Evaluations of areal Mahler measure of multivariable polynomials

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Mahler measure of multivariable rational functions

$P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, the (logarithmic) Mahler measure is:

$$\begin{aligned} \mathrm{m}(P) &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{dx_1}{x_1} \ldots \frac{dx_n}{x_n} \\ &= \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| d\theta_1 \ldots d\theta_n, \end{aligned}$$

where $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in \mathbb{C}^n : |x_i| = 1\}$.

Jensen's formula gives

$$\mathrm{m}(P) = \log |a| + \sum_{|\alpha_j|>1} \log |\alpha_j| \quad \text{if} \quad P(x) = a \prod_j (x - \alpha_j)$$

$$M(P) := \exp(\mathrm{m}(P)).$$

Particular formulas and special values of L-functions

- Smyth (1981)

$$\mathrm{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

- Boyd (1998)

$$\mathrm{m}\left(1+\left(\frac{1-x}{1+x}\right)y\right) = \frac{2}{\pi} L(\chi_{-4}, 2)$$

- L. (2006)

$$\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right) = \frac{93}{\pi^4}\zeta(5)$$

$$\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

- L., Nair & Roy (2023++)

$$\mathrm{m}\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)^2(1+y)z\right) = \frac{21}{2\pi^2}\zeta(3)$$

Particular formulas, special values of L-functions, and some questions

- Rogers & Zudilin (2014)

$$\mathrm{m}\left(x+\frac{1}{x}+y+\frac{1}{y}+1\right)=\frac{15}{4\pi^2}L(E_{15},2)=L'(E_{15},0)$$

1. What analogues of Mahler measure can one obtain by modifying the domain of integration?

2. Do such generalizations produce special values of L-functions? Can we conjecture or prove those formulas?

What if...?

What if we replace the normalized arc length measure on the standard torus with the normalized area measure on the unit disk?

Mahler measure	Areal Mahler measure
$\mathbb{T} = \{x \in \mathbb{C} : \|x\| = 1\}$	$\mathbb{D} = \{x \in \mathbb{C} : \|x\| \leq 1\}$
$\frac{dx}{x}$	$dA(x) = dx$

How much does this change the measure?

The areal Mahler measure

Pritsker (2008) $P \in \mathbb{C}(x_1, \dots, x_n)^{\times}$, the (logarithmic) areal Mahler measure is:

$$\begin{aligned}\mathrm{m}_{\mathbb{D}}(P) =& \frac{1}{\pi^n} \int_{\mathbb{D}^n} \log |P(x_1, \dots, x_n)| dA(x_1) \dots dA(x_n) \\ =& \int_0^1 \cdots \int_0^1 \log |P(\rho_1 e^{2\pi i \theta_1}, \dots, \rho_n e^{2\pi i \theta_n})| \rho_1 \cdots \rho_n \\ & d\rho_1 \dots d\rho_n d\theta_1 \dots d\theta_n,\end{aligned}$$

where $\mathbb{D}^n = \{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1|, \dots, |x_n| \le 1\}$.

Pritsker (2008) If $\quad P(x) = a \prod_{j=1}^d (x - \alpha_j)$,

$$\mathrm{m}_{\mathbb{D}}(P) = \log |a| + \sum_{|\alpha_j| > 1} \log |\alpha_j| + \frac{1}{2} \sum_{|\alpha_j| < 1} \left(|\alpha_j|^2 - 1 \right).$$

Some basic properties of the areal Mahler measure

- $$\mathrm{m}_{\mathbb{D}}(x) = -\frac{1}{2}.$$

- For $P \in \mathbb{C}[x]$,

$$\mathrm{m}(P) - \frac{\deg P}{2} \leq \mathrm{m}_{\mathbb{D}}(P) \leq \mathrm{m}(P).$$

  Equality holds in the lower bound iff $P(z) = a_d z^d$, and in the upper bound iff P does not vanish on $\mathbb{D}$.

- For $P \in \mathbb{Z}[x]$ and $P(0) \neq 0$,

$$\mathrm{m}_{\mathbb{D}}(P) \geq \log |a_0| \geq 0.$$

- Choi and Samuels (2012) For $P \in \mathbb{C}[x]$ and $|P(0)| = 1$

$$\mathrm{m}_{\mathbb{D}}(P) \leq \mathrm{m}(P)^2.$$

Kronecker's Lemma and Lehmer's Question

- Pritsker (2008) Kronecker's Lemma
  If $P \in \mathbb{Z}[x]$ irreducible and $P(0) \neq 0$ then $\mathrm{m}_{\mathbb{D}}(P) = 0$ occurs only if all the roots of P are roots of unity.
- $$\mathrm{m}_{\mathbb{D}}(nx^n - 1) = \log n + \frac{n(n^{-2/n} - 1)}{2}$$
  $$\to 0 \qquad \text{as} \qquad n \to \infty$$
  $$\mathrm{m}_{\mathbb{D}}(x^{2n} + nx^n + 1) = \log\left(\frac{n + \sqrt{n^2 - 4}}{2}\right) + \frac{n}{2}\left(\left(\frac{n - \sqrt{n^2 - 4}}{2}\right)^{2/n} - 1\right)$$
  $$\to 0 \qquad \text{as} \qquad n \to \infty$$

Lehmer's Question has a negative answer!

Multivarible polynomials - The linear binomials

We have $\mathrm{m}(x+y)=\mathrm{m}(x+1)=0$, but, though $\mathrm{m}_{\mathbb{D}}(x+1)=0$,

$$\begin{aligned}\mathrm{m}_{\mathbb{D}}(x+y) =&\frac{1}{\pi^2}\int_{\mathbb{D}^2}\log|x+y|dA(y)dA(x)=\frac{1}{2\pi}\int_{\mathbb{D}}(|x|^2-1)dx\\ =&\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^1(\rho^2-1)\rho d\rho d\theta\\ =&-\frac{1}{4}\\ =&\mathrm{m}(x+y)-\frac{1}{4}.\end{aligned}$$

The linear binomials

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D}}(x_1 \cdots x_m + y) = \frac{1}{2^{m+1}} - \frac{1}{2}.$$

For $m, n \geq 2$,

$$\mathrm{m}_{\mathbb{D}}(x_1 \cdots x_m + y_1 \cdots y_n) = \frac{1}{4} + \binom{m+n-2}{m-1} \frac{1}{2^{m+n}}$$

$$- \frac{1}{2^{m+n}} \sum_{r=0}^{n-1} \binom{m+n-3-r}{m-2} 2^r - \frac{1}{2^{m+n}} \sum_{r=0}^{m-1} \binom{m+n-3-r}{n-2} 2^r$$

$$- \frac{m}{2^{m+n+1}} \sum_{r=0}^{n-1} \binom{m+n-1-r}{m} 2^r - \frac{n}{2^{m+n+1}} \sum_{r=0}^{m-1} \binom{m+n-1-r}{n} 2^r.$$

The linear trinomials

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D}}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3},2) + \frac{1}{6} - \frac{11\sqrt{3}}{16\pi}.$$

Smyth (1981)

$$\mathrm{m}(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3},2).$$

$$\mathrm{m}_{\mathbb{D}}(1+x+y) = \mathrm{m}(1+x+y) + \frac{1}{6} - \frac{11\sqrt{3}}{16\pi}$$

Ideas in the proof

$$\mathrm{m}_{\mathbb{D}}(1+x+y) = \frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}}(|1+x|^2-1)dA(x) + \frac{1}{\pi}\int_{\mathbb{D}\cap\{|1+x|\geq 1\}}\log|1+x|dA(x).$$

If $x=\rho e^{i\theta}$

$$\frac{1}{2\pi}\int_{\mathbb{D}\cap\{|1+x|\leq 1\}}(|1+x|^2-1)dA(x)$$

$$= \frac{1}{\pi}\int_{\frac{2\pi}{3}}^{\pi}\int_0^1(\rho^2+2\rho\cos\theta)\rho d\rho d\theta + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\int_0^{-2\cos\theta}(\rho^2+2\rho\cos\theta)\rho d\rho d\theta$$

$$= -\frac{3\sqrt{3}}{16\pi}.$$

Ideas in the proof

$y=1+x$ and set $y=\rho e^{i\theta}$

$$\begin{aligned}
&\frac{1}{\pi}\int_{\mathbb{D}\cap\{|1+x|\geq 1\}}\log|1+x|dA(x)\\
&=\frac{2}{\pi}\int_0^{\frac{\pi}{3}}\int_1^{2\cos\theta}(\log\rho)\rho d\rho d\theta\\
&=\frac{1}{\pi}\int_0^{\frac{\pi}{3}}\left(4\cos^2\theta\log(2\cos\theta)-2\cos^2\theta+\frac{1}{2}\right)d\theta.
\end{aligned}$$

$$\int_0^{\frac{\pi}{3}}\cos^2\theta\log(2\cos\theta)d\theta=\frac{3\sqrt{3}}{16}L(\chi_{-3},2)+\frac{\pi}{12}-\frac{\sqrt{3}}{16}.$$

A connection with hyperbolic volumes

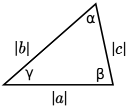
Cassaigne & Maillot (2000)

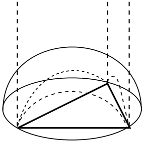
$a, b, c \in \mathbb{C}^*$, $a + bx + cy \in \mathbb{C}[x, y]$

$$\pi \mathrm{m}(a + bx + cy) = \begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha \log|a| + \beta \log|b| + \gamma \log|c| & \triangle, \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle. \end{cases}$$

Bloch–Wigner dilogarithm

$$D(z) := \mathrm{Im}\left(\mathrm{Li}_2(z) + \log(1 - z) \log|z|\right), \quad \mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\log(1-t)}{t} dt$$





The linear trinomials

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D}}(\sqrt{2}+x+y) = \frac{L(\chi_{-4},2)}{\pi} + \mathcal{C}_{\sqrt{2}} + \frac{3}{8} - \frac{3}{2\pi},$$

where

$$\mathcal{C}_{\sqrt{2}} = \frac{\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi^3}}\,{}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{5}{4},\frac{5}{4};1\right) - \frac{\Gamma\left(\frac{1}{4}\right)^2}{72\sqrt{2\pi^3}}\,{}_4F_3\left(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};\frac{3}{2},\frac{7}{4},\frac{7}{4};1\right),$$

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!},$$

$$(a)_0 = 1, \qquad (a)_n = a(a+1)(a+2)\cdots(a+n-1).$$

Cassaigne & Maillot (2000)

$$\mathrm{m}(\sqrt{2}+x+y) = \frac{L(\chi_{-4},2)}{\pi} + \frac{\log 2}{4}.$$

Another example

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D}}\left(y+\left(\frac{1-x}{1+x}\right)\right)=\frac{6}{\pi}L\left(\chi_{-4},2\right)-\log 2-\frac{1}{2}-\frac{1}{\pi}$$

Boyd (1992)

$$\mathrm{m}\left(y+\left(\frac{1-x}{1+x}\right)\right)=\frac{2}{\pi}L(\chi_{-4},2).$$

$$\mathrm{m}_{\mathbb{D}}\left(y+\left(\frac{1-x}{1+x}\right)\right)=3\mathrm{m}\left(y+\left(\frac{1-x}{1+x}\right)\right)-\log 2-\frac{1}{2}-\frac{1}{\pi}$$

Higher Mahler measure

$P \in \mathbb{C}(x_1, \dots, x_n)^\times$, the higher Mahler measure is

$$\mathrm{m}_h(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log^h |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

The higher areal Mahler measure is

$$\mathrm{m}_{\mathbb{D},h}(P) := \frac{1}{\pi^n} \int_{\mathbb{D}^n} \log^h |P(x_1, \dots, x_n)| dA(x_1) \dots dA(x_n).$$

Higher Mahler measure

Sasaki (2015)

$$\mathrm{m}_h\left(\frac{1-x}{1+x}\right) = \frac{|E_h|\pi^h}{2^h} \quad h \text{ even}, \qquad \text{and} \qquad = 0 \quad h \text{ odd}.$$

where E_n denotes the nth Euler number,

$$\frac{s}{e^s-1} = \sum_{n=0}^{\infty} B_n \frac{s^n}{n!} \qquad \frac{2e^s}{e^{2s}+1} = \sum_{n=0}^{\infty} E_n \frac{s^n}{n!}$$

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)!} \quad \text{and} \quad L(\chi_{-4}, 2n+1) = \frac{(-1)^n E_{2n}\pi^{2n+1}}{2^{2n+2}(2n)!}.$$

Higher Mahler measure

L. & Roy (2024) For $h \in \mathbb{Z}_{>0}$ even

$$\mathrm{m}_{\mathbb{D},h}\left(\frac{1-x}{1+x}\right) = \frac{|E_h|\pi^h}{2^h} - \frac{E_{h-2}(\pi i)^{h-2}h(h-1)}{2^{h-2}}\log 2 - \frac{4h!}{2^h}\sum_{m=2}^{h-1}(1-2^{1-m})\zeta(m)\frac{E_{h-m-1}(\pi i)^{h-m-1}}{(h-m-1)!}.$$

For h odd, $\mathrm{m}_{\mathbb{D},h}\left(\frac{1-x}{1+x}\right) = 0$.

Generalized Mahler measure

$P_1, \dots, P_r \in \mathbb{C}(x_1, \dots, x_n)^\times$, the generalized (logarithmic) Maher measure is

$$\mathrm{m}_{\max}(P_1, \dots, P_r) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \max\{\log|P_1|, \dots, \log|P_r|\} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

The generalized (logarithmic) areal Mahler measure is

$$\mathrm{m}_{\mathbb{D},\max}(P_1, \dots, P_r) = \frac{1}{\pi^n} \int_{\mathbb{D}^n} \max\{\log|P_1|, \dots, \log|P_r|\} dA(x_1) \dots dA(x_n).$$

L. & Roy (2024)

$$\mathrm{m}_{\mathbb{D},\max}(x_1, \dots, x_n) = -\frac{1}{2n}.$$

Zeta Mahler measure

$P \in \mathbb{C}(x_1, \ldots, x_n)^\times$, the zeta Mahler measure is

$$Z(s,P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} |P(x_1,\ldots,x_n)|^s \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n},$$

$$Z(s,P) = \sum_{k=0}^{\infty} \frac{\mathrm{m}_k(P) s^k}{k!}.$$

The areal zeta Mahler measure is

$$Z_{\mathbb{D}}(s,P) := \frac{1}{\pi^n} \int_{\mathbb{D}^n} |P(x_1,\ldots,x_n)|^s \, dA(x_1) \ldots dA(x_n).$$

An example of Zeta Mahler measure

L. & Roy (2024)

$$Z_{\mathbb{D}}(s, x+1) = \frac{\Gamma(s+2)}{\Gamma(s/2+2)^2} = \exp\left(\sum_{j=2}^{\infty} \frac{(-1)^j}{j}(1-2^{1-j})(\zeta(j)-1)s^j\right)$$

Akatsuka (2009)

$$Z(s, x+1) = \frac{\Gamma(s+1)}{\Gamma(s/2+1)^2} = \exp\left(\sum_{j=2}^{\infty} \frac{(-1)^j}{j}(1-2^{1-j})\zeta(j)s^j\right).$$

$$Z_{\mathbb{D}}(s, x+1) = \frac{s+1}{(s/2+1)^2} Z(s, x+1).$$

More zeta Mahler measures

L., Nair, Ringeling & Roy (two weeks ago)
For real $s > 0$, not an odd integer,

$$Z_{\mathbb{D}}(k+x+y;s) = \mathrm{Re}\; G(k) - \cot\left(\frac{\pi s}{2}\right) \mathrm{Im}\; G(k),$$

where

$$G(k) = k^s \cdot {}_3F_2\left(-\frac{s}{2}, -\frac{s}{2}, \frac{3}{2}; 2, 3; \frac{4}{k^2}\right).$$

For $k < 2$,

$$\mathrm{m}_{\mathbb{D}}(k+x+y) = -\frac{4k^3}{9\pi} {}_3F_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; \frac{5}{2}, \frac{5}{2}; \frac{k^2}{4}\right) + \frac{k^2}{2} - \frac{1}{4}.$$

$$\mathrm{m}(k+x+y) = \frac{k}{\pi} {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; \frac{k^2}{4}\right)$$

The areal Mahler measure revisited

We get for $k < 2$

$$\mathrm{m}(k+x+y) - \mathrm{m}_{\mathbb{D}}(k+x+y) = \frac{k\sqrt{4-k^2}(10+k^2) + (8-16k^2)\arccos\left(\frac{k}{2}\right)}{16\pi}$$

L., Nair, Ringeling & Roy (two weeks ago)

$$\mathrm{m}_{\mathbb{D}}(k+x+y) = \begin{cases} \frac{1}{\pi} D\left(e^{2i\arcsin(k/2)}\right) + \frac{2}{\pi}\arcsin\left(\frac{k}{2}\right)\log k \\ -\frac{k\sqrt{4-k^2}(10+k^2)+(8-16k^2)\arccos\left(\frac{k}{2}\right)}{16\pi} & k < 2 \\ \log k & k \geq 2 \end{cases}$$

The case $k = \sqrt{2}$ revisited

$$\mathrm{m}_{\mathbb{D}}(\sqrt{2}+x+y) = \frac{L(\chi_{-4},2)}{\pi} + \mathcal{C}_{\sqrt{2}} + \frac{3}{8} - \frac{3}{2\pi},$$

where

$$\mathcal{C}_{\sqrt{2}} = \frac{\Gamma\left(\frac{3}{4}\right)^2}{\sqrt{2\pi^3}}\,{}_4F_3\left(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{5}{4},\frac{5}{4};1\right) - \frac{\Gamma\left(\frac{1}{4}\right)^2}{72\sqrt{2\pi^3}}\,{}_4F_3\left(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};\frac{3}{2},\frac{7}{4},\frac{7}{4};1\right),$$

$$\mathrm{m}(\sqrt{2}+x+y) = \frac{L(\chi_{-4},2)}{\pi} + \frac{\log 2}{4}.$$

$$\mathcal{C}_{\sqrt{2}} = \frac{\log 2}{4}.$$

The transformation $x \mapsto x^r$

$$P(\mathbf{x}) = \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \in \mathbb{C}[x_1, \dots, x_n] \qquad \mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n},$$

Let $A \in M(n \times n, \mathbb{Z})$, $\det(A) \neq 0$.

$$P^{(A)}(\mathbf{x}) := \sum_{\mathbf{m}} c_{\mathbf{m}} \mathbf{x}^{A\mathbf{m}}.$$

Then

$$\mathrm{m}(P) = \mathrm{m}\left(P^{(A)}\right).$$

Particular case: $x \mapsto x^r$.

The transformation $x \mapsto x^r$

L. & Roy (2023+) $r, s \in \mathbb{Z}_{>0}$,

$$\mathrm{m}_{\mathbb{D}}(x^r - a) = \begin{cases} \log^+ |a| & |a| \geq 1, \\ \frac{r}{2}\left(|a|^{\frac{2}{r}} - 1\right) & |a| \leq 1. \end{cases} \qquad \mathrm{m}_{\mathbb{D}}(x^r + y^s) = -\frac{rs}{2(r+s)}.$$

The transformation $x \mapsto x^r$

Polynomials	Mahler measure	Areal Mahler measure
$1 + x^r + y^s$	$\mathrm{m}(1+x+y)$	$\mathrm{m}(1+x+y) - \mathfrak{K}_{r,s}$
$(1+x)^r + y^s$	$r\mathrm{m}(1+x+y)$	$r\mathrm{m}(1+x+y) - \mathfrak{H}_{r,s}$

L. & Roy (2023+) Let $P(x_1, \dots, x_n) \in \mathbb{C}(x_1, \dots, x_n)^\times$ and let $P(0, x_2, \dots, x_n) \in \mathbb{C}(x_2, \dots, x_n)^\times$ be the rational function resulting from P by setting $x_1 = 0$. Let $r \in \mathbb{Z}_{>0}$. Then

$$\lim_{r \to \infty} \mathrm{m}_{\mathbb{D}}(P(x_1^r, x_2, \dots, x_n)) = \mathrm{m}_{\mathbb{D}}(P(0, x_2, \dots, x_n)).$$

Looking ahead

- What types of changes of variables keep the areal Mahler measure invariant?
- Is there a cohomological framework for areal Mahler measure?
- Areal Mahler measure for elliptic curve cases.

$$\mathrm{m}\left(x + \frac{1}{x} + y + \frac{1}{y} + 1\right) = \frac{15}{4\pi^2} L(E_{15}, 2)$$

- Areal Mahler measure of polynomials in more than 2 variables.
- What can we say of $\mathrm{m}_{\mathbb{D}}(P) - \mathrm{m}(P)$?
- Function field analogue (Roy, in progress).

Thanks for your attention!

L. & Roy (2023+)
Let r, s be positive integers. We have

$$
\begin{aligned}
&\mathrm{m}_{\mathbb{D}}\left(1+x^r+y^s\right)\\
&= \frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2) - \frac{r}{6} + \frac{\sqrt{3}r}{12\pi}\left(\zeta\left(1,\frac{r+2}{3r}\right) - \zeta\left(1,\frac{2r+2}{3r}\right) + \zeta\left(1,\frac{r+1}{3r}\right) - \zeta\left(1,\frac{2r+1}{3r}\right)\right)\\
&- \frac{2}{\pi}\sum_{1\le k}\sum_{h=0}^{\lfloor \frac{k}{2}\rfloor}\binom{k}{2h}\frac{(-1)^{h-1}{}_2F_1\left(\frac{1}{2}-h, k-h+\frac{1}{r}+\frac{1}{2}; k-h+\frac{1}{r}+\frac{3}{2}; \frac{1}{4}\right)}{2^{k-2h+1}k(kr+2)\left(2k+\frac{2}{r}-2h+1\right)} + \frac{s}{6}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{1}{kr+1}\\
&- \frac{s\sqrt{3}}{\pi}\sum_{0\le j<k}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\frac{\chi_{-3}(k-j)}{((k+j)r+2)(k-j)} + \frac{s}{4\pi}\sum_{1\le k}\binom{\frac{1}{s}}{k}^2\frac{{}_2F_1\left(\frac{1}{2}, k+\frac{1}{r}+\frac{1}{2}; k+\frac{1}{r}+\frac{3}{2}; \frac{1}{4}\right)}{(kr+1)\left(2k+1+\frac{2}{r}\right)}\\
&+ \frac{s}{\pi}\sum_{0\le j<k}\sum_{h=0}^{\lfloor \frac{k-j}{2}\rfloor}\binom{\frac{1}{s}}{k}\binom{\frac{1}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac{1}{2}-h, k-h+\frac{1}{r}+\frac{1}{2}; k-h+\frac{1}{r}+\frac{3}{2}; \frac{1}{4}\right)}{2^{k-j-2h}((k+j)r+2)\left(2k+\frac{2}{r}-2h+1\right)},
\end{aligned}
$$

where $\zeta(s,x) = \sum_{n=0}^{\infty}\frac{1}{(n+x)^s}$ is the Hurwitz zeta-function.

L. & Roy (2023+)
Let r, s be positive integers. We have

$$
\begin{aligned}
&\mathrm{m}_{\mathbb{D}}((1+x)^r + y^s) \\
&= r\left(\frac{3\sqrt{3}}{4\pi}L(\chi_{-3},2)+\frac{1}{6}-\frac{\sqrt{3}}{2\pi}\right)-\frac{s}{6}+\frac{s}{6}\frac{\Gamma\left(\frac{2r}{s}+2\right)}{\Gamma\left(\frac{r}{s}+2\right)^2} \\
&-\frac{s\sqrt{3}}{\pi}\sum_{0\le j<k}\binom{\frac{r}{s}}{k}\binom{\frac{r}{s}}{j}\frac{\chi_{-3}(k-j)}{(k+j+2)(k-j)}+\frac{s}{4\pi}\sum_{1\le k}\binom{\frac{r}{s}}{k}^2\frac{{}_2F_1\left(\frac{1}{2},k+\frac{3}{2};k+\frac{5}{2};\frac{1}{4}\right)}{(k+1)(2k+3)} \\
&+\frac{s}{\pi}\sum_{0\le j<k}\sum_{h=0}^{\lfloor\frac{k-j}{2}\rfloor}\binom{\frac{r}{s}}{k}\binom{\frac{r}{s}}{j}\binom{k-j}{2h}\frac{(-1)^{k-j+h}{}_2F_1\left(\frac{1}{2}-h,k-h+\frac{3}{2};k-h+\frac{5}{2};\frac{1}{4}\right)}{2^{k-j-2h}(k+j+2)(2k-2h+3)}.
\end{aligned}
$$