Mahler measures of successively exact polynomials

Riccardo Pengo
(based on joint work with **François Brunault**, and also with **Berend Ringeling**)

✉ Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover
pengo@math.uni-hannover.de, **riccardopengo@gmail.com**
https://sites.google.com/view/riccardopengo/

MM(P) - Mahler measures of polynomials
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A tale of many headaches...

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Given $n \in \mathbb{N}$, let $\mathcal{M}_n := m(\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\})$, and let $\mathcal{M} := \bigcup_{n \in \mathbb{N}} \mathcal{M}_n$.

Given $w \in \mathbb{Z}$ let $\mathcal{L}_w := \{L^*(M, 0) \colon M_{/\mathbb{Q}}$ pure motive of weight $w\}$, and $\mathcal{L} := \bigcup_{w \in \mathbb{Z}} \mathcal{L}_w$.

If $X_{/\mathbb{Q}}$ is smooth and projective, then $L^*(H^i(X), j) = L^*(H^i(X)(j), 0) \in \mathcal{L}_{i-2j}$ for every $i, j \in \mathbb{Z}$.

Boyd, **Deninger**, **Rodriguez-Villegas**, ...: How does a given $\mathcal{M}_n$ relate to the different $\mathcal{L}_w$'s?

Chinburg, ...: How does a given $\mathcal{L}_w$ relate to the different $\mathcal{M}_n$'s?

How does the $\mathbb{Q}$-vector space generated by $\mathcal{M}$ relate to the $\mathbb{Q}$-vector space generated by $\mathcal{L}$?

What about the $\mathbb{Q}$-algebras generated by $\mathcal{M}$ and $\mathcal{L}$?

Note that $\mathbb{Q}[\mathcal{M}] \subseteq \mathbb{P}$, where $\mathbb{P} := \mathbb{P}_{\text{eff}}[(2\pi i)^{-1}]$ denotes the $\mathbb{Q}$-algebra of **Kontsevich-Zagier** periods.

According to **Beilinson**'s conjectures, one should also have that $\mathbb{Q}[\mathcal{L}] \subseteq \mathbb{P}$

Therefore, according to **Grothendieck**'s period conjecture, it should be possible to explain any relation between $\mathcal{M}$ and $\mathcal{L}$ by the rules of calculus (additivity, change of variables, Stokes's formula).

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Deninger: For $P \in \mathbb{Z}[x_1, \dots, x_n] \setminus \{0\}$, let $\tilde{P} := P(x_1, \dots, x_{n-1}, 0)$ and $V_P := \{P = 0\} \hookrightarrow \mathbb{G}_m^n$. Then

$$m(P) - m(\tilde{P}) = \frac{(-1)^n}{(2\pi i)^{n-1}} \int_{\gamma_P} \eta_n$$

where $\gamma_P := \{|z_1| = \dots = |z_{n-1}| = 1, |z_n| \leq 1\} \cap V_P(\mathbb{C})$ and $[\eta_n] = \mathrm{reg}_\infty(\{x_1, \dots, x_n\}) \in H_{\mathcal{D}}^{n,n}(\mathbb{G}_m^n)$.

One needs to make sense of $\int_{\gamma_P} \eta_n$. **Deninger** does this by assuming that γ_P is a manifold with boundary.

Beilinson: If $X_{/\mathbb{Q}}$ is smooth and projective, then we should have that $\mathrm{reg}_\infty \colon H_{\mathcal{M}}^{i,j}(X) \otimes \mathbb{R} \xrightarrow{\sim} H_{\mathcal{D}}^{i,j}(X_\mathbb{R})$ and moreover that $\det(\mathrm{reg}_\infty) \in \mathbb{Q}^\times \cdot L^*(H^{i-1}(X), j)$, for every $i, j \in \mathbb{Z}$.

Therefore, if $\partial\gamma_P = \emptyset$ and $\gamma_P \subseteq V_P^{\mathrm{reg}}(\mathbb{C})$, and $\{x_1, \dots, x_n\}\big|_{V_P^{\mathrm{reg}}} \in \iota^*(H_{\mathcal{M}}^{n,n}(X))$ for some smooth and projective compactification $\iota \colon V_P^{\mathrm{reg}} \hookrightarrow X$, then $m(P) - m(\tilde{P}) \rightsquigarrow L^*(H^{n-1}(X), 0)$. To sum up, $\mathcal{M}_n \rightsquigarrow \mathcal{L}_{n-1}$ generically.

Deninger, **Bornhorn**: This is the case for $P = x_1 + x_1^{-1} + x_2 + x_2^{-1} + k$ when $k \in \mathbb{Z} \setminus \{-4, 0, 4\}$.

P.: This is the case for $P = x_1 + x_1^{-1} + x_2 + x_2^{-1} + x_1 x_2^{-1} + x_2 x_1^{-1} + k$ when $k \in \mathbb{Z} \setminus \{-6, 2, 3\}$.

In both cases, $m(P) \overset{?}{\sim}_\mathbb{Q} L^*(E, 0)$ for some elliptic curve E birational to V_P.

Exact polynomials

Ray: If $P = x_1 + x_1^{-1} + x_2 + x_2^{-1} + 4$ then $m(P) = 2 \cdot L^*(\chi_{-4}, -1)$ where $\chi_d(\cdot) := \left(\frac{d}{\cdot}\right)$. Where does this come from?

In this case, $\gamma_P \not\subseteq V_P^{\mathrm{reg}}$, and the blow up of the singular point is defined over $\mathbb{Q}(\sqrt{-4})$.

Smyth: If $P = x_1 + x_2 + 1$ then $m(P) = L^*(\chi_{-3}, -1)$. Where does this come from?

In this case, $\partial\gamma_P \neq \emptyset$, and in fact $\partial\gamma_P$ consists of two points, defined over $\mathbb{Q}(\sqrt{-3})$.

Guilloux & Marché: We call these polynomials **exact**, because, in both cases, $\eta_2|_{V_P^{\mathrm{reg}}}$ is exact.

Maillot, **Lalín**: If $P^* = P(x_1^{-1}, \dots, x_n^{-1})$ and $W_P := V_P \cap V_{P^*}$, then $\partial\gamma_P \subseteq W_P(\mathbb{C})$.

If P is exact, $\gamma_P \subseteq V_P^{\mathrm{reg}}(\mathbb{C})$ and $\partial\gamma_P \subseteq W_P^{\mathrm{reg}}$, we should expect that $m(P) - m(\tilde{P}) \rightsquigarrow L^*(H^{n-2}(X'), -1)$, where X' is some smooth and projective compactification of W_P^{reg} to which we can extend the primitive of $\eta_n|_{W_P^{\mathrm{reg}}}$.

Therefore, $\mathcal{M}_n \rightsquigarrow \mathcal{L}_n \cup \mathcal{L}_{n-1}$ for P generic among all polynomials, or P generic among exact polynomials.

Boyd & Rodriguez-Villegas, **Lalín**, **Brunault**: If $P = (x_1 + 1)(x_2 + 1) + x_3$ then $m(P) = -2L^*(E, -1)$ where $W_P \approx E$.

Boyd, **Bornhorn**: If $P = (x_1^2 + 1)(x_2^2 + 1) + 2x_1x_2$ then $m(P) \overset{?}{=} L^*(E, 0) + L^*(\chi_{-3}, -1)$.

Boyd: If $P = x_2^2 + kx_2(x_1 + 1) + x_1^3$ then $m(P) - \frac{1}{3}\log|k| \overset{?}{\sim}_{\mathbb{Q}} L^*(E, 0)$. In this case, $\{x_1, x_2\}|_{V_P^{\mathrm{reg}}}$ does **not** extend to a smooth compactification. Also, **Smyth** showed that $m((x+y)^2 + 2) = \frac{1}{2}\log(2) + L^*(\chi_{-4}, -1)$.

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General Mahler measures as Deligne periods

Smyth: If $P = x_1 + x_2 + x_3 + 1$ then $m(P) = -14 \cdot \zeta'(-2)$. Where does this come from?

Here P is exact, but $\partial\gamma_P \not\subseteq W_P^{\mathrm{reg}}(\mathbb{C})$. Moreover, W_P is a union of three lines defined over $\mathbb{Q}$.

Lalín: When $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_3^{\pm 1}]$, we can check **double exactness** on V_P and on $\mathrm{Res}_{x_3}(P, P^*) = 0$.

Lalín: *It remains, of course, to find a general algebraic way of describing the successive sets that we obtain by taking boundaries.*

Brunault & P.: For $P \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \setminus \{0\}$, we have that

$$m(P) - m(\tilde{P}) = \frac{(-1)^n}{(2\pi i)^{n-1}} \langle \mathrm{reg}_\infty(\{x_1, \dots, x_n\}|_{V_{PP^*}}), \beta_P \rangle_{V_{PP^*}},$$

where $\beta_P \in H_{n-1}(V_{PP^*}(\mathbb{C}); \mathbb{Z})$ is a symmetrized version of γ_P.

The technical restrictions of **Deninger**'s theorem apply. It should be possible to remove them using **Hironaka**'s triangulations, or the alternative approach taken by **Besser & Deninger**.

Let X be smooth and projective, with a s.n.c.d. $A \cup B \subseteq X$ such that $X \setminus A \twoheadrightarrow V_{PP^*}$ is a resolution of singularities with exceptional divisor $B \setminus A$. Then, $m(P) - m(\tilde{P})$ is a **Deligne period** of $H_{\mathcal{M}}^{n,n}(X \setminus A, B \setminus A)$.

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Successive exactness

Brunault & P.: Using the relative cohomology and weight spectral sequences, we see that

$$m(P) - m(\tilde{P}) \rightsquigarrow \left\{ L^*(H^{n-1-|I|-|J|}(A_I \cap B_J), -|J|) \colon I \subseteq \{1, \dots, r\},\ J \subseteq \{1, \dots, s\} \right\}$$

where $A_I := \bigcap_{i \in I} A_i$ and $B_J := \bigcap_{j \in J} B_j$, with $A = A_1 \cup \cdots \cup A_r$ and $B = B_1 \cup \cdots \cup B_s$.

When only those L-values with $|J| \geq k$ appear, we say that P is at least k-**times exact**.

D'Andrea & Lalín: If $P = (x_1 - 1)(x_2 - 1) - (x_3 - 1)(x_4 - 1)$ then $m(P) = -18 \cdot \zeta'(-2)$. Here, $|I| = 1$ and $|J| = 2$.

Lalín: Let $P = x_1(x_2 - 1) \cdots (x_n - 1) + (x_2 + 1) \cdots (x_n + 1)$. Then $|I| + |J| = n - 1$ and $|I| \in 2 \cdot \{0, 1, \dots, m-1\}$ if $n - 1 = 2m$ (giving $m(P) \in \langle \zeta^*(-2), \dots, \zeta^*(-2m) \rangle_{\mathbb{Q}}$) or $n = 2m$ (giving $m(P) \in \langle L^*_{-4}(-1), \dots, L^*_{-4}(1-2m) \rangle_{\mathbb{Q}}$ where $L_{-4}(s) := L(\chi_{-4}, s)$).

Brunault & P.: If $P = x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + x_1x_2 + x_1x_3 - x_2x_3 - x_2x_4 + x_3x_4 - x_2 + x_3 - x_4 + 1$ then

$$m(P) \stackrel{?}{=} \frac{1}{6} \cdot L^*(E, -2),$$

because $I = \emptyset$, and $J = \{1, 2\}$ gives an elliptic curve $E = B_1 \cap B_2$, of conductor 32.

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Numerical experiments and further mysteries

We found this 4-variable polynomial by looking at P with small coefficients, such that V_P is smooth, $W_P = W_1 \cup W_2$ with W_1, W_2 smooth and rational, and $W_1 \cap W_2 \approx E$. Then, how did we compute $m(P)$?

Rodriguez-Villegas: Note that $2m(P) = \log(k) - \int_0^{1/k} \phi_P(t)dt$, where $k := [P \cdot P^*]_0$, while

$$\phi_P(t) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \frac{Q(z_1, \dots, z_n)}{1 - t \cdot Q(z_1, \dots, z_n)} \frac{dz_1}{z_1} \cdots \frac{dz_n}{z_n}$$

where $Q := P \cdot P^* - k$.

P. & Ringeling: Using **creative telescoping**, or **Lairez**'s algorithm, one can find a polynomial ODE solved by ϕ_P. Finding this can be **slow**, but then it is **very fast** to compute $m(P)$ with very high precision.

Using this algorithm, one can compute $m(1 + x_1 + x_2 + x_3 + x_1x_2 + x_1x_3 - x_2x_3)$ to very high precision. It does **not** seem to be a rational linear combinations of **pure** L-values, and it does **not** fit into **Trieu**'s general framework... Are we facing some **multiple** L-values, in the sense of **Brown**?

Can one use these explicit expressions for $m(P)$ to write it as a **motivic period**, in the sense of **Brown**? Is it **single-valued**, in the sense of **Brown & Dupont**? Are $\mathcal{M}$ or $\mathbb{Q}[\mathcal{M}]$ stable under the **motivic Galois group**?

Given $m_1, \dots, m_r \in \mathcal{M}$, what should $\dim(\langle m_1, \dots, m_r \rangle_{\mathbb{Q}})$ and $\mathrm{trdeg}(\mathbb{Q}(m_1, \dots, m_r)/\mathbb{Q})$ be?

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Thank you very much for your attention!

Je weiter sich die Musik entwickelt,
desto komplizierter wird der Apparat,
den der Komponist aufbietet,
um seine Ideen auszudrücken.

The more the music develops,
the more complex becomes the apparatus
which the composer summons
to express his ideas.

Gustav Mahler
Letter to **Gisela Tolnay-Witt**

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