Hedgehogs in Lehmer's problem

Jan-Willem van Ittersum,
j.w. Berend Ringeling
and Wadim Zudilin

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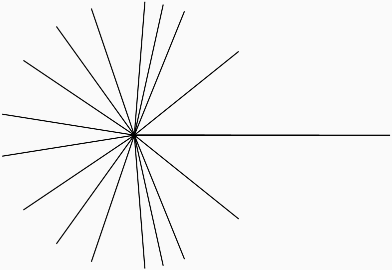
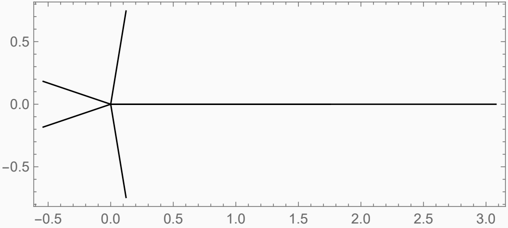
University of Cologne



Hedgehogs

Consider $P(x) = \prod_{j=1}^{d}(x - \alpha_j) \in \mathbb{Z}[x]$ irreducible *non-cyclotomic*.
Let $K = K(\beta_1, \ldots, \beta_n) = \bigcup_{k=1}^{n}[0, \beta_j] = \bigcup_{j=1}^{d}[0, \alpha_j^2] \cup \bigcup_{j=1}^{d}[0, \alpha_j^4]$
be the corresponding hedgehog.

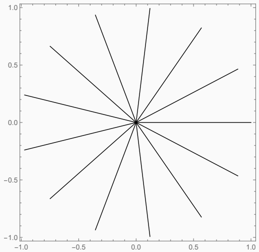


Hedgehog K on Smyth's polynomial $P(x) = x^3 - x - 1$

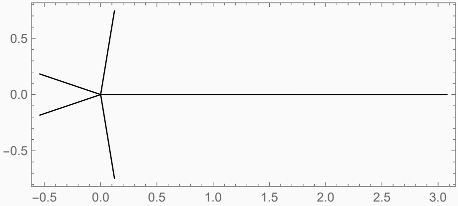
Dubinin: $t(K) \leq 4^{-1/n} \max_j |\beta_j|$; Dimitrov: $t(K) \geq 1$
Combined: *Schinzel–Zassenhaus conjecture* $\max_j |\alpha_j| \geq 2^{1/(4d)}$.

The Mahler measure of a hedgehog

Mahler measure of the hedgehog $M(K) := \prod_{j=1}^{n} \max(1, |\beta_j|)$



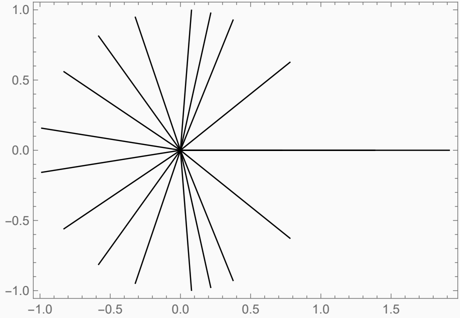
$M(K) = 4$ for a rotationally symmetric hedgehog with $t(K) = 1$



$M(K) = 3.07959562\ldots$ for hedgehog on Smyth's polynomial $P(x) = x^3 - x - 1$

The Mahler measure of a hedgehog

Mahler measure of the hedgehog $M(K) := \prod_{j=1}^{n} \max(1, |\beta_j|)$



$M(K) = 1.91445008\ldots$ for hedgehog on Lehmer's polynomial
$P(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$

Question 1. What is the minimum of $M(K)$ over all hedgehogs K with $t(K) \geq 1$? *W.l.o.g. assume* $t(K) = 1$.

Explicit Riemann mapping

Riemann: there exist a biholomorphic mapping $F : \hat{\mathbb{C}} \backslash \overline{D_1} \to \hat{\mathbb{C}} \backslash K$,

Schmidt: more concretely,

$$F(z) = \prod_{k=1}^{n} \left((z - z_k)(z^{-1} - \overline{z_k}) \right)^{r_k}$$

for some $z_k \in S^1$ and $r_k \geq 0$ with $r_1 + \ldots + r_n = 1$.

In particular, for $K = K(\beta_1, \ldots, \beta_n)$ one finds

$$|\beta_j| = \max_{z \in [z_{j-1}, z_j]} F(z) = \max_{z \in [z_{j-1}, z_j]} \prod_{k=1}^{n} |z - z_k|^{2r_k}.$$

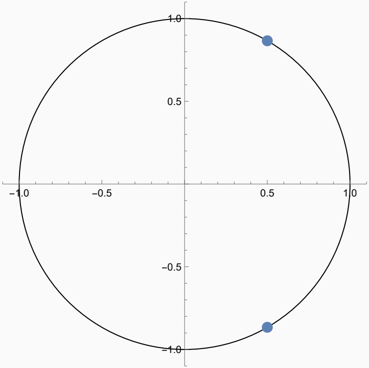
Question 1(c). What is the minimum C_n of

$$\prod_{j=1}^{n} \max_{z \in [z_{j-1}, z_j]} \left\{ 1 \, , \, \prod_{k=1}^{n} |z - z_k|^{1/n} \right\}$$

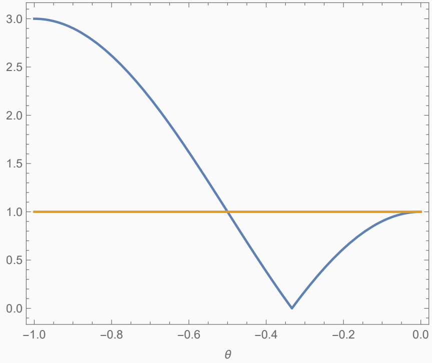
over all $n \geq 1$ and $z_k \in S^1$?

Example: $n = 2$

$$\text{Min } C_2 \text{ of } \prod_{j=1}^{2} \max_{z \in [z_{j-1}, z_j]} \left\{ 1 \, , \, \prod_{k=1}^{2} |z - z_k|^{1/2} \right\} \text{ for } z_k \in S^1.$$



Optimal configuration for z_1 and z_2

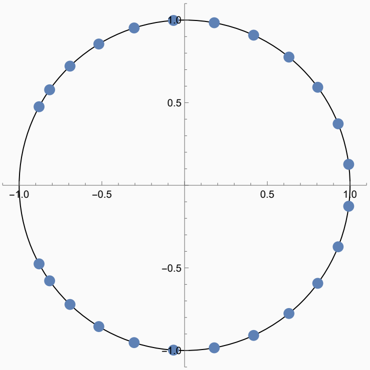


Value of $\prod_{k=1}^{2} |e^{\pi i \theta} - z_k|$ for varying θ

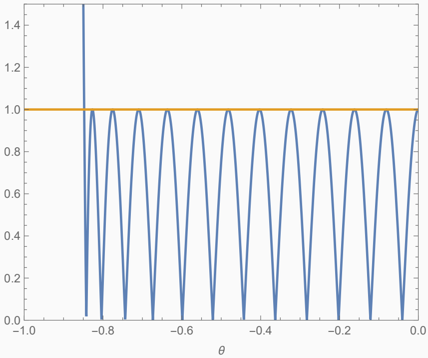
$C_2 = \sqrt{3} = 1.73205\ldots$

Example: $n = 24$

$$\text{Min } C_{24} \text{ of } \prod_{j=1}^{24} \max_{z \in [z_{j-1}, z_j]} \left\{ 1, \prod_{k=1}^{24} |z - z_k|^{1/2} \right\} \text{ for } z_k \in S^1.$$



A configuration for $z_1, \ldots, z_{24}$



Value of $\prod_{k=1}^{24} |e^{\pi i \theta} - z_k|$ for varying θ

$C_{24} \leq 1.236909\ldots$

$C_n \to 1$ as $n \to \infty$

Min C_n of $\displaystyle\prod_{j=1}^{n} \max_{z\in[z_{j-1},z_j]} \left\{1\,,\, \prod_{k=1}^{n} |z - z_k|^{1/n}\right\}$ for $z_k \in S^1$.

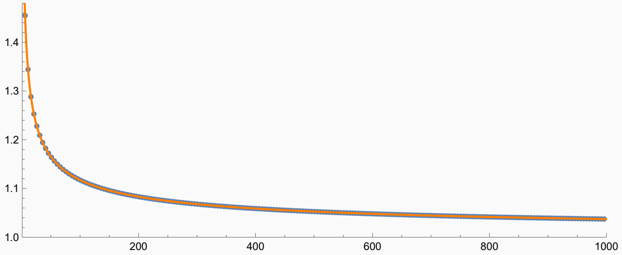
Theorem (I–Ringeling–Zudilin) $C_n \le C_n^* := T_n(2^{1/n})^{1/n}$,

where the *Chebyshev polynomials of the first kind* T_n are the unique polynomials of degree n such that

$$T_n(\cos\theta) = \cos(n\theta).$$

$$(T_n(x))_{n\ge 0} = (1, x, -1 + 2x^2, -3x + 4x^3, 1 - 8x^2 + 8x^4, \ldots)$$

$C_n \to 1$ as $n \to \infty$



The value of C_n^* and the approximation $1 + \nu - \frac{1}{4}\nu^3 + \frac{5}{96}\nu^5 - \frac{1}{128}\nu^7$

Asymptotically, for $\nu = \sqrt{\frac{\log 4}{n}}$,

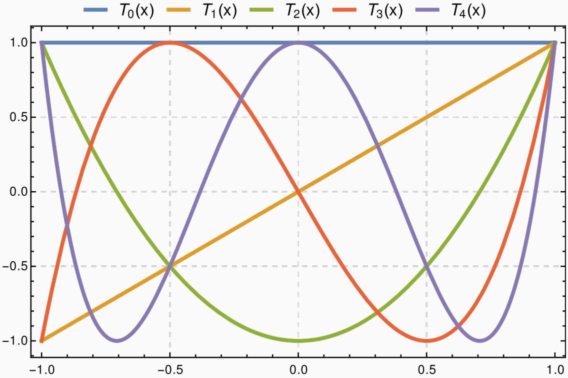
$$C_n^* = 1 + \nu - \frac{1}{4}\nu^3 + \frac{5}{96}\nu^5 - \frac{1}{128}\nu^7 + O(\nu^9) \qquad (n \to \infty),$$

Corollary $(C_n^*)^{\sqrt{n}} \to e^{\sqrt{\log(4)}}$ and $C_n^* \to 1$ as $n \to \infty$.

Key idea in the proof

Take z_i as the roots of

$$(-z)^{n/2}\, T_n\left(2^{1/n-1}\sqrt{2-(z+z^{-1})}\right).$$



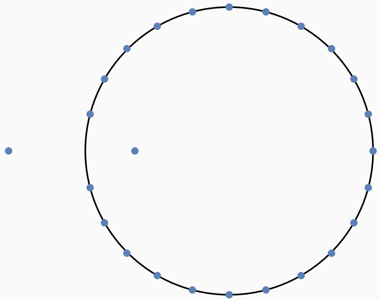
The Chebyshev polynomials

Outlook: integral polynomials and hedgehogs

The minimal polynomial of the β_i for the "Chebyshev hedgehog" $K(\beta_1, \ldots, \beta_{24})$ is

$$f^*(z) = -1 - z + z^{24} + z^{25} - (0.18356\ldots)(z - z^2 + \ldots + z^{23} - z^{24}).$$

Here, $C_{24}^* + (C_{24}^*)^{-1} + 2 = -0.18356\ldots$.



Zeros of f^*