The Mahler measure of Fekete polynomials

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The Mahler measure of a one variable polynomial

- Recall that the Mahler measure of a one variable polynomial P is given by
$$M(P) = \exp\left(\int_0^1 \log|P(e(t))|dt\right),$$
where $e(t) := e^{2\pi it}$.

- We have
$$M(P) = \lim_{q\to 0^+} \|P\|_q = \lim_{q\to 0^+} \left(\int_0^1 |P(e(t))|^q dt\right)^{1/q}.$$

The class of Littlewood polynomials

- A polynomial P is called a **Littlewood polynomial**, if all its coefficients are in $\{-1, 1\}$.
- For a positive integer $n \geq 1$, we will denote by $\mathcal{L}_n$ the class of all Littlewood polynomials of degree n.
- These polynomials have been extensively studied with respect to different aspects : L_q norms and Mahler measure, maximal size, the number of real roots, ...
- By Parseval's formula we have $\|P\|_2 = \sqrt{n+1}$, for any $P \in \mathcal{L}_n$.

Littlewood's flatness Conjecture, 1966 (proved by Balister, Bollobás, Morris, Sahasrabudhe, and Tiba, 2020)

There exists positive constants c_1, c_2 such that for all $n \geq 2$, there exists $P \in \mathcal{L}_n$ such that for all $z \in \mathbb{C}$ with $|z| = 1$ we have

$$c_1\sqrt{n} \leq |P(z)| \leq c_2\sqrt{n}.$$

The Mahler measure of Littlewood polynomials

- Let $P \in \mathcal{L}_n$. By Jensen's inequality we have

$$M(P) \leq \|P\|_2 = \sqrt{n+1}.$$

Mahler's problem for Littlewood's polynomials

Do there exists a positive constant $\varepsilon > 0$, such that for every Littlewood polynomial P of degree n we have

$$M(P) \leq (1-\varepsilon)\sqrt{n}.$$

- The largest known value of $M(P)/\|P\|_2$ for a Littlewood polynomial of of degree $n \geq 1$ is $0.98636...$, achieved by

$$P(x) = x^{12} + x^{11} + x^{10} + x^9 + x^8 - x^7 - x^6 + x^5 + x^4 - x^3 + x^2 - x + 1.$$

Theorem (Choi and Erdélyi, 2014)

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{P\in\mathcal{L}_n} \frac{M(P)}{\sqrt{n}} = e^{-\gamma/2} = 0.749306...,$$

where $\gamma = 0.577215...$ is the Euler-Mascheroni constant.

Choi and Erdélyi (2015)

Constructed an explicit sequence of Littlewood polynomials $\{P_n\}_{n\geq 1}$ such that $\deg(P_n) = n$ and

$$\frac{M(P_n)}{\sqrt{n}} \geq \frac{1}{2} + o(1), \text{ as } n \to \infty.$$

Two important examples: Rudin-Shapiro polynomials and Fekete polynomials

- The Rudin-Shapiro polynomials are defined recursively as follows:
- $P_0(z) = Q_0(z) = 1$.
- $P_{k+1}(z) := P_k(z) + z^{2^k} Q_k(z)$, and $Q_{k+1}(z) := P_k(z) - z^{2^k} Q_k(z)$.
- Note that both P_k and Q_k are Littlewood polynomials of degree $2^k - 1$.
- These polynomials have applications in signal processing and communication systems.

Saffari's conjecture, 1985 (proved by Erdélyi in 2020)

$$\lim_{k\to\infty} \frac{M(P_k)}{2^{k/2}} = \lim_{k\to\infty} \frac{M(Q_k)}{2^{k/2}} = \sqrt{\frac{2}{e}} = 0.857763... > e^{-\gamma/2}.$$

Fekete polynomials

- Let p be a prime number and

$$F_p(z) = \sum_{n=1}^{p-1} \left(\frac{n}{p}\right) z^n$$

be the Fekete polynomial attached to p, where $\left(\frac{n}{p}\right)$ is the usual Legendre symbol modulo p defined by

$$\left(\frac{n}{p}\right) = \begin{cases} 1 & \text{if } n \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } n \text{ is a quadratic non-residue modulo } p, \\ 0 & \text{if } p \mid n. \end{cases}$$

Fekete

If $F_p(x)$ has no zero in $(0,1)$ then the Dirichlet L-function $L\left(s, \left(\frac{\cdot}{p}\right)\right)$ has no real zero $s > 0$.

Fekete Hypothesis (1912)

For p large enough, F_p has no zero in $(0,1)$.

- This was disproved by Pólya in 1919, for a positive proportion of the primes.
- The Fekete Hypothesis was again conjectured by Chowla in 1936 and disproved by Heilbronn in 1937.

Problem (Sarnak)

Do there exist infinitely many p such that F_p has no zero in $(0,1)$?

The Mahler measure of Fekete polynomials

- $F_p(z)/z$ is a Littlewood polynomial whose Mahler measure is the same as the Mahler measure of F_p.
- Littlewood (1966): There exists a constant $\varepsilon_0 > 0$ such that for all primes $p \geq 3$ we have

$$M(F_p) \leq (1 - \varepsilon_0)\sqrt{p}.$$

Theorem (Erdélyi and Lubinsky, 2007)

For every fixed $\varepsilon > 0$, there is a constant c_ε such for all primes $p \geq c_\varepsilon$

$$M(F_p) \geq (1/2 - \varepsilon)\sqrt{p}.$$

Theorem (Erdélyi, 2018)

There exists a constant $\delta > 0$ such that for sufficiently large primes

$$M(F_p) \geq (1/2 + \delta)\sqrt{p}.$$

Erdélyi (Survey, 2020)

"... this problem (of finding an asymptotic for $M(F_p)$ as $p \to \infty$) seems to be beyond reach at the moment. Not even a (published or unpublished) conjecture about the asymptotic seems to be known.

Theorem 1 (Klurman, L, and Munsch, 2023+)

We have

$$M(F_p) \sim \kappa\sqrt{p}, \text{ as } p \to \infty,$$

where $\kappa = 0.74083...$ is an explicit constant.

Note that $\kappa < e^{-\gamma/2} = 0.749306...$, which is the average of $M(P)/\|P\|_2$ over the family of Littlewood polynomials.

A probabilistic random model for F_p

- For $\zeta_p := e(1/p)$ note that $F_p(\zeta_p) = \sum_{n=1}^{p} \left(\frac{n}{p}\right) e(n/p)$ is the Gauss sum attached to the Legendre sum modulo p, and $|F(\zeta_p)| = \sqrt{p}$.
- Then we have

$$\log M(F_p) = \int_0^1 \log|F_p(e(t))|dt = \frac{1}{p}\sum_{k=0}^{p-1}\int_0^1 \log\left|F_p\left(e\left(\frac{k+t}{p}\right)\right)\right|dt$$

$$= \log(\sqrt{p}) + \frac{1}{p}\sum_{k=0}^{p-1}\int_0^1 \log|G_p(k,t)|dt,$$

where for $t \in [0,1]$ we define

$$G_p(k,t) := \frac{F_p\left(e\left(\frac{k+t}{p}\right)\right)}{F_p(\zeta_p)}.$$

Recall

$$G_p(k,t) := \frac{F_p\left(e\left(\frac{k+t}{p}\right)\right)}{F_p(\zeta_p)}.$$

Our goal

Show that

$$\lim_{p\to\infty} \frac{1}{p}\sum_{k=0}^{p-1}\int_0^1 \log|G_p(k,t)|dt = \log\kappa.$$

Elementary identity (Conrey, Granville, Poonen, and Soundararajan, 2000)

We have

$$G_p(k,t) = \sum_{|m|\leq(p-1)/2}\left(\frac{k-m}{p}\right)\frac{e(t)-1}{p(e(\frac{m-t}{p})-1)}.$$

Recall $G_p(k,t) = \sum_{|m|\le(p-1)/2} \left(\frac{k-m}{p}\right) \frac{e(t)-1}{p(e(\frac{m-t}{p})-1)}$.

Heuristic argument

- The shifts $\left\{\left(\frac{k-m}{p}\right)\right\}_{|m|\le(p-1)/2}$ for $1 \le k \le p$ behave like **independent random variables** taking the values ± 1 with probability $1/2$.
- $p(e(\frac{m-t}{p})-1) \approx 2\pi i(m-t)$ if p is large.

Let $G_{\mathbb{X}}$ be the random process on $C[0,1]$ defined by

$$G_{\mathbb{X}}(t) := \sum_{m\in\mathbb{Z}} \frac{e(t)-1}{2\pi i(m-t)} \mathbb{X}(m), \quad t \in [0,1],$$

where $\{\mathbb{X}(m)\}_{m\in\mathbb{Z}}$ are I. I. D. random variables taking the values ± 1 with equal probability $1/2$.

Theorem 2 (Klurman, L, and Munsch, 2023+)

- The sequence of processes $(G_p(k,t))_{t\in[0,1]}$ **converges in law** (in the Banach space $C[0,1]$) to the random process $(G_{\mathbb{X}}(t))_{t\in[0,1]}$.
- For any bounded continuous function $\varphi : C[0,1] \to \mathbb{C}$ we have

$$\lim_{p\to\infty} \frac{1}{p} \sum_{k=0}^{p-1} \varphi\left(G_p(k,t)\right) = \mathbb{E}\varphi(G_{\mathbb{X}}).$$

Steps of the proof

1. Show that the process $(G_{\mathbb{X}}(t))_{t\in[0,1]}$ is a $C[0,1]$-valued random variable (that is $G_{\mathbb{X}}(t)$ is almost surely continuous on $[0,1]$).
2. Prove that the sequence of processes $(G_p(k,t))_{t\in[0,1]}$ converges to the random process $(G_{\mathbb{X}}(t))_{t\in[0,1]}$ in the sense of **convergence in finite distributions**.
3. Prove that the sequence of processes is **tight**, using Kolmogorov's tightness criterion.

Convergence in the sense of finite distributions

- Let $(Y_n)_n$ be a sequence of $C([0,1])$-valued random variables, and let Y be a $C([0,1])$-valued random variable.

Definition of convergence in the sense of finite distributions

We say that $(Y_n)_n$ converges to Y in the sense of **finite distributions** if and only if, for all integers $k \geq 1$, and for all $0 < t_1 < \cdots < t_k < 1$, the random vectors $(Y_n(t_1), ..., Y_n(t_k))$ converge in law to $(Y(t_1), ..., Y(t_k))$.

Proposition (Klurman, L, and Munsch, 2023+)

For all integers $k \geq 1$, and for all $0 < t_1 < \cdots < t_k < 1$, the vectors $(G_p(k,t_1), ..., G_p(k,t))$ converge in law to $(G_{\mathbb{X}}(t_1), ..., (G_{\mathbb{X}}(t_k))$, as $p \to \infty$.

- We use the method of moments.
- The key ingredient is Weil's proof for the Riemann Hypothesis for curves over finite fields.

Prokhorov's Theorem

Suppose such that $(Y_n)_n$ converges to Y in the sense of finite distributions. Then $(Y_n)_n$ converges in law in $C([0,1])$ to Y if and only if $(Y_n)_n$ is **tight**.

Kolmogorov's tightness criterion

The sequence $(Y_n)_n$ is tight if we can show the existence of positive real numbers α, δ, C such that, for any real numbers $0 \leq s < t \leq 1$ and any $n \geq 1$ we have

$$\mathbb{E}\left(|Y_n(t) - Y_n(s)|^\alpha\right) \leq C|t-s|^{1+\delta}.$$

Proposition (Klurman, L, and Munsch, 2023+)

The sequence $(G_p(k,\cdot))_k$ is tight. More precisely, there exists an absolute constant C, such that for odd primes p and all $0 \leq s < t \leq 1$ we have

$$\frac{1}{p}\sum_{k=0}^{p-1} |G_p(k,t) - G_p(k,s)|^2 \leq C|t-s|^{3/2}.$$

Application : L_q norms of Fekete

- Recall $\|P\|_q = \left(\int_0^1 |P(e(t)|^q dt\right)^{1/q}$.
- Erdélyi (2012): For all $q > 0$ we have $\|F_p\|_q \asymp_q \sqrt{p}$.
- Günther and Schmidt (2017): For every integer $k \geq 1$ we have

$$\lim_{p\to\infty} \frac{\|F_p\|_{2k}}{\sqrt{p}} = (\ell_{2k})^{1/2k},$$

where the constant ℓ_{2k} is given via a series of rather complicated recursive combinatorial identities.

Corollary (Klurman, L, and Munsch, 2023+)

For any $0 < q < \infty$, we have

$$\lim_{p\to\infty} \frac{\|F_p\|_q}{\sqrt{p}} = k_q := \left(\int_0^1 \mathbb{E}(|G_{\mathbb{X}}(t)|^q) dt\right)^{1/q}.$$

What about the Mahler measure of Fekete?

$$\log M(F_p) = \log(\sqrt{p}) + \frac{1}{p}\sum_{k=0}^{p-1}\int_0^1 \log|G_p(k,t)|dt,$$

$$G_p(k,t) = \sum_{|m|\le (p-1)/2} \left(\frac{k-m}{p}\right)\frac{e(t)-1}{p(e(\frac{m-t}{p})-1)},$$

$$G_{\mathbb{X}}(t) = \sum_{m\in\mathbb{Z}} \frac{e(t)-1}{2\pi i(m-t)}\mathbb{X}(m).$$

- Our goal is to show that

$$\lim_{p\to\infty}\frac{1}{p}\sum_{k=0}^{p-1}\int_0^1 \log|G_p(k,t)|dt = \log\kappa = \int_0^1 \mathbb{E}\left(\log|G_{\mathbb{X}}(t)|\right)dt.$$

- Big Technical problem : The functional $\ell(\phi) = \int_0^1 \log|\phi(t)|dt$ is **not continuous** on $C([0,1])$ and so we cannot apply Theorem 2 directly.

How to overcome this problem?

- We need to control the logarithmic singularities of $G_p(k,t)$ (and $G_{\mathbb{X}}(t)$).
- Let $H_p(k,t) = G_p(k,t)/(e(t)-1)$ and $H_{\mathbb{X}}(t) = G_{\mathbb{X}}(t)/(e(t)-1)$.
- Let $\varepsilon > 0$ be small and fixed, and consider the functional

$$\widetilde{\ell}_{\varepsilon}(\phi) = \int_{\varepsilon}^{1-\varepsilon} \log(|\phi(t)|)\mathbf{1}_{|\phi(t)|\geq\varepsilon} dt.$$

- By our Theorem 2, the sequence of processes $(H_p(k,t))_{t\in[\varepsilon,1-\varepsilon]}$ converges in law (in the space $C[\varepsilon, 1-\varepsilon]$) to the random process $(H_{\mathbb{X}}(t))_{t\in[\varepsilon,1-\varepsilon]}$.
- Since $\widetilde{\ell}_{\varepsilon}$ is a continuous functional on $C[\varepsilon, 1-\varepsilon]$ we deduce that

$$\lim_{p\to\infty} \frac{1}{p} \sum_{0\leq k\leq p-1} \int_{\varepsilon}^{1-\varepsilon} \log(|H_p(k,t)|)\mathbf{1}_{|H_p(k,t)|\geq\varepsilon} dt$$
$$= \int_{\varepsilon}^{1-\varepsilon} \mathbb{E}\Big(\log(|H_{\mathbb{X}}(t)|)\mathbf{1}_{|H_{\mathbb{X}}(t)|\geq\varepsilon}\Big) dt.$$

- First, we show that if p is large then

$$\frac{1}{p}\sum_{0\le k\le p-1}\left(\int_0^{\varepsilon}+\int_{1-\varepsilon}^{1}\right)\log|H_p(k,t)|\mathbf{1}_{|H_p(k,t)|\ge\varepsilon}dt \ll \varepsilon\log(1/\varepsilon).$$

- Next we show that there is an absolute constant $c>0$ such that for all $0\le k\le p-1$ we have one of the following four cases :

$$\widetilde{H}_p'(k,t)>c,\ \ \widetilde{H}_p'(k,t)<-c,\ \ \widetilde{H}_p''(k,t)>c,\ \ \text{or } \widetilde{H}_p''(k,t)<-c.$$

- We use that to show that for all $0\le k\le p-1$,

$$\int_0^1 \log(|H_p(k,t)|)\mathbf{1}_{|H_p(k,t)|\le\varepsilon}dt \ll \varepsilon^{6/25}.$$

- We establish analogous bounds for the random model $H_{\mathbb{X}}(t)$ using the same ideas.

How to compute the constant κ?

- For any $J \geq 1$, we define the approximate random process

$$G_{\mathbb{X}}^{J}(t) = \sum_{m \in \mathbb{Z}, |m| \leq J} \frac{e(t) - 1}{2\pi i (m - t)} \mathbb{X}(m), \quad t \in [0, 1].$$

- Using the same ideas of the proofs of Theorem 1 and 2 we can show that

$$\mathbb{E}\left(\int_0^1 \log |G_{\mathbb{X}}(t)| dt \right) = \lim_{J \to \infty} \mathbb{E}\left(\int_0^1 \log |G_{\mathbb{X}}^{J}(t)| dt \right).$$

- Then we deduce that

$$\log(\kappa) = -\log(2\pi) + \lim_{J \to \infty} \frac{1}{2^{2J+1}} \sum_{\delta_m \in \{-1,1\}^{2J+1}} \int_0^1 \log \left| \sum_{|m| \leq J} \frac{\delta_m}{m - t} \right| dt,$$

since $\int_0^1 \log |e(t) - 1| dt = 0$.

Thank you very much for your attention!