

1. Motivation
2. Bessel-like setting
3. Extended Bessel-like setting
4. Extended Jacobi/Laguerre-like setting

Motivation

Hermite (1873): $\mathbb{Q}$-linearly independence of $1, e^{c_1 z}, \ldots, e^{c_r z}$ for distinct $c_i, z \in \mathbb{Q}_0$
$\longrightarrow$ transcendence of e

Idea:

1. construct simultaneous rational approximants to $e^{c_1 z}, \ldots, e^{c_r z}$
   $\longrightarrow$ use Hermite-Padé approximation
2. show that their quality is good enough to conclude $\mathbb{Q}$-linearly independence
   $\longrightarrow$ use Nesterenko's criterion

Motivation: Hermite's proof - Hermite-Padé approximation

Idea: **simultaneously approximate** r generating functions f_j of weights w_j at infinity

$$f_j(z) = \sum_{k\geq 0} \frac{m_{j,k}}{z^{k+1}}, \quad m_{j,k} = \int_\Lambda w_j(x) x^k dx$$

Possibilities:

- (type I) approximate a polynomial combination of the f_j by a polynomial

$$\sum_{j=1}^{r} A_{\vec{n},j}(z) f_j(z) - B_{\vec{n}}(z) = O(z^{-|\vec{n}|}), \; z \to \infty$$

$\longrightarrow$ $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$ is a type I MOP w.r.t. $(w_1, \ldots, w_r)$

- (type II) approximate each f_j by a rational function having a common denominator

$$P_{\vec{n}}(z) f_j(z) - Q_{\vec{n},j}(z) = O(z^{-n_j-1}), \; z \to \infty, \; j = 1, \ldots, r$$

$\longrightarrow$ $P_{\vec{n}}$ is a type II MOP w.r.t. $(w_1, \ldots, w_r)$

Motivation: Hermite's proof - Hermite-Padé approximation

Idea: simultaneously approximate r generating functions f_j of weights w_j at infinity

$$f_j(z) = \sum_{k\geq 0} \frac{m_{j,k}}{z^{k+1}}, \quad m_{j,k} = \int_\Lambda w_j(x)x^k dx$$

Possibilities:

- (type I) approximate a polynomial combination of the f_j by a polynomial

$$\sum_{j=1}^{r} A_{\vec{n},j}(z)f_j(z) - B_{\vec{n}}(z) = O(z^{-|\vec{n}|}), \ z \to \infty$$

$\longrightarrow$ $(A_{\vec{n},1}, \ldots, A_{\vec{n},r})$ is a type I MOP w.r.t. $(w_1, \ldots, w_r)$

$\longrightarrow$ use **type I Hermite-Padé approximants** of

$$f_j(z) = \sum_{k\geq 0} \frac{1}{k!} \frac{1}{(c_j z)^{k+1}}$$

Motivation: Hermite's proof - Nesterenko's criterion

Lemma (Nesterenko's criterion)

Let $x_1, \ldots, x_r \in \mathbb{R}$ *and suppose there exists* $(p_{j,n})_{n\in\mathbb{N}} \subset \mathbb{Z}$ *and an increasing function* $\sigma : \mathbb{R} \to (0,\infty)$ *with* $\lim_{t\to\infty} \sigma(t) = \infty$ *and* $\limsup_{t\to\infty} \sigma(t+1)/\sigma(t) = 1$ *such that*

i) $\max\{|p_{j,n}| \mid j = 1,\ldots,r\} \leq e^{\sigma(n)}$,

ii) $c_1 e^{-\tau_1\sigma(n)} \leq \left|\sum_{j=1}^r p_{j,n}x_j - p_{0,n}\right| \leq c_2 e^{-\tau_2\sigma(n)}$ *for some* $c_1, c_2, \tau_1, \tau_2 > 0$.

Then

$$\dim_{\mathbb{Q}} span_{\mathbb{Q}}\{1, x_1, \ldots, x_r\} \geq \frac{\tau_1 + 1}{1 + \tau_1 - \tau_2}.$$

$\longrightarrow$ if $\tau_1 = \tau_2 = r$ then $1, x_1, \ldots, x_r$ are $\mathbb{Q}$-**linearly independent**

Motivation: E-functions

Hermite (1873): $\mathbb{Q}$-linearly independence of $1, e^{c_1 z}, \ldots, e^{c_r z}$ for distinct $c_i, z \in \mathbb{Q}_0$
$\longrightarrow$ can we **generalize Hermite's result**?

Answer: yes, due to work of Siegel/Shidlovskii/Chudnovsky/Zudilin
$\longrightarrow$ can also consider suitable E-functions

$$f(z) = \sum_{k \geq 0} f_k \frac{z^k}{k!}, \quad \begin{cases} f_k \in \mathbb{Q} \text{ for all } k, \\ f_k \text{ grows at most exponentially in } k, \\ \text{common denom of } f_0, \ldots, f_n \text{ grows at most exponentially in } n. \end{cases}$$

Examples: hypergeometric functions $\sum_{k \geq 0} (\vec{a})_k / (\vec{b})_k z^k$ with $\vec{a} \in (\mathbb{Q}_{>0})^p, \vec{b} \in (\mathbb{Q}_{>0})^q$ and $p < q$

Trade-off: approximants are only known implicitly

Motivation: Bessel-like MOP

Hermite (1873): $\mathbb{Q}$-linearly independence of $1, e^{c_1 z}, \ldots, e^{c_r z}$ for distinct $c_i, z \in \mathbb{Q}_0$
$\longrightarrow$ can we **generalize Hermite's construction**?

Answer: yes

- Step 1: construct Bessel-like MOP
  $\longrightarrow$ simultaneous approximants of

$$\sum_{k\geq 0} \frac{(\vec{a}+1)_k}{(\vec{b}+\sum_{i=1}^{j}\vec{e}_i+1)_k} z^k, \quad 1 \leq j \leq q$$

- Step 2: extend Bessel-like MOP
  $\longrightarrow$ simultaneous approximants of

$$\sum_{k\geq 0} \frac{(\vec{a}+1)_k}{(\vec{b}+\sum_{i=1}^{j}\vec{e}_i+1)_k} (c_i z)^k, \quad 1 \leq j \leq q, \quad i = 1, \ldots, m$$

- Step 3: HOPE that Nesterenko's criterion is applicable

Bessel-like setting

Consider q weights with moments of the form

$$\int_\Lambda w_j(z;\vec{a},\vec{b})z^{s-1}dz = \frac{\Gamma(s+\vec{a})}{\Gamma(s+\vec{b}+\vec{e}_j)}, \qquad \vec{a}\in(-1,\infty)^p, \quad \vec{b}\in(-1,\infty)^q, \quad p<q$$

Suppose that $\vec{n}\in\mathcal{N}^q$. The moments of the **type I functions** are given by

$$\int_\Lambda F_{\vec{n}}(z)z^{s-1}dz = \frac{\Gamma(s+\vec{a})}{\Gamma(s+\vec{b}+\vec{n})}(1-s)_{|\vec{n}|-1}$$

The **type I polynomials** are given by

$$A_{\vec{n},j}(z) = \frac{(\vec{a}-b_j)_1}{(\vec{b}-b_j)_1^{*j}}\sum_{J=1}^{q}\sum_{K=0}^{n_J-1}\frac{(b_J+K+1)_{|\vec{n}|-1}}{\prod_{i=1}^{q}(b_i-b_J-K)_{n_i}}\sum_{k=0}^{K-1+\delta_{j,J}}\frac{(\vec{b}-b_J-K)_{k+1}^{*j}(b_j-b_J-K)_k}{(\vec{a}-b_J-K)_{k+1}}z^k$$

The **type II polynomials** are given by

$$P_{\vec{n}}(z) = {}_{q+1}F_p\left(\begin{matrix}-|\vec{n}|\,,\vec{b}+\vec{n}+1\\ \vec{a}+1\end{matrix};z\right)$$

Extended Bessel-like setting

Consider multiple orthogonality w.r.t. $\sqcup_{i=1}^m \vec{w}(c_i z; \vec{a}, \vec{b})$ where $\vec{w}(z; \vec{a}, \vec{b})$ is a B-like system
$\longrightarrow$ mq weights with moments of the form

$$\int_{\Lambda/c_i} w_j(c_i z; \vec{a}, \vec{b}, \vec{c}) z^{s-1} dz = \frac{1}{c_i^s} \frac{\Gamma(s+\vec{a})}{\Gamma(s+\vec{b}+\vec{e}_j)}, \qquad \vec{a} \in (-1,\infty)^p, \quad \vec{b} \in (-1,\infty)^q, \quad p<q$$

Suppose that $\vec{N} = \sqcup_{i=1}^m \vec{n}$. The moments of the **type I functions** are given by

$$\int_{\Lambda^*} F_{\vec{N}}(z) z^{s-1} dx = \frac{\Gamma(s+\vec{a})}{\Gamma(s+\vec{b}+\vec{n})} \oint_{\mathcal{C}} \frac{t^{|\vec{N}|-s-1}}{\prod_{i=1}^m (t-c_i)^{|\vec{n}|}} dt$$

$\longrightarrow$ explicit expressions for type I polynomials
The **type II polynomials** are given by

$$P_{\vec{N}}(z) = {}_{q+1}F_p \left(\begin{array}{c} -|\vec{N}|, \vec{b}+\vec{n}+1 \\ \vec{a}+1 \end{array} ; z \right) \boxtimes_N \prod_{i=1}^m (1-c_i z)^{|\vec{n}|}.$$

Extended Bessel-like setting: quality of approximants

The extended Bessel-like MOP solve the type I Hermite-Padé approximation problem

$$\sum_{i=1}^{m}\sum_{j=1}^{q} A_{\vec{N},i,j}(z)\left[\sum_{k\geq 0}\frac{(\vec{a}+1)_k}{(\vec{b}+\sum_{i=1}^{j}\vec{e}_i+1)_k}\frac{1}{(c_i z)^{k+1}}\right]-B_{\vec{N}}(z)=O(z^{-|\vec{N}|}),\ \ z\to\infty$$

Results for $\vec{N}=(n,\ldots,n)$, $\vec{a}\in\mathbb{Q}^p$, $\vec{b}\in\mathbb{Q}^q$ (with $p<q$), $\vec{c}\in\mathbb{Q}_0^m$ and $z\in\mathbb{Q}_0$:

- (Diophantine quality) As $n\to\infty$,

$$A_{\vec{N},i,j}(z), B_{\vec{N}}(z)\leq C^n n!^{q-p}, \qquad d_n A_{\vec{N},i,j}(z), d_n B_{\vec{N}}(z)\in\mathbb{Z} \text{ for some } (d_n)\subset\mathbb{N} \text{ with } d_n\sim D^n$$

- (Approximation quality) As $n\to\infty$,

$$E_n(z)=E^{n(1+o(1))}n!^{-qm(q-p)}$$

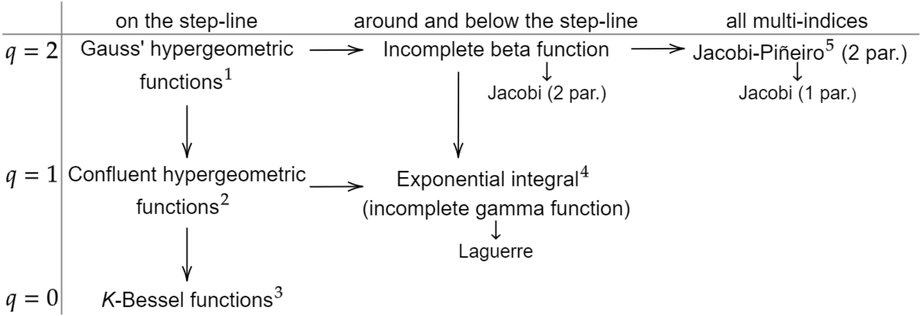
$\longrightarrow$ can apply **Nesterenko's criterion** with $\tau_1=\tau_2=mq$

Jacobi/Laguerre-like setting: appearances in the literature

Consider p slight variations of a weight $w(x; \vec{a}, \vec{b})$ with moments

$$\int_\Lambda w(x; \vec{a}, \vec{b}) x^{s-1} dx = \frac{\Gamma(s+\vec{a})}{\Gamma(s+\vec{b})}, \qquad \vec{a} \in (-1,\infty)^p, \quad \vec{b} \in (-1,\infty)^q, \quad p \geq q$$

Particular examples ($q \leq p = 2$):



1: (Lima-Loureiro, 2021), 2: (Lima-Loureiro, 2020), 3: (Van Assche-Yakubovich, 2000), 4: (Van Assche-W., 2023), 5: (Smet-Van Assche, 2010)

Jacobi/Laguerre-like setting: appearances in the literature

Consider p slight variations of a weight $w(x;\vec{a},\vec{b})$ with moments

$$\int_\Lambda w(x;\vec{a},\vec{b})x^{s-1}dx = \frac{\Gamma(s+\vec{a})}{\Gamma(s+\vec{b})}, \qquad \vec{a}\in(-1,\infty)^p, \quad \vec{b}\in(-1,\infty)^q, \quad p\geq q$$

Other appearances in the literature:

- $(p,0)$-case (Kuijlaars-Zhang, 2014)
  $\longrightarrow$ connection to **random matrices**
- general (p,q)-case (Sokal, 2022)
  $\longrightarrow$ connection to branched continued fractions

Jacobi/Laguerre-like setting: the MOP

Consider p slight variations of a weight $w(x;\vec{a},\vec{b})$ with moments

$$\int_\Lambda w(x;\vec{a},\vec{b})x^{s-1}dx = \frac{\Gamma(s+\vec{a})}{\Gamma(s+\vec{b})}, \qquad \vec{a}\in(-1,\infty)^p, \quad \vec{b}\in(-1,\infty)^q, \quad p\geq q$$

Suppose that $\vec{n}\sqcup\vec{m}\in\mathcal{N}^p$, $\vec{m}\in\mathcal{S}^{p-q}$. The Mellin transform of the **type I functions** is

$$\int_\Lambda F_{\vec{n}\sqcup\vec{m}}(x)x^{s-1}dx = \frac{\Gamma(s+\vec{a})}{\Gamma(s+\vec{b}+\vec{n})}(1-s)_{|\vec{n}|+|\vec{m}|-1}$$

The **type II polynomials** are given by

$$P_{\vec{n}\sqcup\vec{m}}(x) = {}_{q+1}F_p\left(\begin{matrix}-|\vec{n}|-|\vec{m}|\,,\,\vec{b}+\vec{n}+1\\ \vec{a}+1\end{matrix}\;;x\right)$$

$\longrightarrow$ behavior of zeros in (Lima, 2023) or use finite free convolution as in [1]

[1] A. Martínez-Finkelshtein, R. Morales, D. Perales, *Real roots of hypergeometric polynomials via finite free convolution*, Preprint available at arXiv:2309.10970 (2023).

Jacobi/Laguerre-like setting: connection to random matrices

The type II polynomials are given by

$$P_{\vec{n}\sqcup\vec{m}}(x) = {}_{q+1}F_p\left(\begin{matrix} -|\vec{n}|-|\vec{m}|\,,\vec{b}+\vec{n}+1 \\ \vec{a}+1 \end{matrix}\;;x\right)$$

$\longrightarrow$ $P_{\vec{n}\sqcup\vec{m}}(x) = \mathbb{E}[\prod_{j=1}^{N}(x-x_j)]$ taken over the SSV of $X = G_p \dots G_{q+1} T_q \dots T_1$ with

- T_j: $(N+a_j)\times(N+a_{j-1})$ truncation of $(N+n_j+b_j)\times(N+n_j+b_j)$ random unitary matrix
- G_j: $(N+a_j)\times(N+a_{j-1})$ Ginibre matrix

In fact, the SSV of X are a **MOP ensemble** associated with $\vec{w}(x;\vec{a},\vec{b})$

Conclusion: SSV of $G_p \dots G_{q+1} T_q \dots T_1 \leftrightarrow$ zeros of J/L-like MOP
$\longrightarrow$ natural to study zeros of MOP with finite free convolution

Extended Jacobi/Laguerre-like setting

Consider multiple orthogonality w.r.t. $\sqcup_{i=1}^m \vec{w}(c_i x; \vec{a}, \vec{b})$ where $\vec{w}(x; \vec{a}, \vec{b})$ is a J/L-like system

Particular examples:

- $\vec{a} = (\alpha), \vec{b} = ()$: $(c_i x)^\alpha e^{-c_i x}$ on $(0, \infty)/c_i$ for $i = 1, \ldots, m$
  $\longrightarrow$ **multiple Laguerre polynomials of 2nd kind** if all $c_i > 0$
- $\vec{a} = (\alpha), \vec{b} = (\alpha + \beta)$: $(c_i x)^\alpha (1 - c_i x)^\beta$ on $(0, 1)/c_i$ for $i = 1, \ldots, m$
  $\longrightarrow$ *multiple Jacobi polynomials of 2nd kind* if all $c_i > 0$

Suppose that $\vec{N} = \sqcup_{i=1}^m (\vec{n} \sqcup \vec{m})$. The moments of the **type I functions** are given by

$$\int_{\Lambda^*} F_{\vec{N}}(x) x^{s-1} dx = \frac{\Gamma(s + \vec{a})}{\Gamma(s + \vec{b} + \vec{n})} \oint_{\mathcal{C}} \frac{t^{|\vec{N}| - s - 1}}{\prod_{i=1}^m (t - c_i)^{|\vec{n}| + |\vec{m}|}} dt.$$

The **type II polynomials** are given by

$$P_{\vec{N}}(x) = {}_{q+1}F_p \left(\begin{array}{c} -|\vec{N}|, \vec{b} + \vec{n} + 1 \\ \vec{a} + 1 \end{array} ; x \right) \boxtimes_N \prod_{i=1}^m (1 - c_i x)^{|\vec{n}| + |\vec{m}|}.$$

Extended Jacobi/Laguerre-like setting: connection to random matrices

The type II polynomials are given by

$$P_{\vec{N}}(x) = {}_{q+1}F_p\left(\begin{matrix} -N, \vec{b}+\vec{n}+1 \\ \vec{a}+1 \end{matrix} ; x \right) \boxtimes_N \prod_{i=1}^{m} (1-c_i x)^{|\vec{n}|+|\vec{m}|}$$

$\longrightarrow P_{\vec{N}}(x) = \mathbb{E}[\prod_{j=1}^{N}(x-x_j)]$ taken over the SSV of XC with

- $X = G_p \ldots G_{q+1} T_q \ldots T_1$ similarly as before
- C: $(N+a_0) \times N$ (deterministic) matrix with SSV $c_1^{-1}, \ldots, c_m^{-1}$ each of multiplicity $|\vec{n}|+|\vec{m}|$

In fact, the SSV of XC are a **MOP ensemble** associated with $\sqcup_{i=1}^{m} \vec{w}(c_i x; \vec{a}, \vec{b})$

Conclusion: SSV of $G_p \ldots G_{q+1} T_q \ldots T_1 C \leftrightarrow$ zeros of extended J/L-like MOP

Open problems

Open problems:

- nature of extended system of weights: AT-system, Nikishin system, something else?
- further classification of known multiple orthogonal polynomials?
  - multiple Laguerre polynomials of 1st kind?
  - MOP associated with I-Bessel functions?

Any questions?