HOPE: Hypergeometric and Orthogonal Polynomials Event

An international conference

Radboud University Nijmegen

1–3 May 2024

Orthogonal polynomials, in particular those that come as special cases of generalized hypergeometric functions, are a unique enterprise that unifies all areas of mathematics.

We HOPE that discussing recent developments of the topic, with aspects meeting our personal (Nijmegen-infused) standards and tastes, will lead to its further advances.

This conference is sponsored by the NWO — Dutch Research Council (project OCENW.KLEIN.006), the PWN — Platform Wiskunde Nederland, the GQT — Geometry and Quantum Theory cluster and the IMAPP — Institute for Mathematics, Astrophysics and Particle Physics, Radboud University Nijmegen.

Wednesday 1 May 2024, room HG 00.071	
$09^{05} - 09^{20}$	*opening & registration*
$09^{20} - 10^{00}$	Christian Krattenthaler: Uvarov's formula — a thorough discussion
$10^{10} - 10^{30}$	Pedro Ribeiro: Humbert functions and sums of squares
$11^{00} - 11^{20}$	Max van Horssen: Shift operators for nonsymmetric Jacobi polynomials of type BC_1 and their norms
$11^{30} - 12^{10}$	Tom Koornwinder: Symmetric and nonsymmetric Askey–Wilson functions and symmetries of the Askey–Wilson DAHA
$15^{00} - 15^{40}$	Andrei Martínez-Finkelshtein: Zeros of hypergeometric polynomials and free probability
$15^{50} - 16^{10}$	Berend Ringeling: Zeta Mahler functions
$16^{40} - 17^{20}$	Pablo Román: Matrix-valued orthogonal polynomials, Fourier algebras and non-abelian Toda-type equations

Thursday 2 May 2024, room HG 00.071	
$09^{20} - 09^{40}$	Carel Wagenaar: Markov dualities and orthogonal polynomials
$09^{45} - 10^{05}$	Dominik Brennecken: The Laplace transform, Riesz distributions and zeta distributions in the Dunkl setting
$10^{10} - 10^{30}$	Philip Schlösser: Intermediate Macdonald polynomials
$11^{00} - 11^{20}$	Gabriele Bogo: Supersingular abelian surfaces and orthogonal polynomials
$11^{30} - 12^{10}$	Jacopo Gandini: On the multiplication of spherical functions of reductive spherical pairs of type A
$15^{00} - 15^{40}$	Grzegorz Świderski: On some Jacobi matrices with a trace class resolvent
$15^{50} - 16^{10}$	Stein Meereboer: Relative Weyl group symmetries of quantum spherical functions
$16^{40} - 17^{20}$	Ana Loureiro: A journey through the generalised symmetric Freud weights
$17^{30} - 18^{00}$	*problem session*
$18^{30} - 21^{30}$	*HOPE dinner at Valdin*

Friday 3 May 2024, room HG 00.071	
$09^{20} - 10^{00}$	Hjalmar Rosengren: Elliptic hypergeometric functions and the Ruijsenaars model
$10^{10} - 10^{30}$	Wolter Groenevelt: Multivariate Meixner polynomials
	HOPE photo
$11^{00} - 11^{20}$	Thomas Wolfs: Multiple orthogonal polynomials with hypergeometric moment generating functions
$11^{30} - 12^{10}$	Ana Foulquié Moreno: Type I classical multiple orthogonal polynomials
$15^{00} - 15^{40}$	Jasper Stokman: Quasi-polynomial analogs of Koornwinder polynomials
$15^{50} - 16^{10}$	Maarten van Pruijssen: Shift operators for a class of intermediate Jacobi polynomials of type BC_2
$16^{40} - 17^{20}$	Walter Van Assche: Hermite–Padé approximation to Catalan's constant
	closing

Abstracts

Christian Krattenthaler (*Universität Wien*): Uvarov's formula — a thorough discussion

Christoffel's classical theorem provides a determinantal formula for the orthogonal polynomials corresponding to a polynomial deformation of a given measure in terms of the original orthogonal polynomials. Uvarov's formula from around 1960 provides a generalisation to the case of a rational deformation of a given measure. While the formula has been widely used since then, it seems that it has never been rigorously proved. After reviewing some history, I shall present a determinant identity that implies Uvarov's formula. To the best of my knowledge, this constitutes the first rigorous proof of the formula.

Pedro Ribeiro (*Universidade do Porto*): Humbert functions and sums of squares

Let $r_k(n)$ denote the number of ways in which n can be represented as a sum of k squares. Inspired by the work of Koshliakov and Popov, Berndt et al. [*Adv. Math.* **338** (2018)] proved a very general formula involving $r_k(n)$ and the product of two Bessel functions, $I_\nu(Y\sqrt{n})\,K_\nu(X\sqrt{n})$. The method of these authors, which combines Voronoï's formula with the evaluation of a Hankel transform due to Koshliakov, only works when the two Bessel functions have the same index.

Motivated by a problem concerning the number of zeros of shifted combinations of Dirichlet series [`arXiv:2401.02813 [math.NT]`], we shall discuss how the aforementioned formula can be extended to include products of Bessel functions having different indices, that is, $I_\mu(Y\sqrt{n})K_\nu(X\sqrt{n})$. As we shall see, one crucial ingredient in our discussion is a connection formula between the Humbert hypergeometric functions Φ_3 and Ψ_2.

Part of this talk is based on joint work with Semyon Yakubovich (*Universidade do Porto*).

Max van Horssen (*Radboud Universiteit*): Shift operators for nonsymmetric Jacobi polynomials of type BC_1 and their norms

We introduce differential-reflection operators that act as shift operators for the nonsymmetric Jacobi polynomials of type BC_1. The construction relies heavily on the connection with matrix-valued orthogonal polynomials as introduced by M. van Pruijssen, which was studied in rank one by M. van Pruijssen and the speaker. We develop some theory for these differential-reflection shift operators, which is in great analogy with the theory for the classical shift operators in rank one, as developed by G. Heckman and E. Opdam. Notably, there are four fundamental differential-reflection operators that symmetrize to the fundamental classical shift operators, and they can be used to compute the norms of the nonsymmetric Jacobi polynomials of type BC_1.

Tom Koornwinder (*Universiteit van Amsterdam*): Symmetric and nonsymmetric Askey–Wilson functions and symmetries of the Askey–Wilson DAHA

The concept of a double affine Hecke algebra (DAHA) was introduced by Cherednik. A DAHA, associated with a certain root system of rank n, has a representation (the basic representation) on the space of Laurent polynomials of n variables.There are n commuting so-called Y-elements of the DAHA, acting as q-difference-reflection operators in the basic representation, which have non-symmetric Macdonald polynomials as joint eigenfunctions. Sahi extended this to root system BC_n. For $n = 1$ the DAHA has a very simple structure and the eigenfunctions of the Y-element are the nonsymmetric Askey–Wilson polynomials. This DAHA has many symmetries, some more obvious than others. Questions are then how these symmetries occur in the basic representation and how the symmetries are visible in the eigenfunctions. One may also take the spherical subalgebra of the DAHA, which yields the Zhedanov algebra. Then the symmetrization of Y yields the second order q-difference operator which has the Askey–Wilson polynomials as eigenfunctions.

For some symmetries, to make them visible in the eigenfucntions, we have to consider transcendental eigenfunctions: the (symmetric) Askey–Wilson functions, first considered by Suslov and in more detail, also from the point of view of harmonic analysis, by Koelink and Stokman. In a subsequent paper Stokman defined nonsymmetric Askey–Wilson functions. In the lecture we give an alternative construction of the nonsymmetric Askey–Wilson functions, starting from the symmetric ones, and we discuss symmetries of these functions.

This is joint work with Marta Mazzocco (*University of Birmingham*).

Andrei Martínez-Finkelshtein (*Baylor University* & *University of Almería*): Zeros of hypergeometric polynomials and free probability

The concept of finite free convolution of polynomials arises within the framework of free probability theory. Recently it has gained attention due to its applications in the study of hypergeometric polynomials. Specifically, these polynomials can be represented as a finite free convolution of more elementary building blocks. This representation, combined with the preservation of real zeros and interlacing properties through free convolutions, provides an effective tool for analyzing when all roots of a particular hypergeometric polynomial are real. Consequently, this approach offers a fresh perspective on the zero properties of hypergeometric polynomials.

Furthermore, this representation remains valid even in the asymptotic regime, allowing us to express the limit zero distribution of generalized hypergeometric polynomials as a free convolution of more "elementary" measures. We demonstrate these results through applications to various families of multiple orthogonal polynomials.

This is joint work with Rafael Morales (Baylor University) and Daniel Perales (Texas A&M University).

Berend Ringeling: Zeta Mahler functions

In this talk we discuss the zeta Mahler function (ZMF), which is closely related to the Mahler measure. Often these ZMFs are expressible through hypergeometric functions. We discuss examples of these functions and establish some properties of the zeros, such as an RH-type phenomenon.

Pablo Román (*Universidad Nacional de Cordoba*): Matrix-valued orthogonal polynomials, Fourier algebras and non-abelian Toda-type equations

In this talk we will discuss the Fourier algebras of differential and difference operators related to a sequence of matrix-valued orthogonal polynomials (MVOPs) and some of their applications. We will show that certain symmetric operators within Fourier algebra of a family of MVOPs can be used to construct a one-parameter deformation of the family. We will establish that the recurrence coefficients associated with these operators satisfy generalizations of the non-Abelian lattice equations. Additionally, will provide a Lax pair formulation for these equations, and show examples of deformed Hermite and Laguerre-type MVOPs.

Carel Wagenaar (*Technische Universiteit Delft*): Markov dualities and orthogonal polynomials

Orthogonal polynomials are a useful tool in analyzing Markov processes. They appear as so-called "duality functions" between certain Markov processes. In this talk, I will introduce the concept of Markov duality and show that Krawtchouk polynomials appear as duality functions for the Symmetric Exclusion Process. Lastly, I will give an overview of the more general relation between hypergeometric orthogonal polynomials and Markov dualities.

Dominik Brennecken (*Universität Paderborn*): The Laplace transform, Riesz distributions and zeta distributions in the Dunkl setting

The Dunkl setting related to root systems of type A generalizes many classical results from the radial analysis on symmetric cones. Concepts such as the Laplace transform, Riesz distributions, spherical functions and hypergeometric functions have a natural extension to the Dunkl setting. In this talk we shall present Laplace transform identities involving the Cherednik kernel and the Heckman–Opdam hypergeometric function of type A. These formulas are in line with the corresponding formulas on symmetric cones where the Cherednik kernel and hypergeometric function take the role of the generalized power functions and the spherical functions. We present that the Riesz distributions form a group under Dunkl convolution. We will further show how type B Dunkl theory naturally comes in and how zeta distributions are connected to Riesz distributions.

Philip Schlösser (*Radboud Universiteit*): Intermediate Macdonald polynomials

Macdonald's theory of orthogonal polynomials associates with every (affine) root system S a family of orthogonal Laurent polynomials that are invariant under the finite Weyl group W_0 of S (*symmetric* Macdonald polynomials), and a family of polynomials that does not have any particular symmetry (*non-symmetric* Macdonald polynomials). I will discuss how this formalism can be extended to accomodate polynomials invariant under a parabolic subgroup $W_I \leq W_0$ (*intermediate* Macdonald polynomials), and how these can be related to vectors of W_0-invariant polynomials.

As an application, I will discuss the rank 1 case of Askey–Wilson polynomials with $I = \emptyset$, and how this provides a way to relate non-symmetric and symmetric Askey–Wilson polynomials to each other in a way reminiscent of work of Bouzeffour, Koornwinder, and Mazzocco.

Gabriele Bogo (*Universität Bielefeld*): Supersingular abelian surfaces and orthogonal polynomials

I will discuss the relation between orthogonal polynomials and a problem in arithmetic geometry, namely, the determination of certain varieties with exceptionally big endomorphism ring. More precisely, I will consider one-parameter families of abelian surfaces in characteristic $p > 0$ and show that the supersingular locus of such families is described by the zeros of orthogonal polynomials, generalizing the work of Atkin and Kaneko–Zagier on supersingular elliptic curves. In certain cases, these polynomials can be recovered, modulo p, by truncated hypergeometric series.

Jacopo Gandini (*Università di Bologna*): On the multiplication of spherical functions of reductive spherical pairs of type A

Let $X = G/K$ be an affine homogeneous spherical space for a reductive group G (that is, K is a reductive subgroup, and the coordinate ring $\mathbb{C}[G/K]$ is a multiplicity-free G-module). An important combinatorial object attached to X is the spherical root datum R_X, which generalizes the restricted root system of a symmetric space. Every irreducible submodule of $\mathbb{C}[X]$ yields a zonal spherical function (uniquely determined up to scalar factors). When X is a symmetric variety, the zonal spherical functions are specializations of Jacobi polynomials (or equivalently Jack polynomials, when the restricted root system is of type A) at suitable parameters associated to X. However, in general a theory of zonal spherical functions in terms of Jacobi polynomials is not available.

In the talk (based on joint work with Paolo Bravi) I will consider the problem of decomposing the product of spherical functions on X, bringing into the picture the spherical root datum R_X and the associated reductive group G_X. When R_X is of type A, relying on a conjecture of Stanley on the multiplication of Jack polynomials, I will rephrase the problem into a branching problem for G_X.

Grzegorz Świderski (*Institute of Mathematics of the Polish Academy of Sciences*): On some Jacobi matrices with a trace class resolvent

In the talk I will describe some properties of orthogonal polynomials $(P_n)_{n=0}^{\infty}$ with respect to determinate measures μ on the real line such that $\operatorname{supp}\mu = \{\lambda_j\}_{j=1}^{\infty}$, where $0 < \lambda_1 < \lambda_2 < \cdots$ and $\lambda_j \to \infty$. If

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} < \infty, \tag{1}$$

then the asymptotic behaviour of $(P_n)_{n=0}^{\infty}$ is particularly simple. I will present a criterion for (1) in the special case when $(P_n)_{n=0}^{\infty}$ are the so-called *birth–death polynomials*. Proofs are based on analysis of the associated Jacobi matrices. As an application I will consider properties of a modification of q-Laguerre polynomials leading to a determinate moment problem. This is joint work in progress with Pavel Šťovíček (*Czech Technical University in Prague*).

Stein Meereboer (*Radboud Universiteit*): Relative Weyl group symmetries of quantum spherical functions

There is a close connection between zonal spherical functions of compact symmetric pairs and orthogonal polynomials. On the level of quantum groups this connection lifts to a correspondence between quantum zonal spherical functions and q-orthogonal polynomials. A celebrated result of Koornwinder shows that zonal spherical functions related to the quantum analog of the pair $(SU(2), U(1))$ can be identified with Askey–Wilson polynomials. Subsequently, Letzter extended this result for non-reduced root systems demonstrating that zonal spherical functions for quantum symmetric pairs may be identified with Macdonald polynomials. Letzter's analysis rests upon two key observations:

(1) the zonal spherical functions are relative Weyl group invariant, and

(2) the zonal spherical functions are joint eigenfunctions of the Macdonald q-difference operator.

In the talk I want to highlight property (1) and discuss the generalization to general characters of quantum symmetric spaces.

Ana Loureiro (*University of Kent*): A journey through the generalised symmetric Freud weights

Orthogonal polynomials with respect to generalised symmetric Freud weights will be the theme of this talk. The associated recurrence coefficients satisfy a system of nonlinear equations. These sit on the discrete Painlevé I hierarchy, and they satisfy an array of intriguing properties. In this talk I HOPE to walk the audience through some fascinating and intriguing structures inherent to these systems.

Hjalmar Rosengren (*Chalmers tekniska högskola och Göteborgs universitet*): Elliptic hypergeometric functions and the Ruijsenaars model

We will discuss some relations between elliptic hypergeometric functions and the Ruijsenaars model. In particular, we describe a family of integral operators that commute with the Ruijsenaars Hamiltonian. The mutual commutativity of these operators is equivalent to an integral transformation conjectured by Gadde, Rastelli, Razamat and Yan. We will sketch a proof of this conjecture. The talk is based on ongoing joint work with Eric Rains.

Wolter Groenevelt (*Technische Universiteit Delft*): Multivariate Meixner polynomials

In this talk I show how Griffiths' multivariate Meixner polynomials occur as matrix coefficients of holomorphic discrete series representations of the group $SU(1, d)$. From this interpretation several properties of the Meixner polynomials can be derived, for example orthogonality relations and a duality property. This is joint work with Joop Vermeulen.

Thomas Wolfs (*KU Leuven*): Multiple orthogonal polynomials with hypergeometric moment generating functions

I will discuss three families of multiple orthogonal polynomials associated with weights for which the moment generating functions are hypergeometric series with slightly varying parameters. The weights are supported on the unit interval, the positive real line or the unit circle, and the multiple orthogonal polynomials can be seen as generalizations of the Jacobi, Laguerre or Bessel orthogonal polynomials. The resulting polynomials are always hypergeometric in nature (they arise as a hypergeometric polynomial or as a combination of them) and they appear naturally in problems in random matrix theory and Diophantine approximation. I will explain how a further extension of the Bessel-like polynomials can be used to obtain a generalization of Hermite's classical result on the transcendence of Euler's number e.

Ana Foulquié Moreno (*Universidade de Aveiro*): Type I classical multiple orthogonal polynomials

Recently we have obtained, for the case of two measures, explicit expressions for the Hahn multiple polynomials of type I, in terms of Kampé de Fériet hypergeometric series. From these results, part of the Askey scheme for multiple orthogonal polynomials type I was completed. In terms of generalized hypergeometric series and Kampé de Fériet hypergeometric series, we have obtained explicit expressions for the multiple orthogonal polynomials of type I for the Jacobi–Piñeiro, Meixner I, Meixner II, Kravchuk, Laguerre I, Laguerre II and Charlier families.

Moreover, for the case of p measures we have obtained explicit expressions for the Jacobi–Piñeiro multiple orthogonal polynomials of type I, and through limit relations we have obtained explicit hypergeometric expression for the Laguerre I, Laguerre II and Hermite multiple orthogonal polynomials of type I. We also get explicitly the coefficients of the recurrence relation that these multiple type I polynomials verify.

The talk is based on joint work with Amílcar Branquinho, Juan EF Díaz and Manuel Mañas on the results obtained in [*J. Math. Anal. Appl.* **528** (2023), 1277471], [`arXiv:2310.18294 [math.CA]`, to appear in *Proc. Amer. Math. Soc.*] and [`arXiv:2404.13958 [math.CA]`].

Jasper Stokman (*Universiteit van Amsterdam*): Quasi-polynomial analogs of Koornwinder polynomials

Quasi-polynomials are linear combinations of monomials with possibly nonintegral exponents. In this talk I introduce natural families of quasi-polynomial generalisations of the Koornwinder polynomials. I will discuss their representation-theoretic background and their interpretation in terms of vector-valued analogs of Koornwinder polynomials, with focus on the Askey–Wilson (rank one) case.

Maarten van Pruijssen (*Radboud Universiteit*): Shift operators for a class of intermediate Jacobi polynomials of type BC_2

Intermediate Jacobi polynomials for a root system R are polynomials that are invariant under a reflection subgroup of the Weyl group. The two extreme cases are the non-symmetric Jacobi polynomials and the Heckman–Opdam polynomials. In this talk we discuss polynomials that are invariant under a reflection subgroup of index two of the Weyl group of type BC_2. We recover a shift operator for the Heckman–Opdam polynomials of type BC_2 and we obtain shift operators for the intermediate Jacobi polynomials. This is joint work with Max van Horssen.

Walter Van Assche (*KU Leuven*): Hermite–Padé approximation to Catalan's constant

We will give some explicit rational approximations to π and Catalan's constant G which are based on Hermite–Padé approximation (type I and type II) for two and four Markov functions. The corresponding multiple orthogonal polynomials satisfy a Rodrigues-type formula, and explicit expressions can be found for the Hermite–Padé approximants and the error of approximation. The analytic properties are quite good but in order to have integers in the rational approximation one needs to multiply by large integers and this obstructs a proof of irrationality. Nevertheless, the rational approximants allow one to compute the constants with high accuracy.

Silent participants

Edward Berengoltz (*Universiteit van Amsterdam*)

Amílcar Branquinho (*Universidade de Coimbra*)

Juan E. F. Díaz (*Universidade de Aveiro*)

Enno Diekema

Dmitrii Karp (*Holon Institute of Technology*)

Erik Koelink (*Radboud University Nijmegen*)

Stefan Kolb (*Newcastle University*)

Lukas Langen (*Universität Paderborn*)

Wadim Zudilin (*Radboud University Nijmegen*)