Mahler measure of polynomials defining singular $K3$ surfaces

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Introduction

Introduced by Mahler in 1962,
the logarithmic Mahler measure of a polynomial P is

$$m(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log \mid P(x_1, \cdots, x_n) \mid \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}$$

and its Mahler measure

$$M(P) = \exp(m(P))$$

where

$$\mathbb{T}^n = \{(x_1, \cdots, x_n) \in \mathbb{C}^n / |x_1| = \cdots = |x_n| = 1\}.$$

Remarks

- $n = 1$
  By Jensen's formula, if $P \in \mathbb{Z}[X]$ is monic, then

$$M(P) = \prod_{P(\alpha)=0} \max(|\,\alpha\,|, 1).$$

So it is related to Lehmer's question (1933)
Does there exist $P \in \mathbb{Z}[X]$, monic, non cyclotomic, satisfying

$$1 < M(P) < M(P_0) = 1.1762\cdots ?$$

The polynomial

$$P_0(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

is the Lehmer polynomial, in fact a Salem polynomial.

Lehmer's problem is still open.
A partial answer by Smyth (1971)

$$M(P) \geq 1.32 \cdots$$

if P is non reciprocal.

First explicit Mahler measures

- $m(1+x) = 0$ (by Jensen's formula)
- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) =: L'(\chi_{-3}, -1) \quad \text{Smyth (1980)}$$
- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3) \quad \text{Smyth (1980)}$$

$$L(\chi_{-3}, s) = \sum_{n\geq 1} \frac{\chi_{-3}(n)}{n^s}$$

$$d_f = L'(\chi_{-f}, -1) = \frac{f^{3/2}}{4\pi} L(\chi_{-f}, 2) \qquad \text{Boyd's notation}$$

Deninger (1996) conjectured

$$m(x+\frac{1}{x}+y+\frac{1}{y}+1) \stackrel{?}{=} \frac{15}{4\pi^2}L(E,2) =: L'(E,0)$$

E elliptic curve of conductor 15 defined by the polynomial
This conjecture was proved (May 2011) by Rogers and Zudilin thanks to a previous result due to Lalin.
Deninger's guess comes from Beilinson's Conjectures.

Villegas's results (1998)

$$m(x+1/x+y+1/y-k) = \frac{1}{2}\Re[-2\pi i\tau + 4\sum_{n=1}^{\infty}\sum_{d|n}\chi(d)d^2\frac{q^n}{n}]$$

or in terms of Eisenstein's series

$$\Re[\frac{16\Im(\tau)}{\pi^2}\sum_{m,n\in\mathbb{Z}}\chi(n)\frac{1}{(m4\tau+n)^2(m4\bar{\tau}+n)}]$$

where $q = \exp 2\pi i\tau$ and $\chi(n) = \left(\frac{n}{4}\right)$

$$k^2 = 1/\mu(\tau) \qquad \mu = q - 8q^2 + 44q^3 - 192q^4 + \ldots$$

When k defines a CM elliptic curve, namely $k = 4\sqrt{2}$ defining

$$A : y^2 = x^3 - 44x + 112 \qquad \text{with conductor} \qquad 64$$

it follows

$$m(x + 1/x + y + 1/y - 4\sqrt{2}) = \frac{64}{4\pi^2} L(A, 2)$$

Also, if $k = 4/\sqrt{2}$ defining

$$B : y^2 = x^3 + 4x \qquad \text{with conductor} \qquad 32$$

it follows

$$m(x + 1/x + y + 1/y - 4/\sqrt{2}) = \frac{32}{4\pi^2} L(B, 2)$$

Finally for $k = 3\sqrt{2}$ we get the modular elliptic curve $X_0(24)$ and using Beilinson's theorem it is possible to get a formula of the same type for the Mahler measure.

A similar result was proved by Benferhat (2009) (one of my former students) concerning the family

$$x + 1/x + y + 1/y + x/y + y/x - k = 0$$

written as

$$1/xy[(x + y + 1)(xy + y + x) - (k + 3)xy] = 0$$

Hints of proof

From Verrill we know that putting $k + 3 = 1/t$, it defines an elliptic modular surface for the congruence group $\Gamma_1(6)$ with Picard-Fuchs equation near 0 (satisfied by the periods)

$$t(t - 1)(9t - 1)f'' + (27t^2 - 20t + 1)f' + 3(3t - 1)f = 0$$

with two properties

- For the Hauptmodul $t = \frac{\eta(6\tau)^8\eta(\tau)^4}{\eta(3\tau)^4\eta(2\tau)^8} = q - 4q^2 + 10q^3 - 20q^4 + 39q^5 + \ldots$
- the solution near 0 is expressed as

$$f = \frac{\eta(2\tau)^6\eta(3\tau)}{\eta(\tau)^3\eta(6\tau)^2}$$

- With $k + 3 = 1/t$ it follows that

$$\tilde{m}'(k) = \frac{1}{2i(\pi)^2}\int_{(\mathbb{T})^2} \frac{t}{-1 + \frac{(x+y+1)(xy+y+x)}{xy}} \frac{dx}{x}\frac{dy}{y}$$

is a period of the elliptic curve. Hence it satisfies the Picard-Fuchs equation; moreover it can be identified with the solution near 0. Thus

$$\tilde{m}'(k) = -tf \qquad d\tilde{m} = -f\frac{dt}{t} = -f\frac{t'(q)dq}{t}$$

$$-f(t)\frac{q\frac{dt}{dq}}{t} = 1 + L(q) + 8L(q^2) \qquad L(q) = \sum_{n\geq 1}(\sum_{d|n}\chi(d)d^2)q^n$$

Finally by integration we get

$$
\begin{aligned}
m(k) = \;& \Re\left(-2i\pi\tau + \sum_{n\geq 1}\left(\sum_{d|n}\chi(d)d^2\right)\frac{\exp 2i\pi n\tau}{n}\right)\\
& +8\left(\Re\sum_{n\geq 1}\left(\sum_{d|n}\chi(d)d^2\right)\frac{\exp 4i\pi n\tau}{2n}\right)
\end{aligned}
$$

and in terms of Eisenstein-Kronecker series

$$
\begin{aligned}
m(k) = \;& \Re\left(\frac{9\sqrt{3}\Im\tau}{4\pi^2}\sum_{(m,n)\neq(0,0)}\frac{\chi(n)}{(3m\tau+n)^2(3m\bar{\tau}+n)}\right)\\
& +8\Re\left(\frac{9\sqrt{3}\Im\tau}{4\pi^2}\sum_{(m,n)\neq(0,0)}\frac{\chi(n)}{(6m\tau+n)^2(6m\bar{\tau}+n)}\right)
\end{aligned}
$$

For $k=0$ the elliptic curve is CM with conductor 36 more precisely $36a1$ with τ imaginary quadratic and we can recover $m(0) = 2L'(E_{36}, 0)$.

CM elliptic curves and elliptic modular curves are rare in these families. Other people Mellit, Zudilin, Brunault used other techniques as parallel lines or parametrization of elliptic curves with modular units.

From elliptic curves to $K3$ surfaces

So replace E by a surface X which is also a Calabi-Yau variety, i.e. a $K3$-surface and try to answer the questions:
What are the analog of Deninger, Boyd, R-Villegas 's results and conjectures?

Our results concern polynomials of two families, namely

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining $K3$-surfaces Y_k and

$$Q_k = (x + y + z + 1)(xy + xz + yz + xyz) - (k + 4)xyz$$

defining $K3$-surfaces Z_k.

Basic facts on K3-surfaces

What's a K3-surface?

It is a smooth surface X satisfying

- $H^1(X, \mathcal{O}_X) = 0$ i.e. X simply connected
- $K_X = 0$ i.e. the canonical bundle is trivial i.e. there exists a unique, up to scalars, holomorphic 2-form ω on X.

A double covering branched along a plane sextic for example defines a K3-surface X.
It is the case of polynomials P_k put in the form

$$(2z+x+\frac{1}{x}+y+\frac{1}{y}-k)^2=(x+\frac{1}{x}+y+\frac{1}{y}-k)^2-4$$

Main properties

- $H_2(X,\mathbb{Z})$ is a free group of rank 22.

Main properties (continued)

- With the intersection pairing, $H_2(X, \mathbb{Z})$ is a lattice and

$$H_2(X, \mathbb{Z}) \simeq U_2^3 \perp (-E_8)^2 := \mathcal{L}$$

$\mathcal{L}$ is the $K3$-lattice, U_2 the hyperbolic lattice of rank 2, E_8 the unimodular lattice of rank 8.

- $$Pic(X) \subset H_2(X, \mathbb{Z}) \simeq Hom(H^2(X, \mathbb{Z}), \mathbb{Z})$$

where $Pic(X)$ is the group of divisors modulo linear equivalence, parametrized by the algebraic cycles (since for $K3$ surfaces linear and algebraic equivalence are the same).

- $$Pic(X) \simeq \mathbb{Z}^{\rho(X)}$$

$$\rho(X) := \text{Picard number of } X$$

$$1 \leq \rho(X) \leq 20$$

- $$T(X) := (Pic(X))^{\perp}$$

  is the transcendental lattice of dimension $22 - \rho(X)$
- If $\{\gamma_1, \cdots, \gamma_{22}\}$ is a $\mathbb{Z}$-basis of $H_2(X, \mathbb{Z})$ and ω the holomorphic 2-form,

  $$\int_{\gamma_i} \omega$$

  is called a period of X and

  $$\int_{\gamma} \omega = 0 \text{ for } \gamma \in Pic(X).$$

- If $\{X_z\}$ is a family of $K3$ surfaces, $z \in \mathbb{P}^1$ with generic Picard number ρ and ω_z the corresponding holomorphic 2-form, then the periods of X_z satisfy a Picard-Fuchs differential equation of order $k = 22 - \rho$. For our family $k = 3$.

Modular pencils of $K3$ surfaces are quite interesting to apply the technique recalled before. They are provided by Peters & Stienstra, Verrill. Namely the family P_k defining $K3$ surfaces Y_k

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

and the family of polynomials (Q'_k) defining (X_k) with generic Picard number 19 and generic transcendental lattice $U \oplus \langle 6 \rangle$ defined by

$$Q'_k = (x + xy + xyz + 1)(1 + z + zy + zxy) - (k+4)xyz, \qquad k \in \mathbb{C}.$$

The Mahler measure of Q_k is in fact the same as the Mahler measure of Q'_k since the change variables $x = X$, $y = XY$, $z = XYZ$ transforms Q_k into Q'_k and thus gets the Mahler measure inchanged. Hence we can deduce that the generic Picard number of Z_k is 19 and that Z_k and X_k are singular $K3$ surfaces for the same values of k. These values and those corresponding to singular $K3$ surfaces Y_k have been computed long ago by Boyd.
Why did I choose Q_k instead of Q'_k for evaluating the Mahler measure? Because of my complete ignorance concerning $K3$ surfaces and their L-series.

The talk in Paris of Professor Shiga about K3 surfaces motivated me. Verrill's results allowed me to express the Mahler measure in terms of L-series of a modular form.

But how to compute the L-series of the $K3$-surface? My unique model was Peters, Top, van der Vlugt in their paper "The Hasse zeta function of a $K3$ surface related to the number of words of weight 5 in the Mela's codes" (1992).

And precisely their $K3$ surface was Q_{-3}.

Theorem

(B. 2005) Let $k = t + \frac{1}{t}$ *and*

$$t = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^6, \quad \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n\geq 1}(1 - e^{2\pi i n\tau}), \quad q = \exp 2\pi i\tau$$

$$\begin{aligned} m(P_k) =& \frac{\Im\tau}{8\pi^3}\Big\{\sum_{m,\kappa}{}' \Big(-4\Big(2\Re\frac{1}{(m\tau+\kappa)^3(m\bar{\tau}+\kappa)} + \frac{1}{(m\tau+\kappa)^2(m\bar{\tau}+\kappa)^2}\Big) \\ & + 16\Big(2\Re\frac{1}{(2m\tau+\kappa)^3(2m\bar{\tau}+\kappa)} + \frac{1}{(2m\tau+\kappa)^2(2m\bar{\tau}+\kappa)^2}\Big) \\ & - 36\Big(2\Re\frac{1}{(3m\tau+\kappa)^3(3m\bar{\tau}+\kappa)} + \frac{1}{(3m\tau+\kappa)^2(3m\bar{\tau}+\kappa)^2}\Big) \\ & + 144\Big(2\Re\frac{1}{(6m\tau+\kappa)^3(6m\bar{\tau}+\kappa)} + \frac{1}{(6m\tau+\kappa)^2(6m\bar{\tau}+\kappa)^2}\Big)\Big)\Big\} \end{aligned}$$

Sketch of proof

Let

$$P_k = x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} - k$$

defining the family (X_k) of $K3$-surfaces.

- For $k \in \mathbb{P}^1$, generically $\rho = 19$.
- The family is $\mathcal{M}_k$-polarized with

$$\mathcal{M}_k \simeq U_2 \perp (-E_8)^2 \perp \langle -12 \rangle$$

- Its transcendental lattice satisfies

$$T_k \simeq U_2 \perp \langle 12 \rangle$$

- The Picard-Fuchs differential equation is

$$(k^2-4)(k^2-36)y''' + 6k(k^2-20)y'' + (7k^2-48)y' + ky = 0$$

(Peters and Stienstra's results)

- The family is modular in the following sense
  if $k = t + \frac{1}{t}$, $\tau \in \mathcal{H}$ and τ as in the theorem

$$t(\frac{a\tau + b}{c\tau + d}) = t(\tau) \ \forall \ (\begin{matrix} a & b \\ c & d \end{matrix}) \in \Gamma_1(6,2)^* \subset \Gamma_0(12)^* + 12$$

where

$$\Gamma_1(6) = \{(\begin{matrix} a & b \\ c & d \end{matrix}) \in Sl_2(\mathbb{Z}) \ / \ a \equiv d \equiv 1 \ (6) \ \ c \equiv 0 \ (6)\}$$

$$\Gamma_1(6,2) = \{(\begin{matrix} a & b \\ c & d \end{matrix}) \in \Gamma_1(6) \ \ c \equiv 6b \ \ (12)\}$$

and

$$\Gamma_1(6,2)^* = \langle \Gamma_1(6,2), w_6 \rangle$$

.

- The P-F equation has a basis of solutions $G(\tau)$, $\tau G(\tau)$, $\tau^2 G(\tau)$ with

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$$

  satisfying

$$G(\tau) = F(t(\tau)), \quad F(t) = \sum_{n\geq 0} v_n t^{2n+1}, \quad v_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

- $\frac{dm(P_k)}{dk}$ is a period, hence satisfies the P-F equation
- $$\frac{dm(P_k)}{dk} = G(\tau)$$
- $$dm(P_k) = -G(\tau)\frac{dt}{t}\frac{1-t^2}{t}$$

  is a weight 4 modular form for $\Gamma_1(6,2)^*$
- so can be expressed as a combination of $E_4(n\tau)$ for $n = 1, 2, 3, 6$

- By integration you get

$$m(P_k) = \Re(-\pi i\tau + \sum_{n\geq 1}(\sum_{d|n} d^3)(4\frac{q^n}{n} - 8\frac{q^{2n}}{2n} + 12\frac{q^{3n}}{3n} - 24\frac{q^{6n}}{6n}))$$

- Then using a Fourier development one deduces the expression of the Mahler measure in terms of an Eisenstein-Kronecker series

$$\begin{aligned} m(P_k) = \tfrac{\Im\tau}{8\pi^3} \textstyle\sum_{m,k}' \quad & [-4\tfrac{(m(\tau+\bar\tau)+2k)^2}{D_\tau^3} + \tfrac{4}{D_\tau^2} \\ & +16\tfrac{(2m(\tau+\bar\tau)+2k)^2}{D_{2\tau}^3} - \tfrac{16}{D_{2\tau}^2} \\ & -36\tfrac{(3m(\tau+\bar\tau)+2k)^2}{D_{3\tau}^3} + \tfrac{36}{D_{3\tau}^2} \\ & +144\tfrac{(6m(\tau+\bar\tau)+2k)^2}{D_{6\tau}^3} - \tfrac{144}{D_{6\tau}^2}] \end{aligned}$$

where

$$D_{j\tau} = (mj\tau + k)(mj\bar\tau + k)$$

The singular $K3$ surfaces of the Apéry-Fermi's family (Y_k) correspond to imaginary quadratic τ such that

$$t = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau\eta(3\tau)}\right)^6, \qquad k = t + \frac{1}{t}.$$

They have been computed by Boyd.

k	τ	Equation of τ
0	$\frac{-3+\sqrt{-3}}{6}$	$3\tau^2+3\tau+1=0$
2	$\frac{-2+\sqrt{-2}}{6}$	$6\tau^2+4\tau+1=0$
3	$\frac{-3+\sqrt{-15}}{12}$	$6\tau^2+3\tau+1=0$
6	$\frac{\sqrt{-6}}{6}$	$6\tau^2+1=0$
10	$\frac{\sqrt{-2}}{2}$	$2\tau^2+1=0$
18	$\frac{\sqrt{-30}}{6}$	$6\tau^2+5=0$
102	$\frac{\sqrt{-6\times 13}}{6}$	$6\tau^2+13=0$
198	$\frac{\sqrt{-17\times 6}}{6}$	$6\tau^2+17=0$
$2\sqrt{5}$	$\frac{-1+\sqrt{-5}}{6}$	$6\tau^2+2\tau+1=0$
$3\sqrt{6}$	$\frac{\sqrt{-3}}{3}$	$3\tau^2+1=0$
$2\sqrt{-3}$	$\frac{-1+\sqrt{-1}}{2}$	$2\tau^2+2\tau+1=0$
$3\sqrt{-5}$	$\frac{-3+\sqrt{-15}}{6}$	$3\tau^2+3\tau+2=0$

$m(P_0) =$	d_3 Boyd, B. (2005)	
$m(P_2) =$	$\frac{8\sqrt{8}}{\pi^3} L(f_8, 3)$ B. '09	$L(Y_2) = L(f_8, 3)$
$m(P_3) =$	$\frac{15\sqrt{15}}{2\pi^3} L(f_{15}, 3)$ (BFFLM)	$L(Y_3) = L(f_{15}, 3)$
$m(P_6) =$	$\frac{24\sqrt{24}}{2\pi^3} L(f_{24}, 3)$ (BFFLM)	$L(Y_6) = L(f_{24}, 3)$
$m(P_{10}) =$	$\frac{72\sqrt{72}}{9\pi^3} L(f_8, 3) + 2d_3$ B. 2010	$L(Y_{10}) = L(f_8, 3)$
$m(P_{18}) =$	$\frac{120\sqrt{120}}{9\pi^3} L(f_{120}, 3) + \frac{14}{5} d_3$	$L(Y_{18}) = L(f_{120}, 3)$
	(BFFLM)	(BFFLM)
$m(P_{102}) =$	$\frac{(312)^{3/2}}{13\times 4\pi^3} L(f_{312.b}, 3) + \frac{2}{13} d_{24}$	$L(Y_{102}) \stackrel{?}{=} L(f_{312}, 3)$
	B. (2022)	B. (2022)
$m(P_{198}) =$	$\frac{(408)^{3/2}}{17\times 4\pi^3} L(f_{408.b}, 3) + \frac{23\times 4}{17} d_3$	$L(Y_{198}) \stackrel{?}{=} L(f_{408}, 3)$
	B. (2022)	B. (2022)
$m(P_{2\sqrt{5}})=$	$2.\frac{20\sqrt{20}}{4\pi^3} L(f_{20.3.d.a}, 3)$ B. '20	$L(Y_{2\sqrt{5}}) = L(f_{20}, 3)$
$m(P_{3\sqrt{-5}}) =$	$\frac{6}{5}\frac{15\sqrt{15}}{2\pi^3} L(f_{15}, 3) + \frac{d_{15}}{10}$ B. '20	$L(Y_{3\sqrt{-5}}) = L(f_{15} \otimes \chi_5, 3)$
$m(P_{k^2=54}) =$	$\frac{48^{3/2}}{32\pi^3} L(f_{48}, 3) + \frac{11}{8} d_3$ B. '23	$L(Y_{k^2=54}) = L(f_{12}, 3)$
$m(P_{k^2=-12}) =$	$\frac{36}{\pi^3} L(f_{36.3.d.a}, 3) + \frac{4}{3} d_3$ B. '22	$L(Y_{2\sqrt{-3}}) = L(f_{36} \otimes \chi_3, 3)$

(BFFLM) is for Bertin, Feaver, Fuselier, Lalin, Manes (2011) (WIN2, Banff)
Similar results are obtained concerning the family Q_k defining $K3$ surfaces Z_k (S. is for Samart).

$m(Q_0) =$	$2\frac{12\sqrt{12}}{4\pi^3}L(f_{12},3)$(B. 2005)	$L(Z_0,3) = L(f_{12},3)$
$m(Q_{12}) =$	$8\frac{12\sqrt{12}}{4\pi^3}L(f_{12},3)$ (B. 2005)	$L(Z_{12},3) = L(f_{12},3)$
$m(Q_{-3}) =$	$\frac{8}{5}d_3$ (B. 2008)	$L(Z_{-3},3) = L(f_{15}\otimes\chi_{-3},3)$
$m(Q_{-36}) =$	$8\frac{12\sqrt{12}}{4\pi^3}L(f_{12},3) + 2d_4$ (S.2012)	$L(Z_{-36},3) \overset{?1}{=} L(f_{12},3)$
$m(Q_{-6}) =$	$\frac{7}{2}\frac{12\sqrt{12}}{4\pi^3}L(f_{12},3) + d_4$(S. 2012)	$L(Z_{-6},3) = L(f_{12},3)$
$m(Q_4) =$	$\frac{5}{\pi^3}8\sqrt{2}L(f_8,3)$	$L(Z_4,3) = L(f_8,3)$
$m(Q_{60}) =$	$\frac{21\sqrt{15}}{2\pi^3}L(f_{15}\otimes\chi_{-3},3) + 4d_3$	$L(Z_{60},3) \overset{?2}{=} L(f_{15}\otimes\chi_{-3},3)$
$m(Q_{-12}) =$	$\frac{3}{2}\frac{24\sqrt{24}}{4\pi^3}L(f_{24},3) + \frac{5}{2}d_3$	$L(Z_{-12},3) = L(f_{24},3)$

?1 if the infinite section is defined over $\mathbb{Q}(I)$
?2 if the infinite section is defined over $\mathbb{Q}(\sqrt{-3})$

The L-series: Livné's modularity theorem

Theorem

Let S be a $K3$-surface defined over $\mathbb{Q}$, with Picard number 20 and discriminant N. Its transcendental lattice $T(S)$ is a dimension 2 $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-module thus defines a L series, $L(T(S), s)$.
There exists a weight 3 modular form, f, CM over $\mathbb{Q}(\sqrt{-N})$ satisfying

$$L(T(S), s) \doteq L(f, s) = \sum_{n \geq 1} \frac{A_n}{n^s}.$$

The discriminant N is the determinant of the Gram matrix of the transcendental lattice.

How to compute the A_n of the L-series

Lemma

(B. 2010) Let Y an elliptic K3-surface defined over $\mathbb{Q}$ by a Weierstrass equation $Y(t)$. If rank $(Y(t)) = r$ and the r infinite sections generating the Mordell-Weil lattice are defined respectively over $\mathbb{Q}(\sqrt{d_i})$,$i = 1, \ldots, r$, then

$$A_p = -\sum_{t \in \mathbb{P}^1(\mathbb{F}_p),\quad Y(t) \quad smooth} a_p(t) - \sum_{t \in \mathbb{P}^1(\mathbb{F}_p),\quad Y(t) \quad singular} \epsilon_p(t) - \sum_{i=1}^{r} \left(\frac{d_i}{p}\right)$$

where

$$a_p(t) = p + 1 - \#Y(t)(\mathbb{F}_p)$$

$$\epsilon_p(t) = \begin{cases} 0, & \text{if the reduction of } Y(t) \text{ is additive} \\ 1, & \text{if the reduction of } Y(t) \text{ is split multiplicative} \\ -1, & \text{if the reduction of } Y(t) \text{ is non split multiplicative} \end{cases}.$$

The infinite sections are not always defined on $\mathbb{Q}(\sqrt{d_i})$
It is the case for $L(Y_{102})$ and $L(Y_{198})$ where one of the infinite sections generating the Mordell-Weil lattice is probably defined on $\mathbb{Q}(\sqrt{-7}), \sqrt{-11})$.
But we may conjecture the result from Schütt's classification and a new Bertin & Lecacheux result.

Theorem

(Schütt's classification) Consider the following classification of singular $K3$-surfaces over $\mathbb{Q}$

1. *by the discriminant d of the transcendental lattice of the surface up to squares,*
2. *by the associated newform up to twisting,*
3. *by the level of the associated newform up to squares,*
4. *by the CM-field $\mathbb{Q}(\sqrt{-d})$ of the associated newform.*

Then, all these classifications are equivalent. In particuliar, $\mathbb{Q}(\sqrt{-d})$ has exponent 1 or 2.

Theorem

(B. & Lecacheux (2022)) The transcendental lattices of the singular members of the previous families are given in the following table.

Y_0	$[4\quad 2\quad 4]$	Z_{-36}	$?[6\quad 0\quad 8]$	X_{-36}	$[2\quad 0\quad 6]$
Y_2	$[2\quad 0\quad 4]$	Z_{-12}	$[10\quad 2\quad 10]$	X_{-12}	$[4\quad 0\quad 6]$
Y_3	$[2\quad 1\quad 8]$	Z_{-6}	$[8\quad 4\quad 8]$	X_{-6}	$[6\quad 0\quad 8]$
Y_6	$[2\quad 0\quad 12]$	Z_{-3}	$[4\quad 1\quad 4]$	X_{-3}	$[6\quad 0\quad 10]$
Y_{10}	$[6\quad 0\quad 12]$	Z_0	$[2\quad 0\quad 6]$	X_0	$[2\quad 0\quad 6]$
Y_{18}	$[10\quad 0\quad 12]$	Z_4	$[2\quad 0\quad 16]$	X_4	$[2\quad 0\quad 4]$
Y_{102}	$[12\quad 0\quad 26]$	Z_{12}	$[2\quad 0\quad 24]$	X_{12}	$[2\quad 1\quad 2]$
Y_{198}	$[12\quad 0\quad 34]$	Z_{60}	$?[6\quad 0\quad 10]$	X_{60}	$[4\quad 1\quad 4]$
$Y_{k^2=20}$	$[2\quad 0\quad 10]$				
$Y_{k^2=54}$	$[4\quad 0\quad 12]$				
$Y_{k^2=-12}$	$[6\quad 0\quad 6]$				
$Y_{k^2=-45}$	$[8\quad 2\quad 8]$				

Y_k is the desingularization of the set of zeroes of P_k.
Z_k is the desingularization of the set of zeroes of Q_k.
X_k is the desingularization of the set of zeroes of Q'_k.
Shimada and Zhang 's notation:

$$[a \quad b \quad c] := \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

Quite recently, O. Lecacheux proved the two last results marked with ?.

Examples of getting L-series: $L(Y_{k^2=-45})$ and $L(Y_{k^2=-12})$

$$L(Y_{k^2=-45}, 3) = L(f_{15} \otimes \chi_5, 3) \tag{1}$$

$$L(Y_{k^2=-12}, 3) = L(f_{36} \otimes \chi_3, 3) \tag{2}$$

In general to compute these L-series we apply the lemma. But we need

- an elliptic fibration with Weierstrass equation defined over $\mathbb{Q}$;
- the r infinite sections generating the Mordell-Weil lattice

For both Weierstrass equations defining $Y_{k^2=-45}$ and $Y_{k^2=-12}$ we get $r=2$. For both, from generic results of Bertin and Lecacheux, we obtain one infinite section. But we need another infinite section!
In the first case, $Y_{k^2=-45}$ is the Kummer surface of another surface Z_{-3} since $T_{Z_{-3}} = [4 \quad 1 \quad 4]$. Thus there exists a 2-isogeny between the surface and its Kummer surface which preserves the L-series.
Since Z_{-3} has an elliptic fibration with $r=0$ its L-series can be easily computed and gives (1).

Proof of (2)

$Y_{k^2=-12}$ has an elliptic fibration with Weierstrass equation

$$y^2 = x^3 - (t^3 + 3t^2 - 6t + 4)x^2 + t^3 x$$

with two infinite sections

$$(1, (t-1)\sqrt{-3}) \text{ from(B-L),}(\left(\frac{t-4}{t+2}\right)^2, \frac{3(t^2-16)t(t-1)}{(t+2)^3})\text{(Sage)}$$

One infinite section defined over $\mathbb{Q}(\sqrt{-3})$ and the other over $\mathbb{Q}$.
The $A(p)$ are computed using the Pari order

$$A(p) = -sum(t = 2, p-1, ellak(e(t), p)) - \left(\frac{-3}{p}\right) p - p - \left(\frac{-1}{p}\right)$$

Proof of (2)

Now we must compare to the $\alpha(p)$ given by the CM newform of level 36 and weight 3 (36.3.d.a in LMFDB)

$$f_{36}(q) = q - 2q^2 + 4q^4 + 8q^5 - 8q^8 - 16q^{10} - 10q^{13} + 16q^{16} - 16q^{17} + 32q^{20} + 39q^{25}$$

p	5	11	13	17	19	23	29	31	37	41	43	47	53
$\alpha(p)$	-8	0	-10	-16	0	0	-40	0	-70	80	0	0	56
A(p)	-8	0	-10	16	0	0	40	0	-70	-80	0	0	-56

Remark

So results on Algebraic Geometry (essentially Livné's and Schütt's) lead to results on Number Theory (Mahler measure).

In the opposite direction, an observation on Mahler measures leads to an algebraic geometry result.

We observed that polynomials Q_k defining Z_k and polynomials Q'_k defining X_k have the same Mahler measure. What is the relation between the $K3$ surfaces Z_k and X_k?

Indeed, Bertin and Lecacheux proved the following theorem.

Theorem

1) The transcendental lattice of the generic member Z_k is $U \oplus \langle 24 \rangle$.
2) There is a genus 1 fibration of Z_k whose Jacobian surface J_k is a $K3$ surface of the Verrill's family with generic transcendental lattice $U \oplus \langle 6 \rangle$.

Y_0	$[4 \quad 2 \quad 4]$	Z_{-36}	$[6 \quad 0 \quad 8]$	X_{-36}	$[2 \quad 0 \quad 6]$	J_{-36}	$[6 \quad 0 \quad 8]$
Y_2	$[2 \quad 0 \quad 4]$	Z_{-12}	$[10 \quad 2 \quad 10]$	X_{-12}	$[4 \quad 0 \quad 6]$	J_{-12}	$[4 \quad 0 \quad 6]$
Y_3	$[2 \quad 1 \quad 8]$	Z_{-6}	$[8 \quad 4 \quad 8]$	X_{-6}	$[6 \quad 0 \quad 8]$	J_{-6}	$[2 \quad 0 \quad 6]$
Y_6	$[2 \quad 0 \quad 12]$	Z_{-3}	$[4 \quad 1 \quad 4]$	X_{-3}	$[6 \quad 0 \quad 10]$	J_{-3}	$[4 \quad 1 \quad 4]$
Y_{10}	$[6 \quad 0 \quad 12]$	Z_0	$[2 \quad 0 \quad 6]$	X_0	$[2 \quad 0 \quad 6]$	J_0	$[2 \quad 1 \quad 2]$
Y_{18}	$[10 \quad 0 \quad 12]$	Z_4	$[2 \quad 0 \quad 16]$	X_4	$[2 \quad 0 \quad 4]$	J_4	$[2 \quad 0 \quad 4]$
Y_{102}	$[12 \quad 0 \quad 26]$	Z_{12}	$[2 \quad 0 \quad 24]$	X_{12}	$[2 \quad 1 \quad 2]$	J_{12}	$[2 \quad 0 \quad 6]$
Y_{198}	$[12 \quad 0 \quad 34]$	Z_{60}	$[6 \quad 0 \quad 10]$	X_{60}	$[4 \quad 1 \quad 4]$	J_{60}	$[6 \quad 0 \quad 10]$
$Y_{k^2=20}$	$[2 \quad 0 \quad 10]$						
$Y_{k^2=54}$	$[4 \quad 0 \quad 12]$						
$Y_{k^2=-12}$	$[6 \quad 0 \quad 6]$						
$Y_{k^2=-45}$	$[8 \quad 2 \quad 8]$						

Computing the Dirichlet part: an example

Evaluating

$$\sum_{k,m}{}' \left(-\frac{1}{(13m^2+6k^2)^2} + \frac{1}{(26m^2+3k^2)^2} - \frac{1}{(39m^2+2k^2)^2} + \frac{1}{(78m^2+k^2)^2} \right)$$

needs to use a formula by Huard, Kaplan and Williams counting the number of representations of a positive integer by a representative system of inequivalent binary definite positive quadratic forms of given discriminant.
Needs also much care since for example if $n = 6k^2 + 13m^2$, $2n = 3(2k)^2 + 26m^2$.
It can be formulated in terms of Epstein functions and is related to a Zagier's conjecture.

Denote (a, b, c) the quadratic primitive positive definite form

$$Q(x, y) = ax^2 + bxy + cy^2, \qquad a, b, c \quad \text{integers}$$

and $d = b^2 - 4ac < 0$ its discriminant, $d \equiv 0 \quad \text{ou} \quad 1 \quad \text{modulo} \quad 4.$
The associate Epstein function is as follows

$$\zeta_Q(s) := \zeta_{(a,b,c)}(s) := \sum_{m,n}{}' \frac{1}{(am^2 + bmn + cn^2)^s}.$$

where Σ' means $(m, n) \neq (0, 0)$ and completed as

$$\tilde{\zeta_Q}(s) := \mid \operatorname{disc}(Q) \mid^{-1/2} \pi^{-s} \zeta_Q(s)$$

Let us recall Zagier's conjecture.

Conjecture *For all $s \geq 2$, $\tilde{\zeta_Q}(s)$ is a $\mathbb{Q}$-linear combination of values of the s-th polylogarithm in algebraic numbers.*

Example of results obtained

For D fundamental discriminant of a quadratic field, define

$$L_D(s) := \sum_{n>0} \frac{\left(\frac{D}{n}\right)}{n^s}, \left(\frac{D}{n}\right) \text{ Kronecker symbol.}$$

Theorem

Denote respectively f_1, f_2, f_3, f_4 the quadratic forms $f_1 = (6, 0, 13)$, $f_2 = (2, 0, 39)$, $f_3 = (1, 0, 78)$ and $f_4 = (3, 0, 26)$. Then

$$\zeta_{f_1}(s) = \frac{1}{2}(\zeta(s)L_{-312}(s) + L_{-3}(s)L_{104}(s) - L_{13}L_{-24}(s) - L_8(s)L_{-39}(s))$$

$$\zeta_{f_2}(s) = \frac{1}{2}(\zeta(s)L_{-312}(s) - L_{-3}(s)L_{104}(s) - L_{13}(s)L_{-24}(s) + L_8(s)L_{-39}(s))$$

$$*\zeta_{f_3}(s) = \frac{1}{2}(\zeta(s)L_{-312}(s) + L_{-3}(s)L_{104}(s) + L_{13}(s)L_{-24}(s) + L_8(s)L_{-39}(s))$$

$$\zeta_{f_4}(s) = \frac{1}{2}(\zeta(s)L_{-312}(s) - L_{-3}(s)L_{104}(s) + L_{13}(s)L_{-24}(s) - L_8(s)L_{-39}(s))$$

The result * was already known (see p. 60 of the book “Lattice sums then and now” by Borwein, Glasser, McPhedran, Wan and Zucker).

Comments and questions

Polynomials Q_k

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + xy + \frac{1}{xy} + xz + \frac{1}{xz} + yz + \frac{1}{yz} + k$$

are in the same class of polyhedron as reflexive polytope 1529.

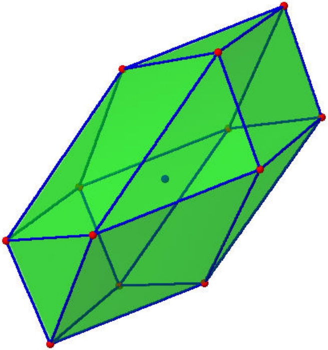

Figure: Reflexive polytope 1529

With Sage we can show that this polyhedron has

1. 8 facets of Mahler measure d_3 defined by polynomials of type $1 + X + Y$
2. 6 facets of Mahler measure 0 defined by polynomials of type $1 + X + Y + XY$

   Question: Is there a link with the fact that the Dirichlet parts of the Mahler measures are proportional either to d_3 or to 0?

   Other question: What should be the Mahler measures of faces of Newton polyhedron to find such expressions of the Mahler measure?