Ordinary primes > 2 in $\leqslant 2$ minutes

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For a normalised cusp eigenform $f(\tau) = \sum_{n=1}^{\infty} b(n)q^n \in \mathbb{Z}[[q]]$ (with $q = e^{2\pi i\tau}$), consider the question of nonvanishing $b(p)$ modulo p. The primes for which such nonvanishing takes place are known as *ordinary primes* (for the newform).

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The following result for the weight 4 newform $\eta(2\tau)^4 \eta(4\tau)^4 = q \prod_{m=1}^{\infty} (1 - q^{2m})^4 (1 - q^{4m})^4$ implies a significantly stronger divisibility property than just $p \mid b(p)$, thus giving a heuristical argument why they are unlikely to show up ‘too often’.

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Theorem. A prime $p>2$ is non-ordinary for the newform if and only if the degree $4(p-1)$ polynomial

$$2^{4(p-1)}(a+1)_{p-1}^4\cdot\sum_{k=0}^{p-1}\frac{(a+\frac{1}{2})_k^4}{(a+1)_k^4}$$

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in $\mathbb{Z}[a]$ has *all* its coefficients divisible by p.

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$$2^{4(p-1)}(a+1)_{p-1}^4\cdot\sum_{k=0}^{p-1}\frac{(a+\frac12)_k^4}{(a+1)_k^4}=\sum_{k=0}^{p-1}2^{4(p-1-k)}\prod_{j=1}^{k}(2a+2j-1)^4\prod_{j=k+1}^{p-1}(a+j)^4$$

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Proof: Hypergeometric magic.