MM(P): Mahler Measures of Polynomials

Problems from an international conference
on the occasions of Mahler's 120th birthday
and 90 years of Lehmer's problem

Radboud University Nijmegen

Below we list the problems from the *problem session* held on the afternoon of Thursday 26 October 2023. The notation $\mathrm{m}(P)$ stands for the logarithmic Mahler measure of a non-zero Laurent polynomial $P(x_1,\dots,x_r)$:

$$\mathrm{m}\big(P(x_1,\dots,x_r)\big) = \frac{1}{(2\pi i)^r} \int\!\!\cdots\!\!\int_{|x_1|=\cdots=|x_r|=1} \log|P(x_1,\dots,x_r)|\,\frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_r}{x_r}.$$

Problem 1 (posed by Chris Smyth)**.** For a two-variable polynomial $F(x,y)$ we know by work of Boyd and Lawton that $\mathrm{m}(F(x,x^n))$ tends to $\mathrm{m}(F(x,y))$ as $n\to\infty$. Here F should be irreducible, with its Newton polytope being 2-dimensional. Find a positive *lower* bound for the modulus of the difference, when it exists. This would make $\mathrm{m}(F(x,y))$ a genuine limit point of one-variable polynomial Mahler measures. A nonzero asymptotic expansion for $\mathrm{m}(F(x,x^n))-\mathrm{m}(F(x,y))$ would also guarantee that $\mathrm{m}(F(x,y))$ is a genuine limit point. But examples such as $F(x,y)=x+y+2$ show that the sequence $\mathrm{m}(F(x,x^n))-\mathrm{m}(F(x,y))$ can be zero for all $n\in\mathbb{N}$. This 'genuine limit point' issue was first raised by Boyd in his *Speculations ...* paper of 1981.

A further question is whether $\mathrm{m}(F(x,y)$ is a two-sided limit of the sequence $\{\mathrm{m}(F(x,x^n))\}_{n\in\mathbb{N}}$. This was shown to be true for the example $F(x,y)=x+y+1$ by Boyd in Appendix 2 of the same paper.

Problem 2 (posed by François Brunault)**.** There is a fundamental notion of *exact* polynomials that come naturally from the works of C. Deninger, F. Rodrigues Villegas and M. Lalín, and formally defined and investigated in 2021 by A. Guilloux and J. Marché. In the one-variable case, a Laurent polynomial $P(x)\in\mathbb{C}[x^{\pm1}]$ is exact when all its non-zero roots have modulus one. In the two-variable case, a Laurent polynomial $P(x,y)\in\mathbb{C}[x^{\pm1},y^{\pm1}]$ vanishing on a curve

$C \subset (\mathbb{C}^*)^2$ is said to be exact if there is a 'volume' function $V(x,y)\colon C \to \mathbb{R}$ such that $\mathrm{d}V = \eta_2\big|_C$, where $\eta_2(x,y) = \log|y|\,\mathrm{d}\arg x - \log|x|\,\mathrm{d}\arg y$. The definition extends to the multi-variable case by replacing the differential form η_2 with an explicit regulator $(m-1)$-form $\eta_m(x_1,\dots,x_m)$ on the zero locus of the polynomial (a related discussion appears in the talk by R. Pengo).

Experimentally we observe that, given two exact polynomials A and B in $\mathbb{C}[x_1,\dots,x_m]$, normalised so that their constant terms have modulus 1, the polynomial $A + By$ in $m+1$ variables $x_1,\dots,x_m,y$ is exact. This can be shown rigorously when $m = 1$ (in this case, this amounts to the existence of the Bloch–Wigner dilogarithm $D\colon \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$) but no argument is known for $m > 1$.

As an immediate reaction from the audience, the following question was raised.

Problem 3 (posed by Pavlo Yatsyna)**.** Given two Laurent polynomials $A, B \in \mathbb{C}[x_1^{\pm 1},\dots,x_m^{\pm 1}]$ with $m \ge 2$, assume that the polynomial $A + By$ in $m+1$ variables $x_1,\dots,x_m,y$ is exact. Is then true that the polynomials $A(x_1,\dots,x_m)$ and $B(x_1,\dots,x_m)$ are exact?

Problem 4 (posed by Riccardo Pengo and Fabien Pazuki)**.** For a number field K with r_1 real embeddings and r_2 conjugate pairs of complex embeddings, denote by $r = r_1 + r_2 - 1$ the rank of its group of units and by σ_j, where $j = 1,\dots,r,r+1$, the corresponding embeddings (with only one representative from each pair of complex embeddings). Choose a set $\{\alpha_i\}_{i=1}^r$ of generators for the unit group of K modulo roots of unity and consider the regulator

$$\operatorname{Reg}(K) = \det\big(m_j \log|\sigma_j(\alpha_i)|\big)_{i,j=1}^r,$$

where $m_j = 1$ or 2 depending on whether σ_j is real or complex. Is it true that there is a set of polynomials $P_1,\dots,P_r \in \mathbb{Q}[x]$ such that

$$\frac{\mathrm{m}(P_1)\cdots\mathrm{m}(P_r)}{\operatorname{Reg}(K)} \in \mathbb{Q}^\times\ ?$$

A weaker form of the question is as follows: Is it true that there is a set of polynomials $P_{ij} \in \mathbb{Q}[x]$, where $i,j = 1,\dots,r$, such that

$$\frac{\det\big(\mathrm{m}(P_{ij})\big)_{i,j=1}^r}{\operatorname{Reg}(K)} \in \mathbb{Q}^\times\ ?$$

Problem 5 (posed by Matilde Lalín)**.** There is a nice general formula for the logarithmic Mahler measure of the $(m+1)$-variable 'polynomial'

$$P(x_1,\dots,x_m,x) = 1 + \left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_m}{1+x_m}\right)x;$$

though it is not quite a polynomial, the measure does not change after multiplying this rational function by $(1+x_1)\cdots(1+x_m)$, thus transforming it into a polynomial in $\mathbb{Z}[x_1,\dots,x_m,x]$. Is it possible to compute its *areal* Mahler measure $\mathrm{m}_{\mathbb{D}}(P)$? The latter is defined in Lalín's talk and means the integration over each torus $|x_j|=1$ against $\mathrm{d}x_j/x_j$ replaced with the one over the unit disk $|x_j|\le 1$ against $\mathrm{d}x_j$.

Problem 6 (communicated by Wadim Zudilin, following observations of Chris Smyth)**.** The recent book *Around the Unit Circle: Mahler Measure, Integer Matrices and Roots of Unity* by J. McKee and C. Smyth contains Table D.2 of known small Mahler measures of two-variable polynomials (which extends an earlier table from the 2005 paper of D. Boyd with M. Mossinghoff). The table starts with[1]

$$
\begin{aligned}
\mathrm{m}(Q_1(x,y)) &= 0.22748\ldots, \quad Q_1(x,y) = yx^4 - x^3 - y - \frac{1}{y} - \frac{1}{x^3} + \frac{1}{yx^4},\\
\mathrm{m}(Q_2(x,y)) &= 0.25133\ldots, \quad Q_2(x,y) = x + y + 1 + \frac{1}{y} + \frac{1}{x},\\
\mathrm{m}(Q_3(x,y)) &= 0.26933\ldots, \quad Q_3(x,y) = yx^2 + x^2 + yx - 1 + \frac{1}{yx} + \frac{1}{x^2} + \frac{1}{yx^2},\\
\mathrm{m}(Q_4(x,y)) &= 0.27436\ldots, \quad Q_4(x,y) = yx^6 + x^5 + y + \frac{1}{y} + \frac{1}{x^5} + \frac{1}{yx^6}.
\end{aligned}
$$

Some of these instances are numerically identified as L-values of elliptic curves in the works of Boyd and of Boyd and Mossinghoff; for example, $\mathrm{m}(Q_1)=\mathrm{m}(\tilde{Q}_1)=L'(E_{14},0)$ and $\mathrm{m}(Q_2)=L'(E_{15},0)$ where the conductor 14 and 15 elliptic curves are given by $\tilde{Q}_1(x,y)=xy+y+x+1+1/x+1/y+1/(xy)=0$ and $Q_2(x,y)=0$, respectively. Some entries in the table are known to be the Mahler measures of the Alexander polynomials of links.

The table contains shortest known Laurent-polynomial representation of the corresponding Mahler measures; the zero locus does not necessarily correspond to a curve of smallest possible genus. The relation

$$
Q_1(x,y) = \frac{(x^2+1)(x^6y^2 - x^5y - x^4y^2 + x^3y - x^2 - xy + 1)}{yx^4} = 0
$$

corresponds to a genus 3 curve $x^6y^2 - x^5y - x^4y^2 + x^3y - x^2 - xy + 1 = 0$, and the latter polynomial is also $x^3y\tilde{Q}_1(-x^2,-xy)$. The zero loci of $Q_3(x,y)$ and of $Q_4(x,y)$ are genus 7 curves; the latter variety can be also identified with $\tilde{Q}_4(-x^2,-x^3y)=0$, where

$$
\tilde{Q}_4(x,y) = xy + y + \frac{y}{x} + x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{x}{y} + \frac{1}{y} + \frac{1}{xy}
$$

[1] We replace the Mahler measures with the logarithmic Mahler measures here.

has 11 terms in the Laurent expansion but corresponds to a genus 3 curve.

It would be interesting to have a complete identification of the entries of Table D.2 from the book in terms of L-values, at least to understand a geometric significance of these Mahler measures.

Problem 7 (communicated by Wadim Zudilin, stated in the talk by Mahya Mehrabdollahei)**.** In a 1984 preprint T. Chinburg has speculated that, for every odd quadratic character χ_{-D}, there exists a non-zero polynomial $P = P_D(x, y) \in \mathbb{Z}[x, y]$ such that $\mathrm{m}(P_D)/L'(\chi_{-D}, -1) \in \mathbb{Q}^\times$. This is known in the literature as Chinburg's conjecture, and for a finite list of positive $D \equiv 0, 3 \pmod 4$ such polynomials have been explicitly constructed.

Chinburg also claimed to establish a weaker form of his expectation, namely that, given an odd quadratic character χ_{-D}, there is a *rational* function $R = R_D(x, y) = P_D(x, y)/Q_D(x, y)$ with $P_D, Q_D \in \mathbb{Z}[x, y]$ such that

$$\frac{\mathrm{m}(R_D)}{L'(\chi_{-D}, -1)} = \frac{\mathrm{m}(P_D) - \mathrm{m}(Q_D)}{L'(\chi_{-D}, -1)} \in \mathbb{Q}^\times.$$

However an unrepairable mistake makes his argument invalid and leaves the claim an open problem as well. The latter is known as the weak (version of) Chinburg's conjecture.