Some results on Mahler measures of curves parametrized by modular units

[MM(P) conference]

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Introduction

Definition (Mahler (1962))

Let $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \setminus \{0\}$. The *(logarithmic) Mahler measure* of P is

$$
\begin{aligned}
m(P) &:= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} \\
&= \int_0^1 \cdots \int_0^1 \log \left| P\left(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}\right) \right| d\theta_1 \cdots d\theta_n.
\end{aligned}
$$

In this talk, we focus on MM of families of bivariate polynomials defining elliptic curves.

Introduction

Boyd(1998) numerically computed MM of a number of bivariate polynomials, including the following families:

$$x + \frac{1}{x} + y + \frac{1}{y} + \sqrt{\alpha},$$
$$(x+y)(x+1)(y+1) - \alpha xy,$$
$$x^3 + y^3 + 1 - \sqrt[3]{\alpha}xy$$

and conjectured that for all, but finitely many, $\alpha \in \mathbb{Z}$, their MM satisfy an identity of the form

$$m(P) \stackrel{?}{=} cL'(E,0),$$

where $c \in \mathbb{Q}^\times$ and E is an elliptic curve.

Introduction

Theorem (Deninger (1997))

Let $P \in \mathbb{C}[x, y]$ be irreducible. Suppose $P = 0$ defines an elliptic curve E and the closure of the *Deninger path* γ associated to P is a finite union of smooth paths in E. Then

$$m(P) - m(P^*) = -r(\{x, y\})[\gamma] := -\frac{1}{2\pi}\int_\gamma \eta(x, y),$$

where $P^*(x)$ is leading coefficient of $P(x, y)$ (seen as a polynomial in y) and $\eta(f, g) = \log|f| d\arg g - \log|g| d\arg f$.

Introduction

Conjecture (Bloch-Beilinson)

Let E be an elliptic curve defined over $\mathbb{Q}$ and $\mathcal{E}$ a Néron model of E. Then $\operatorname{rank}(K_2(\mathcal{E})) = 1$ and for $\alpha \in K_2(\mathcal{E})\backslash K_2(\mathcal{E})_{\text{tor}}$

$$\frac{r(\alpha)}{L'(E,0)} \in \mathbb{Q}^{\times},$$

where $r : K_2(\mathcal{E}) \to \mathbb{R}$ is the associated regulator map.

Numerical evidence due to Boyd and Rodriguez Villegas for the three families above suggests that for sufficiently large $|\alpha|$

$$\alpha \in \mathbb{Z} \Rightarrow \{x, y\}^M \in K_2(\mathcal{E}_\alpha) \text{ for some } M \in \mathbb{N}.$$

Hence their Mahler measures should be related to L-values via the Bloch-Beilinson conjecture.

Elliptic curves parametrized by modular units

Under favorable conditions, we can relate the regulator integral to an L-value using the following result.

Theorem (Brunault-Mellit-Zudilin, 2014)

Let N be a positive integer and define

$$g_a(\tau) = q^{NB_2(a/N)/2} \prod_{\substack{n\geq 1 \\ n\equiv a \bmod N}} (1-q^n) \prod_{\substack{n\geq 1 \\ n\equiv -a \bmod N}} (1-q^n),\ q := e^{2\pi i\tau},$$

where $B_2(x) = \{x\}^2 - \{x\} + 1/6$. Then for any $a, b, c \in \mathbb{Z}$ such that $N \nmid ac$ and $N \nmid bc$,

$$\int_{c/N}^{i\infty} \eta(g_a, g_b) = \frac{1}{4\pi} L(f(\tau) - f(i\infty), 2),$$

where $f(\tau) = f_{a,b;c}(\tau)$ is a weight 2 modular form which can be defined explicitly.

Elliptic curves parametrized by modular units

To apply B-M-Z formula, one needs a curve which can be parametrized by modular units (e.g. modular functions written as products/quotients of $g_a(\tau)$.)

Theorem (Brunault, 2016)

There are only finitely many elliptic curves over $\mathbb{Q}$ which can be parametrized by modular units.

The proof of Brunault's theorem relies on Watkin's bound for the modular degree of an elliptic curve.

Elliptic curves parametrized by modular units

$11a3$		$20a2$		$32a4$
$14a1$		$21a1$		$35a3$
$14a4$		$24a1$		$36a1$
$14a6$		$24a3$		$36a2$
$15a1$		$24a4$		$40a3$
$15a3$		$26a3$		$44a1$
$15a8$		$27a3$		$54a3$
$17a4$		$27a4$		$56a1$
$19a3$		$30a1$		$92a1$
$20a1$		$32a1$		$108a1$

Elliptic curves over $\mathbb{Q}$ of conductor ≤ 1000 parametrized by modular units supported on the torsion points (Brunault, 2016)

The family $x + \frac{1}{x} + y + \frac{1}{y} + \sqrt{\alpha}$

E	α	c	E	α	c	E	α	c
$11a3$			$20a2$			$32a4$	-16	2
$14a1$			$21a1$			$35a3$		
$14a4$			$24a1$	-2	$3/2$	$36a1$		
$14a6$			$24a3$	64	4	$36a2$		
$15a1$			$24a4$	4	1	$40a3$	-4	1
$15a3$	25	6	$26a3$			$44a1$		
$15a8$	1	1	$27a3$			$54a3$		
$17a4$	-1	2	$27a4$			$56a1$	4	$1/4$
$19a3$			$30a1$			$92a1$		
$20a1$			$32a1$	8	1	$108a1$		

Mellit (2011): $40a3$
Zudilin (2014): $15a8$, $17a4$, $24a4$, $56a1$
The rest follow from results of Rodriguez Villegas (1999), Lalín-Rogers (2007), Lalín (2010), Rogers-Zudilin (2012).

The family $(x+y)(x+1)(y+1) - \alpha xy$

E	α	c	E	α	c	E	α	c
$11a3$			$20a2$	-2	3	$32a4$	-16	2
$14a1$	7	6	$21a1$			$35a3$		
$14a4$	1	1	$24a1$	-2	$3/2$	$36a1$	2	$1/2$
$14a6$	-8	10	$24a3$	64	4	$36a2$	-4	2
$15a1$			$24a4$	4	1	$40a3$	-4	1
$15a3$	25	6	$26a3$			$44a1$		
$15a8$	1	1	$27a3$			$54a3$		
$17a4$	-1	2	$27a4$			$56a1$	4	$1/4$
$19a3$			$30a1$			$92a1$		
$20a1$	4	2	$32a1$	8	1	$108a1$		

Mellit (2012, 2019): $14a1$, $14a4$, $14a6$
Rogers-Zidilin (2012), **Bertin (unpublished)**: $20a1$, $20a2$
Rodriguez Villegas (1999), Benferhat (2009), Rogers (2011): $36a1$, $36a2$

Appearance of the remaining curves in other families

E	α	c	E	α	c	E	α	c
$11a3$	2	5	$20a2$	-2	3	$32a4$	-16	2
$14a1$	7	6	$21a1$			$35a3$		
$14a4$	1	1	$24a1$	-2	$3/2$	$36a1$	2	$1/2$
$14a6$	-8	10	$24a3$	64	4	$36a2$	-4	2
$15a1$			$24a4$	4	1	$40a3$	-4	1
$15a3$	25	6	$26a3$			$44a1$		
$15a8$	1	1	$27a3$			$54a3$		
$17a4$	-1	2	$27a4$			$56a1$	4	$1/4$
$19a3$			$30a1$			$92a1$		
$20a1$	4	2	$32a1$	8	1	$108a1$		

The curve $11a3$ appears in the family $y^2 + (x^2 + \alpha x - 1)y + x^3$ $(\alpha = 2)$ and its MM formula was proven by Brunault (2006).

Appearance of the remaining curves in other families

E	α	c	E	α	c	E	α	c
$11a3$	2	5	$20a2$	-2	3	$32a4$	-16	2
$14a1$	7	6	$21a1$			$35a3$		
$14a4$	1	1	$24a1$	-2	$3/2$	$36a1$	2	$1/2$
$14a6$	-8	10	$24a3$	64	4	$36a2$	-4	2
$15a1$			$24a4$	4	1	$40a3$	-4	1
$15a3$	25	6	$26a3$			$44a1$		
$15a8$	1	1	$27a3$			$54a3$		
$17a4$	-1	2	$27a4$			$56a1$	4	$1/4$
$19a3$			$30a1$			$92a1$		
$20a1$	4	2	$32a1$	8	1	$108a1$		

The curves $15a1$, $21a1$, and $30a1$ appear in the 2-parameter family $\alpha(x+1/x)+y+1/y+\beta$, which is *non-tempered* in general.

Appearance of the remaining curves in other families

Modular (unit) parametrizations for the curves $15a1$ and $21a1$ were apparently known to Ramanujan:

Entry 62 (p. 324). *Let*

$$P = \frac{f(-q)}{q^{1/12}f(-q^3)} \quad \text{and} \quad Q = \frac{f(-q^5)}{q^{5/12}f(-q^{15})}.$$

Then

$$(PQ)^2 + 5 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3.$$

Entry 68 (p. 323). *Let*

$$P = \frac{f(-q)}{q^{1/4}f(-q^7)} \quad \text{and} \quad Q = \frac{f(-q^3)}{q^{3/4}f(-q^{21})}.$$

Then

$$PQ + \frac{7}{PQ} = \left(\frac{Q}{P}\right)^2 - 3 + \left(\frac{P}{Q}\right)^2.$$

(Berndt, Ramanujan's notebooks Part IV)

Appearance of the remaining curves in other families

Theorem 1 (Lalín-S.-Zudilin (2015))

Let $P_{\alpha,\beta} = \alpha\left(x + \frac{1}{x}\right) + y + \frac{1}{y} + \beta$.
Write $yP_{\alpha,\beta} = (y - y_+(x))(y - y_-(x))$ *and let*

$$m^{\pm}(P_{\alpha,\beta}) = \frac{1}{2\pi i}\int_{|x|=1} \log^+ |y_{\pm}(x)| \frac{dx}{x}.$$

Then

$$m^{\pm}(P_{\sqrt{7},3}) = \mp\frac{1}{2}L'(f_{21},0) + \frac{(2\mp 1)}{8}\log 7,$$
$$m(P_{1,3}) = m^{-}(P_{\sqrt{7},3}) - 3m^{+}(P_{\sqrt{7},3}),$$
$$m(P_{1,3}) = 2L'(f_{21},0),$$

where f_{21} *is the normalized weight* 2 *newform of level* 21.

Appearance of the remaining curves in other families

Theorem 2 (Meemark-S. (2020))

Let $P_{\alpha,\beta} = \alpha\left(x + \frac{1}{x}\right) + y + \frac{1}{y} + \beta$. *Then the curve* $P_{2,3} = 0$ *can be parametrized by*

$$
\begin{aligned}
x(\tau) &= 2\frac{\eta(2\tau)\eta(6\tau)\eta(10\tau)\eta(30\tau)}{\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau)} \\
&= \frac{2}{g_1(\tau)g_3^2(\tau)g_5^2(\tau)g_7(\tau)g_9^2(\tau)g_{11}(\tau)g_{13}(\tau)g_{15}^2(\tau)}, \\
y(\tau) &= -\left(\frac{\eta(\tau)\eta(5\tau)\eta(6\tau)\eta(30\tau)}{\eta(2\tau)\eta(3\tau)\eta(10\tau)\eta(15\tau)}\right)^2 \\
&= -g_1^2(\tau)g_5^4(\tau)g_7^2(\tau)g_{11}^2(\tau)g_{13}^2(\tau).
\end{aligned}
$$

Moreover,
$m((x+y)(x+1)(y+1) - 3xy) = \frac{3}{2}m(P_{2,3}) - \log 2 = L'(f_{30}, 0).$

Appearance of the remaining curves in other families

E	α	c	E	α	c	E	α	c
$11a3$	2	5	$20a2$	-2	3	$32a4$	-16	2
$14a1$	7	6	$21a1$			$35a3$		
$14a4$	1	1	$24a1$	-2	$3/2$	$36a1$	2	$1/2$
$14a6$	-8	10	$24a3$	64	4	$36a2$	-4	2
$15a1$			$24a4$	4	1	$40a3$	-4	1
$15a3$	25	6	$26a3$			$44a1$		
$15a8$	1	1	$27a3$	-216	3	$54a3$		
$17a4$	-1	2	$27a4$			$56a1$	4	$1/4$
$19a3$			$30a1$			$92a1$		
$20a1$	4	2	$32a1$	8	1	$108a1$		

The curve $27a3$ comes from $x^3 + y^3 + 1 + 6xy$, whose MM formula was proven by Rodriguez Villegas (1999).

Appearance of the remaining curves in other families

E	α	c	E	α	c	E	α	c
$11a3$	2	5	$20a2$	-2	3	$32a4$	-16	2
$14a1$	7	6	$21a1$			$35a3$	-8	1
$14a4$	1	1	$24a1$	-2	$3/2$	$36a1$	2	$1/2$
$14a6$	-8	10	$24a3$	64	4	$36a2$	-4	2
$15a1$			$24a4$	4	1	$40a3$	-4	1
$15a3$	25	6	$26a3$	1	?	$44a1$	16	?
$15a8$	1	1	$27a3$	-216	3	$54a3$	-27	1
$17a4$	-1	2	$27a4$	-216	3	$56a1$	4	$1/4$
$19a3$	8	?	$30a1$			$92a1$	4	?
$20a1$	4	2	$32a1$	8	1	$108a1$		

The curves $19a3$, $26a3$, $27a4$, $35a3$, $44a1$, $54a3$, and $92a1$ appear in the family $y^2 + (x^2 - \sqrt[3]{\alpha}x)y + x$, which is 3-isogeneous to $x^3 + y^3 + 1 - \sqrt[3]{\alpha}xy$.

Non-reciprocal families of elliptic curves

Consider polynomials of the form

$$P_k(x,y) = A(x)y^2 + B(x)y + C(x).$$

Define

$$\begin{aligned} Z_k &= \{(x,y) \in \mathbb{C}^2 \mid P_k(x,y) = 0\} \\ K &= \{k \in \mathbb{C} \mid Z_k \cap \mathbb{T}^2 \neq \emptyset\} \\ G_\infty &= \text{the unbounded component of } \mathbb{C}\backslash K \end{aligned}$$

If $P_k(x,y)$ is reciprocal, then $K \subseteq \mathbb{R}$, implying $\overline{G}_\infty = \mathbb{C}$. By continuity, one could expect that their Mahler measures are rational multiples of (elliptic curve or Dirichlet) L-values *for all* $k \in \mathbb{Z}$.

Non-reciprocal families of elliptic curves

Consider the family

$$Q_k(x,y) = y^2 + (x^2 - kx)y + x.$$

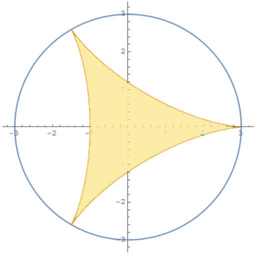
For $k \neq 0, 3$, $E_k : Q_k = 0$ defines an elliptic curve which is isogeneous to the Hessian curve $P_k(x,y) := x^3 + y^3 + 1 - kxy = 0$. By the transformation,

$$(x^2y)^3 P_k\left(\frac{y}{x^2}, \frac{1}{xy}\right) = Q_k(x^3, y^3),$$

we have that $m(P_k) = m(Q_k) =: n(k)$.

Non-reciprocal families of elliptic curves

Note that P_k and Q_k are both tempered but non-reciprocal, so the set K associated with this family has non-empty interior.



Boyds and Rodriguez Villegas verified numerically that for many $k \in \mathbb{R}\backslash(-1,3)$ such that $k^3 \in \mathbb{Z}$

$$n(k) \stackrel{?}{=} r_k L'(E_k, 0). \qquad (1)$$

Non-reciprocal families of elliptic curves

Question: How does $n(k)$ behave when $k \in (-1, 3)$?

k^3	$n(k)/L'(E_k, 0)$
-3	$0.1111111111111\ldots$
-2	$0.1666666666666\ldots$
-1	$2.0000000000000\ldots$
1	$0.77029121013793\ldots$
2	$1.10425002440073\ldots$
3	$0.40982233187650\ldots$
$\vdots$	$\vdots$
25	$0.834010932792831\ldots$
26	$0.083356155544972\ldots$
28	$0.166666666666666\ldots$
29	$0.041666666666666\ldots$

Numerical values of $n(k)/L'(E_k, 0)$

Non-reciprocal families of elliptic curves

Recall from Deninger's result that

$$n(k) = -\frac{1}{2\pi}\int_{\overline{\gamma}_k} \eta(x,y),$$

where $\gamma_k = \{(x,y) \in \mathbb{C}^2 \mid |x| = 1, |y| > 1, Q_k(x,y) = 0\}$, called the *Deninger path* on E_k.
If Q_k does not vanish on $\mathbb{T}^2$, then $\gamma_k \in H_1(E_k,\mathbb{Z})$, so one could expect that $n(k)/L'(E_k,0) \in \mathbb{Q}^\times$, provided k satisfies a suitable integrality condition (i.e., $k^3 \in \mathbb{Z}$), by Bloch-Beilinson conjecture.
However, this is not the case for $k \in (-1,3)$.

Non-reciprocal families of elliptic curves

Let us first factorize Q_k as

$$Q_k(x,y) = y^2 + (x^2 - kx)y + x = (y - y_+(x))(y - y_-(x)),$$

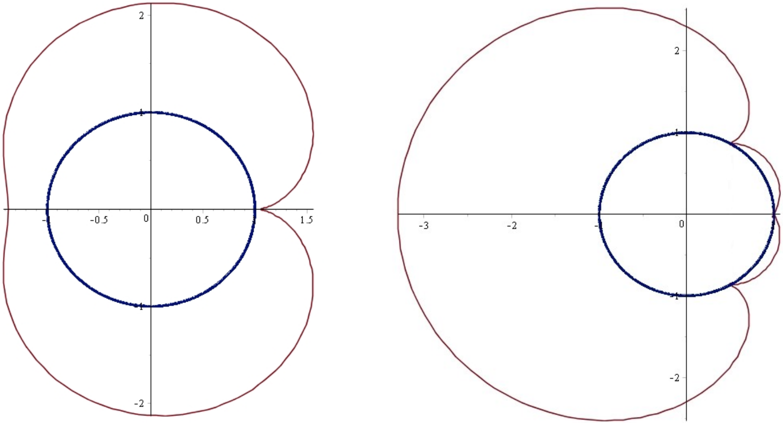
where

$$y_\pm(x) = -(x^2 - kx)\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{x(x-k)^2}}\right).$$

It can be shown that if $|x| = 1$, then $|y_-(x)| \leq 1 \leq |y_+(x)|$, so by Jensen's formula,

$$n(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|y_+(e^{i\theta})|d\theta = \frac{1}{\pi}\int_0^{\pi} \log|y_+(e^{i\theta})|d\theta.$$

Non-reciprocal families of elliptic curves



Graphs of $y_+(e^{i\theta})$ for $k = -1.1$ (left) and $k = 2$ (right)

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Non-reciprocal families of elliptic curves

Proposition (S.)

For $k \in (-1, 3)$, we have

$$E_k \cap \mathbb{T}^2 = \left\{ \left(e^{i\theta}, y_\pm(e^{i\theta}) \right) \mid \theta = 0, \pm \cos^{-1}\left(\frac{k-1}{2} \right) \right\}.$$

Note that we can write $n(k) = I(k) + J(k)$, where

$$I(k) = \frac{1}{2\pi} \int_{-c(k)}^{c(k)} \log |y_+(e^{i\theta})| d\theta = \frac{1}{\pi} \int_{0}^{c(k)} \log |y_+(e^{i\theta})| d\theta,$$

$$J(k) = \frac{1}{2\pi} \int_{c(k)}^{2\pi - c(k)} \log |y_+(e^{i\theta})| d\theta = \frac{1}{\pi} \int_{c(k)}^{\pi} \log |y_+(e^{i\theta})| d\theta,$$

and $c(k) = \cos^{-1}\left(\frac{k-1}{2} \right)$.

Non-reciprocal families of elliptic curves

Maybe we could find a linear combination of $I(k)$ and $J(k)$ for which the underlying path is closed. Using `PSLQ` algorithm, we find (numerically) that for $k \in (0,3)$ such that $k^3 \in \mathbb{Z}$

$$\tilde{n}(k) := I(k) - 2J(k) = n(k) - 3J(k) \stackrel{?}{=} r_k L'(E_k, 0). \qquad (2)$$

k^3	Cremona label of E_k	r_k	k^3	Cremona label of E_k	r_k
1	$26a3$	-1	14	$2548d1$	$1/36$
2	$20a1$	$-5/3$	15	$1350i1$	$1/18$
3	$54a1$	$-2/3$	16	$44a1$	$-4/3$
4	$92a1$	$-1/3$	17	$2890e1$	$-1/27$
5	$550d1$	$-1/9$	18	$324b1$	$-1/6$
6	$756f1$	$-1/18$	19	$722a1$	$1/9$
7	$490a1$	$1/9$	20	$700i1$	$-1/9$
8	$19a3$	-3	21	$2464k1$	$-1/27$
9	$162c1$	$-1/3$	22	$2420d1$	$1/26$
10	$1700c1$	$1/36$	23	$1058b1$	$-1/12$
11	$242b1$	$-1/3$	24	$27a1$	-3
12	$540d1$	$1/9$	25	$50a1$	$-5/3$
13	$2366d1$	$-1/45$	26	$676c1$	$-1/6$

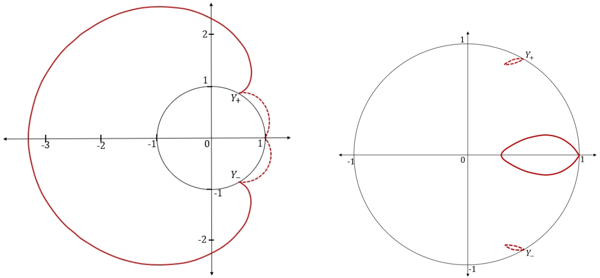
Non-reciprocal families of elliptic curves

Lemma (S.)

For $k \in (-1, 3)$,

$$\tilde{n}(k) = -\frac{1}{2\pi} \int_{\tilde{\gamma}_k} \eta(x, y)$$

for some $\tilde{\gamma}_k \in H_1^-(E_k, \mathbb{Z})$.



Graphs of $y_+(e^{i\theta})$ (left) and $y_-(e^{i\theta})$ (right) for $k = 2$

Non-reciprocal families of elliptic curves

Theorem (S.,2023)

We have

$$\tilde{n}(1) = -L'(f_{26}, 0),$$
$$\tilde{n}(\sqrt[3]{2}) = -\frac{5}{3}L'(f_{20}, 0),$$
$$\tilde{n}(\sqrt[3]{4}) = -\frac{1}{3}L'(f_{92}, 0),$$
$$\tilde{n}(2) = -3L'(f_{19}, 0),$$
$$\tilde{n}(\sqrt[3]{16}) = -\frac{4}{3}L'(f_{44}, 0),$$

where $f_N \in S_2(\Gamma_0(N))$.

Non-reciprocal families of elliptic curves

Proof.

Since $E_2 \cong 19a3$, it admits a modular parametrization $\varphi : X_1(19) \to E_2$ and there is a weight 2 newform f_2 associated to it. By some computations, we find that $\varphi_*\{4/19, -4/19\} = \tilde{\gamma}_2$. Moreover, E_2 can be parametrized by (with $N = 19$)

$$x(\tau) = -\frac{g_1 g_7 g_8}{g_2 g_3 g_5}, \quad y(\tau) = \frac{g_1 g_7 g_8}{g_4 g_6 g_9}.$$

Hence by B-M-Z,

$$\tilde{n}(2) = \frac{1}{2\pi}\int_{-4/19}^{4/19} \eta(x(\tau), y(\tau)) = -\frac{1}{4\pi^2} L(57 f_2, 2) = -3L'(f_2, 0).$$

Non-reciprocal families of elliptic curves

Proof (continued).

The remaining formulas can be proven in a similar manner using the following modular unit parametrizations ($N = 20, 26, 44, 92$ resp.):

$$x(\tau) = -\frac{1}{2^{\frac{2}{3}}}\frac{g_1g_3g_7g_9g_{10}^2}{g_2g_5^4g_6}, \qquad y(\tau) = -\frac{1}{2^{\frac{4}{3}}}\frac{g_{10}^2}{g_2g_6},$$

$$x(\tau) = -\frac{g_3g_8g_{11}g_{12}}{g_4g_6g_7g_9}, \qquad y(\tau) = \frac{g_1g_5g_8g_{12}}{g_2g_7g_9g_{10}},$$

$$x(\tau) = -\frac{1}{2^{\frac{2}{3}}}\left(g_{11}\prod_{n=0}^{10} g_{2n+1}\right)^2, \qquad y(\tau) = -\frac{1}{2^{\frac{4}{3}}}(g_2g_6g_{10}g_{14}g_{18}g_{22})^2,$$

$$x(\tau) = -\frac{1}{2^{\frac{2}{3}}}g_{23}\prod_{n=0}^{22} g_{2n+1}, \qquad y(\tau) = -\frac{1}{2^{\frac{4}{3}}}\prod_{n=0}^{11} g_{4n+2}.$$

Non-reciprocal families of elliptic curves

We also have the following general formula for $\tilde{n}(k)$.

Theorem (S.,2023)

For $k \in (-1,3)\backslash\{0\}$, the following identity is true:

$$\tilde{n}(k) = \frac{4}{1-3\,\mathrm{sgn}(k)}\,\mathrm{Re}\left(\log k - \frac{2}{k^3}\,{}_4F_3\left(\begin{matrix}\frac{4}{3},\ \frac{5}{3},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix}\ \middle|\ \frac{27}{k^3}\right)\right).$$

Proof sketch: Write $\frac{\mathrm{d}}{\mathrm{d}k}\tilde{n}(k)$ in terms of an elliptic integral, which can be easily transformed in to ${}_2F_1$-hypergeometric function. Then integrate both sides and apply boundary conditions obtained from the known formula for $k \in \mathbb{C}\backslash K$.

Thank you for your attention!