MM(P): Mahler Measures of Polynomials

An international conference
on the occasions of Mahler's 120th birthday
and 90 years of Lehmer's problem

Radboud University Nijmegen

The first conference dedicated exclusively to the MM and titled "The Many Aspects of Mahler's Measure" was organised 20 years ago at BIRS (Canada). The report from the event is available online in pdf.

A few years later there was another successful workshop on the topic, "Variations on Mahler's Measure" at CIRM Luminy, see the archive copy; then another event "Mahler Measure in Mobile" — the programme of the conference as well as pictures from it are still accessible.

There were later several workshops featuring the MM as a subtheme; there was one MM summer school; there were attempts to make another MM conference.

It finally materialises: we are happy to measure progress of recent MM developments in Nijmegen.

Citing the last passage from the 2003 BIRS report, "It is to be hoped that the research inspired by such workshops will continue to unravel the mysteries of Mahler's marvellous measure."

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Radboud University

IN DEI NOMINE FELICITER

Tuesday 24 October 2023, room HG 00.303	
$09^{30} - 09^{45}$	*Opening*
$09^{50} - 10^{50}$	François Brunault (*keynote*): Mahler measures and modular forms
$11^{15} - 11^{45}$	Mahya Mehrabdollahei: The Mahler measure of certain exact polynomial families
$11^{50} - 12^{20}$	Riccardo Pengo: Mahler measures of successively exact polynomials
$15^{15} - 15^{45}$	Marie José Bertin: Mahler measure of polynomials defining singular K3 surfaces
$15^{50} - 16^{20}$	Thu Hà Trieu: The Mahler measure exact polynomials in three variables
$16^{40} - 17^{10}$	Jan-Willem van Ittersum: Hedgehogs in Lehmer's problem
$17^{15} - 17^{45}$	Michael Mossinghoff: The Lind–Mahler measure and integer group determinants

Wednesday 25 October 2023, room HG 00.303	
$09^{15} - 09^{45}$	Youness Lamzouri: On the Mahler measure of Fekete polynomials
$09^{50} - 10^{50}$	Artūras Dubickas (*keynote*): Mahler measures, their quotients and differences
$11^{15} - 11^{45}$	Berend Ringeling: Analytic continuation of the zeta Mahler function
$11^{50} - 12^{20}$	Harry Schmidt: Lower bounds for the canonical height of polynomials with tame monodromy

Thursday 26 October 2023, room HG 00.303	
$09^{15} - 09^{45}$	Fabien Pazuki: Explicit bounds on the coefficients of modular polynomials and the size of $X_0(N)$
$09^{50} - 10^{50}$	Matilde Lalín (*keynote*): Evaluations of areal Mahler measure of multivariable polynomials
$11^{15} - 11^{45}$	Detchat Samart: Some recent results on Mahler measures of curves parametrized by modular units
$11^{50} - 12^{20}$	Hourong Qin: L-values of congruent number elliptic curves
$15^{15} - 15^{45}$	Chris Smyth: Thue sets
$16^{00} - 17^{00}$	Philipp Habegger (*keynote*): The transfinite diameter of finite trees and dynamical Schinzel–Zassenhaus bounds
$17^{05} - 18^{00}$	*Problem session*
$18^{30} - 22^{00}$	*Conference dinner*

Friday 27 October 2023, room HG 00.303	
$09^{15} - 09^{45}$	Hang Liu: Mahler measure of $\mathbb{Q}$-curves
$09^{50} - 10^{20}$	Siva Sankar Nair: The Mahler measure of a family of polynomials with arbitrarily many variables
$10^{25} - 10^{55}$	Subham Roy: Generalized Mahler measure of Laurent polynomials
$11^{15} - 11^{45}$	Lola Thompson: Mahler Measure and Manifolds
$11^{50} - 12^{20}$	Gabriel Dill: On the frequency of height values

Abstracts

François Brunault (*ÉNS Lyon*): Mahler measures and modular forms

I will give an overview of the (mostly conjectural) links between Mahler measures of polynomials and L-functions of modular forms. In particular, I will report on the proof of the conjecture relating the measure of $(1+x)(1+y)+z$ and the L-value at $s=3$ of an elliptic curve of conductor 15. The proof is based on joint work with Wadim Zudilin on modular regulators. It suggests a possible link between Mahler measures and multiple L-values of modular forms (specifically, Eisenstein series), which sometimes unexpectedly simplify to classical L-values of cusp forms.

Mahya Mehrabdollahei (*Göttingen University*): The Mahler measure of certain exact polynomial families

The Mahler measure of a polynomial is a way to quantify its complexity or "size" in the context of number theory. The study of univariate Mahler measures and their properties has gained significance, particularly in light of the still-open problem posed by Lehmer. Boyd's research on Lehmer's problems led to Boyd's conjecture, which, when resolved, also answers Lehmer's conjecture. An approach to tackle Boyd's conjecture involves studying sequences of multivariate Mahler measures and their asymptotic behavior. Unlike the univariate case, there is no general closed formula for computing multivariate Mahler measures. However, there exists a family of bivariate polynomials known as "regular exact" for which the Mahler measure can be computed through finite sums. The Mahler measure of an exact polynomial often reveals connections to special values of L-functions, rendering this family especially intriguing for exploration. An important conjecture by Chinburg explores the link between special values of Dirichlet L-functions and multivariate Mahler measures. In this talk, we provide an overview of the Mahler measures of specific sequences of exact polynomials and their relevance to Boyd's and Chinburg's conjectures.

Riccardo Pengo (*Leibniz Universität Hannover*): Mahler measures of successively exact polynomials

Since the pioneering works of Smyth and Ray, followed by the seminal works of Boyd, Deninger and Rodriguez-Villegas, Mahler measures have been related to special values of L-functions. In this talk, based on joint work with François Brunault, I will explain why the singularities and the boundary of the union of the hypersurfaces cut out in the algebraic torus by a polynomial and its reciprocal are the right objects to look at, if one aims at finding all the possible L-values related to a given Mahler measure. In particular, I will introduce the notion of successively exact polynomials, which generalizes a notion studied by Lalín and Guilloux-Marché. I will also complement my talk with various examples, and with a discussion of the numerical aspects involved in these investigations.

Marie José Bertin (*Sorbonne Université*): Mahler measure of polynomials defining singular K3 surfaces

I shall give the Mahler measure of many three-variable polynomials defining singular K3 surfaces. This Mahler measure is the sum of a modular part and a Dirichlet part. The modular part is expressed as the corresponding L-series of the singular K3 surface whilst the Dirichlet part may have a link with the Mahler measure of facets of the Newton polyhedron of the polynomial.

Thu Hà Trieu (*ÉNS Lyon*): The Mahler measure exact polynomials in three variables

We prove that the Mahler measure of a three-variable exact polynomial, under certain explicit conditions, can be expressed in terms of elliptic curve L-functions and values of the Bloch–Wigner dilogarithm, conditionally on Beilinson's conjecture. In some cases, these dilogarithmic values simplify to Dirichlet L-values. This generalises a result of M. Lalín for the polynomial $z+(x+1)(y+1)$. We apply our method to several other Mahler measure identities conjectured by D. Boyd and F. Brunault.

Jan-Willem van Ittersum (*University of Cologne*): Hedgehogs in Lehmer's problem

Motivated by a famous question of Lehmer about the Mahler measure we study and solve its analytic analogue.

Michael Mossinghoff (*Center for Communications Research*): The Lind–Mahler measure and integer group determinants

In 2005, Lind interpreted the Mahler measure of a polynomial in $\mathbb{Z}[x]$ as an integral involving an integer combination of characters for the circle group $\mathbb{T}^1$, and he proposed an analogous quantity for an arbitrary compact abelian group G. The resulting *Lind–Mahler measure*, and corresponding Lehmer problem, has since been investigated for a number of groups. Separately, in 1977 Taussky-Todd asked if one could characterize the determinants of the integral circulant matrices of a given size. This effectively coincides with the problem of Lind when G is a finite cyclic group: the Lind–Mahler measure of a polynomial in this case is essentially the determinant of a corresponding circulant matrix. We present an overview of recent work on the Lind–Mahler measure and the corresponding problems for integer group determinants, and report on some recent joint work with Chris Pinner and others in this area for various finite abelian groups.

Youness Lamzouri (*Université de Lorraine in Nancy* & *Institut Universitaire de France*): On the Mahler measure of Fekete polynomials

The Fekete polynomial F_p attached to an odd prime p is the polynomial of degree $p-1$ whose coefficients are the values of the Legendre symbol modulo p. These polynomials were initially considered by Fekete, Chowla and others in relation to real zeros of Dirichlet L-functions. Fekete polynomials are also an important example of the class of Littlewood polynomials, and their Mahler measure was extensively studied by several authors. In particular, an old open problem asks to determine an asymptotic formula for the Mahler measure of F_p as $p \to \infty$. In this talk, I will present a recent work, joint with Klurman and Munsch, where we resolve this problem. Our method relies on showing that the distribution of the values of Fekete polynomials on the unit circle are governed by an explicit limiting (non-Gaussian) random point process.

Artūras Dubickas (*Vilnius University*): Mahler measures, their quotients and differences

Let $\mathcal{M}$ be the set of Mahler measures of algebraic numbers. The most known open question related to the set $\mathcal{M}$ is that of Lehmer (1933) on whether there is a constant $c > 1$ such that the intersection $\mathcal{M} \cap (1, c)$ is empty. There are also other interesting problems related to the structure of $\mathcal{M}$. For example, sometimes it not clear whether a given algebraic number $\alpha > 1$ belongs to the set $\mathcal{M}$ or not. In 2004, Dixon and the speaker established a criterion that allows to determine whether a given unit $\alpha > 1$ belongs to $\mathcal{M}$ or not. On the other hand, the problem is open already for some quadratic algebraic numbers which are not units, say, for $\alpha = 1 + \sqrt{17}$ (Schinzel, 2004). In order to see how rich is the set $\mathcal{M}$ one can also study the sets of algebraic numbers expressible as sums, differences, quotients, etc. of two elements of $\mathcal{M}$. Since Pisot and Salem numbers belong to $\mathcal{M}$, similar questions arise for their sumsets, difference sets, quotient sets, etc.

In my talk I will present some old and new results on the above mentioned and similar questions.

Berend Ringeling (*Radboud University Nijmegen*): Analytic continuation of the zeta Mahler function

In this talk we discuss the zeta Mahler function, which is closely related to the Mahler measure. We show that every such function can be meromorphically continued to the whole complex plane. We give many examples, and explain the computational aspect of this problem.

Harry Schmidt (*University of Warwick*): Lower bounds for the canonical height of polynomials with tame monodromy

I will present joint work with Philipp Habegger in which we prove lower bounds for the local canonical height of certain polynomials that satisfy what we call the quill hypothesis. This is a geometric condition on the set of critical points of iterations of these polynomials and related to considerations of monodromy. Our proof follows the strategy of Dimitrov's proof of the Schinzel–Zassenhaus conjecture and gives the first polynomial lower bounds for dynamical canonical heights that are not Lattès.

Fabien Pazuki (*University of Copenhagen* and *Université de Bordeaux*): Explicit bounds on the coefficients of modular polynomials and the size of $X_0(N)$

We give sharp upper and lower bounds on the size of the coefficients of the modular polynomials for the elliptic j-function. These bounds make explicit the best previously known asymptotic bounds. The proof relies on a careful study of the Mahler measure of a family of specializations of the modular polynomial. We also give an asymptotic comparison between the Faltings height of the modular curve $X_0(N)$ and the height of this modular polynomial, giving a link between these two ways of measuring the "size" of the modular curve. The talk is based on joint work with Florian Breuer and Desirée Gijon Gomez.

Matilde Lalín (*Université de Montréal*): Evaluations of areal Mahler measure of multivariable polynomials

In 2008 Pritsker defined a natural counterpart of the Mahler measure, which is obtained by replacing the normalized arc length measure on the standard n-torus by the normalized area measure on the product of n open unit disks. In this talk, we will investigate some similarities and differences between the two versions of Mahler measure. We will also discuss some evaluations of the areal Mahler measure of multivariable polynomials, which also yields special values of L-functions. This is joint work with my PhD student Subham Roy.

Detchat Samart (*Burapha University*): Some recent results on Mahler measures of curves parametrized by modular units

Boyd conjectured that Mahler measures of some bivariate polynomials $P \in \overline{\mathbb{Q}}[X^{\pm 1}, Y^{\pm 1}]$ are, under certain arithmetic conditions, rational multiples of $L'(E, 0)$, where E is the elliptic curve defined by the zero locus of P. An elegant method introduced by Brunault, Mellit, and Zudilin can be used to verify these identities when the underlying curve $P(X, Y) = 0$ admits a modular unit parametrization; known results include those of conductor 14, 15, 17, 21, 24, 35, 40, and 56. In this talk, we report some recent (and not so recent) progress in this direction for curves of conductor 19, 20, 26, 30, 44, and 92, which appear in Brunault's list of elliptic curves parametrized by modular units. These examples arise in our ongoing study of Mahler measures of certain families whose favorable condition such as temperedness or reciprocality is violated.

Hourong Qin (*Nanjing University*): L-values of congruent number elliptic curves

Let n be a positive square-free integer. The elliptic curve $y^2 = x^3 - n^2x$ is called a congruent number elliptic curve, since it is closely related to the classical congruent number problem. In this talk, I will present and explain our recent results on the L-values of congruent number elliptic curves.

Chris Smyth (*Edinburgh University*): Thue sets

I define Thue sets to be those sets of positive real numbers having a specific order type, and give two examples of such sets. I then discuss some (limited) evidence for the set of all Mahler measures of polynomials with integer coefficients also being a Thue set.

Philipp Habegger (*University of Basel*): The transfinite diameter of finite trees and dynamical Schinzel–Zassenhaus bounds

A few years ago, Dimitrov proved the Schinzel–Zassenhaus Conjecture. Harry Schmidt and I showed how his general strategy can be adapted to cover some dynamical variants of this conjecture. One common tool in both results is Dubinin's Theorem on the transfinite diameter of hedgehogs. Motivated by Mahler's work on root separation, I gave an elementary proof of Dubinin's Theorem, albeit with a worse numerical constant. In this talk I will report on joint work in progress with Harry Schmidt. We find new upper bounds for the transfinite diameter of some finite topological trees. These trees can often replace the hedgehog in our earlier work and are more attuned to the dynamical setting.

Hang Liu (*Shenzhen University*): Mahler measure of $\mathbb{Q}$-curves

We extend Boyd's conjecture on the Mahler measure of 2-variable polynomials to several families of polynomials with coefficients in real quadratic fields. This includes Deninger's family and Hesse's family. In order to attack these conjectures, we develop an algorithm which finds $\mathbb{Q}$-curves parametrised by modular units. Using our algorithm and the Rogers–Zudilin method, we prove our conjecture on the Mahler measure for several polynomials in these families, corresponding to newforms of level 24, 39, 40 and 56. We also consider two families of tempered polynomials defined over the simplest cubic fields and simplest quartic fields respectively. We conjecture the Mahler measures of these families are linear combinations of L-functions of newforms of weight 2 and prove one example in each family. The talk is based on joint work with François Brunault and Wang Haixu.

Siva Sankar Nair (*Université de Montréal*): The Mahler measure of a family of polynomials with arbitrarily many variables

In this talk, we will calculate the exact Mahler measure of a family of polynomials that contains, for every integer $n > 1$, an n-variable polynomial. This is done by extending a method of Lalín, which uses an iterative process to reduce the Mahler measure computation to the evaluation of polylogarithm functions at roots of unity. As a result, we obtain the Mahler measure as a combination of values of the Riemann zeta function and the Dirichlet L-function associated with the character of conductor 3.

We will present an exact formula for the Mahler measure of an infinite family of polynomials with arbitrarily many variables. The formula is obtained by manipulating the Mahler measure integral using certain transformations, followed by an iterative process that reduces this computation to the evaluation of certain polylogarithm functions at sixth roots of unity. This yields values of the Riemann zeta function and the Dirichlet L-function associated to the character of conductor 3.

Subham Roy (*Université de Montréal*): Generalized Mahler measure of Laurent polynomials

The (logarithmic) Mahler measure of a non-zero rational polynomial P in n variables is defined as the mean of $\log|P|$ restricted to the standard n-torus $\mathbb{T}^n = \{(x_1, \ldots, x_n) \in (\mathbb{C}^*)^n : |x_i| = 1 \text{ for } 1 \leq i \leq n\}$. The Mahler measure has been related to special values of L-functions, and this has been explained in terms of regulators. In 2018, Lalín and Mittal considered the generalized Mahler measure (where the mean of $\log|P|$ is restricted to arbitrary n-torus) to obtain relations between certain polynomials mentioned in Boyd's paper. In this talk, we will investigate the definition of the generalized Mahler measure for all Laurent polynomials in two variables when they do not vanish on the integration torus, and discuss few results we obtained involving the relation between the standard Mahler measure and the generalized Mahler measure of such polynomials.

Lola Thompson (*Utrecht University*): Mahler Measure and Manifolds

We will discuss how Mahler measure and related concepts are connected to problems about lengths of geodesics on arithmetic hyperbolic manifolds. As a result, by solving problems using tools from number theory, we are able to answer quantitative questions in spectral geometry. This talk is based on joint work with Mikhail Belolipetsky, Matilde Lalín, and Plinio G. P. Murillo; and with Benjamin Linowitz, D. B. McReynolds, and Paul Pollack.

Gabriel Dill (*Universität Bonn*): On the frequency of height values

Asymptotics for the number of algebraic numbers of bounded height contained in a given number field or of a given degree are due to Schanuel and Masser–Vaaler respectively. But what happens if we consider algebraic numbers of a given degree d and fixed height? And what about the number of values up to a given threshold that the height takes on algebraic numbers of a given degree? Similarly precise asymptotics are too much to hope for in general, but maybe the rough order of growth can be determined? We report on a recent work in which we have tried (with some success) to answer these questions.

Silent participants

Finn Bartsch (*Radboud University Nijmegen*)

Frits Beukers (*Utrecht University*)

Sebastián Carrillo Santana (*Utrecht University*)

Gunther Cornelissen (*Utrecht University*)

Paulius Drungilas (*Vilnius University*)

Jan-Hendrik Evertse (*Leiden University*)

Samira le Grand (*Utrecht University*)

Xuejun Guo (*Nanjing University*)

Harald Andrés Helfgott (*CNRS/IMJ*)

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