Arithmetic differential equations of second order

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A beautiful formula

It is not surprising that interactions of many physics systems can be described in terms of linear differential equations. Some of them appear to possess remarkable arithmetic properties, which often can be explained through bridging them with modular and hypergeometric functions.
My talk is about an investigation of an innocent generating function of the squares of Legendre polynomials has brought to life an interesting family of equations of second order whose solutions are hyperelliptic integrals.

Here is a beautiful formula:

$$\sum_{n=0}^{\infty}\left(16-5\sqrt{10}+60n\right)u_n P_n\left(\frac{5\sqrt{2}+17\sqrt{5}}{45}\right)\left(\frac{5\sqrt{2}-3\sqrt{5}}{5}\right)^n = \frac{135\sqrt{2}+81\sqrt{5}}{\pi\sqrt{2}}.$$

But what are these u_n and $P_n(x)$ floating around?

Legendre polynomials

The latter designate the Legendre polynomials

$$P_n(x) = \sum_{m=0}^{n} \binom{n}{m}^2 \left(\frac{x-1}{2}\right)^m \left(\frac{x+1}{2}\right)^{n-m};$$

they can also be given by a recurrence equation

$$(n+1)P_{n+1}(y) - (2n+1)yP_n(y) + nP_{n-1}(y) = 0 \quad \text{for } n \geq 0,$$
$$\text{with } P_0(y) = 1 \text{ and } P_{-1}(y) = 0,$$

or read off from a simple generating series

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) z^n.$$

Apéry-like recursions

The sequence u_n is one of the two famous Apéry sequences,

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n} \quad \text{for } n = 0, 1, 2, \ldots;$$

it shows up as the denominator sequence of good rational approximations to $\zeta(2)$. It can be generated by the recurrence equation

$$(n+1)^2 u_{n+1} = (an^2 + an + b)u_n - cn^2 u_{n-1} \quad \text{for } n = 0, 1, 2, \ldots,$$
$$\text{with } u_{-1} = 0,\ u_0 = 1,$$

with the choice $(a, b, c) = (11, 3, -1)$.
But there are other choices of the parameters $(a, b, c) \in \mathbb{Z}^3$ for which the sequence $u_n = u_n(a, b, c)$ is integer-valued; all those were (conjecturally!) classified by Zagier in his work on Apéry-like recursions.

Reminder of the beauty

So here is the beautiful formula again:

$$\sum_{n=0}^{\infty}\bigl(16-5\sqrt{10}+60n\bigr)\,u_nP_n\left(\frac{5\sqrt{2}+17\sqrt{5}}{45}\right)\left(\frac{5\sqrt{2}-3\sqrt{5}}{5}\right)^n = \frac{135\sqrt{2}+81\sqrt{5}}{\pi\sqrt{2}}.$$

In disguise, this is a formula for $1/\pi$, and there are many of this type currently known — not all proved however.

The formula originates from the identity

$$\sum_{n=0}^{\infty} u_n P_n\left(\frac{(X+Y)(1+cXY)-2aXY}{(Y-X)(1-cXY)}\right)\left(\frac{Y-X}{1-cXY}\right)^n$$
$$= (1-cXY)\left\{\sum_{n=0}^{\infty} u_n X^n\right\}\left\{\sum_{n=0}^{\infty} u_n Y^n\right\}$$

(joint work with Wan), valid in a neighbourhood of $X = Y = 0$, and from a modular parametrisation of the generating series $\sum_{n=0}^{\infty} u_n X^n$ (which depends on the choice of (a, b, c) on Zagier's list).
This identity for a 'non-standard' generating series $f(z) = \sum_{n=0}^{\infty} u_n P_n(y) z^n$ of Legendre polynomials (but one can also view it as a function in y rather than z) and its several cousins are principal sources of formulae for $1/\pi$. Namely, the function $f(z)$ is (up to an algebraic factor) a product of $f_1(z)$ and $f_2(z)$, each an algebraic pullback of a very classical ${}_2F_1$ hypergeometric function (in every single case representing an elliptic integral). In simplest situations one even get $f_1(z) = f_2(z)$ (Clausen-type formulae).

Nonstandards

But there happen to be less standard situations, discovered experimentally by (Zhi-Wei) Sun around 2011. His list contained examples where $f(z)$ is

$$F(y,z)=\sum_{n=0}^{\infty}\binom{2n}{n}P_n(y)^2z^n,$$

for which no similar product decompositions were possible to produce. The function satisfies a fourth-order linear differential equation, viewed as a function of either y or z, and both equations have (the Zariski closure of) the monodromy group O_4.
What we find out in our joint work with Mark van Hoeij and Duco van Straten that the $F(y,z)$ also decomposes into a product of two functions, each satisfying a second order linear differential equation. But these functions are neither elliptic nor ${}_2F_1$ hypergeometric.

Main result

For the function $F(y,z)$, the following decomposition is valid:

$$F(y,z) = w\, I_+(4z, w^2)\, I_-(4z, w^2),$$

where $w = \sqrt{(1+4z)^2 - 16y^2 z} + 4y\sqrt{-z}$ and

$$I_\pm(u,x) = \frac{1}{\pi}\int_0^1 \frac{1 - uv \pm v\sqrt{2u^2-2u}}{\sqrt{v(1-v)((1-v)(1-u^2v)(1+uv)^2 + x\,v(1-uv)^2)}}\,\mathrm{d}v$$

are generically *hyperelliptic* integrals.
Furthermore, for each $u \in \mathbb{C}$, the function $I_\pm(u,x)$ satisfies a second-order differential equation with coefficients from $\mathcal{L}[x]$, where $\mathcal{L} = \mathbb{Q}(u, \sqrt{2u^2-2u})$. Surprisingly enough, such second-order equations (and there are infinitely many of them because of the extra parameter u), not reducible to elliptic integrals, were not recorded in the literature; they are reasonably simple counterexamples to a 1990 conjecture of Dwork (which is already disproved *finitely* many times through examples based on Shimura and Teichmüller curves defined over quadratic extensions of $\mathbb{Q}$).

Parametrisation

$$y = (x + (t^2 - 1)^2)\sqrt{z/x}, \quad z = \frac{1}{4(1 - 2t^2)}$$

rationalises the square root in the decomposition of $F(y, z)$, namely, $\sqrt{2u^2 - 2u} = 2tu$ and $u = 1/(1 - 2t^2)$, so that $\mathcal{L} = \mathbb{Q}(t)$. One of the most remarkable aspects is that the hyperelliptic integrals $I_\pm(u, x)$ satisfy second order equations corresponding to two conjugate operators $L_\pm^{\mathrm{PF}} \in \mathcal{L}(x)[\mathrm{d}/\mathrm{d}x]$. This is unusual, as the Picard–Fuchs equation for a hyperelliptic integral of genus g is expected to have order $2g$. For such a reduction to lower order to happen, the corresponding hyperelliptic curve and differential must have special properties. One of these, encountered in our monodromy computation, is multiplication by $\sqrt{2}$. For the underlying family of curves $C = C^{u,x} : Y^2 = H(x, u, v)$, where

$$H = v(1 - v)\big((1 - v)(1 - u^2 v)(1 + uv)^2 + x\, v(1 - uv)^2\big) \in \mathbb{Q}(u, x)[v],$$

we verify this property with a Humbert relation.

One expects special properties of a curve to be reflected in special properties of its Jacobian $\mathrm{Jac}(C) = H^0(\Omega_C)^*/H_1(C,\mathbb{Z})$, which is a two dimensional complex torus attached to C. If we fix a symplectic basis $\alpha_1, \beta_1, \alpha_2, \beta_2$ for the homology group $H_1(C,\mathbb{Z})$ and a basis for the space $H^0(\Omega_C)$ of differentials of the first kind

$$\omega_1 = \frac{\mathrm{d}v}{\sqrt{H}}, \quad \omega_2 = \frac{v\,\mathrm{d}v}{\sqrt{H}},$$

one can identify $\mathrm{Jac}(C)$ with the torus $\mathbb{C}^2/\Lambda$, where Λ is the period lattice, generated over $\mathbb{Z}$ by the columns of the period matrix

$$\begin{pmatrix} \int_{\alpha_1}\omega_1 & \int_{\alpha_2}\omega_1 & \int_{\beta_1}\omega_1 & \int_{\beta_2}\omega_1 \\ \int_{\alpha_1}\omega_2 & \int_{\alpha_2}\omega_2 & \int_{\beta_1}\omega_2 & \int_{\beta_2}\omega_2 \end{pmatrix},$$

which by appropriate normalisation can be brought in the form

$$\begin{pmatrix} 1 & 0 & \tau_1 & \tau_2 \\ 0 & 1 & \tau_2 & \tau_3 \end{pmatrix}.$$

Humbert surface

If the Jacobian has non-trivial endomorphisms, this is reflected in a *Humbert relation* between τ_1, τ_2 and τ_3, which is of the form

$$A\tau_1 + B\tau_2 + C\tau_3 + D(\tau_2^2 - \tau_1\tau_3) + E = 0.$$

Inside the three dimensional moduli space $\mathcal{M}_3$ of all curves of genus 2 or, what amounts to the same, the space of $\mathcal{A}_2$ of (pricipally polarised) abelian surfaces, the curves that satisfy such a Humbert relation form a *Humbert surface* H_Δ of discriminant $\Delta = B^2 - 4AC - 4DE$. For a general curve belonging to H_Δ the set of all endomorphisms of the Jacobian is an order in the real quadratic field with discriminant Δ, and we say that the curve (or its Jacobian) has *real multiplication*.

Humbert relation

By numerical evaluation of the period matrix, after picking numerical values for u, x, one can check that our curves $C^{u,x}$ belong to the Humbert surface H_8: $\mathbb{Z}[\sqrt{2}] \subset \mathrm{End}(\mathrm{Jac}(C^{u,x}))$.
However, one can do better than approximate verification. The insight of Humbert (1899) was that for each Δ there is a polynomial condition on the coefficients a_1, a_2, a_3 of the genus two curve, written in the form

$$Y^2 = v(v - a_1)(v - a_2)(v - a_3)(v - a_4)$$

to be on the Humbert surface H_Δ. For $\Delta = 8$ the condition, already found by Humbert in 1899, reads

$$4a_1a_2a_3a_4((a_1 + a_3)(a_2 + a_4) - 2a_1a_3 - 2a_2a_4)^2 \\ = (a_2 - a_4)^2(a_1 - a_3)^2(a_1a_3 + a_2a_4)^2.$$

Using this, it is trivial to verify that the smooth curves $C^{u,x}$ admit real multiplication by $\sqrt{2}$, i.e., the endomorphism ring of their Jacobians contains $\mathbb{Z}[\sqrt{2}]$.

Apéry-type recursion

As a somewhat more arithmetic outcome of our proof, the following infinite family of Apéry-type recursions shows up: Define degree $4n$ polynomials $u_n = u_n(t)$ by $u_n = 0$ for $n < 0$, $u_0 = 1$ and

$$\begin{aligned}
&(n+1)^2 u_{n+1} \\
&\quad - 2^2(16(t^4 - 6t^3 - 4t^2 + 6t - 1)(n^2 + n) + 4t^4 - 24t^3 - 12t^2 + 20t - 3)u_n \\
&\quad - 2^{11} t(t-1)^3(t+1)(8(t^2 + 2t - 1)n^2 - 2t^2 - 6t + 3)u_{n-1} \\
&\quad + 2^{18} t^2(t-1)^6(t+1)^2(2n+1)(2n-3)u_{n-2} = 0 \quad \text{for } n = 0, 1, 2, \ldots .
\end{aligned}$$

Then $u_n \in \mathbb{Z}[t]$.
This recursion corresponds to a second order linear differential equation which cannot be pullbacked to a ${}_2F_1$ hypergeometric function, nor to a Heun equation.

How general things are

A task of recognising the innocent-looking generating function $F(y,z)$ of the squares of Legendre polynomials in a closed form has turned out to produce families of hyperelliptic integrals that satisfy second order Picard–Fuchs differential equations, with the latter having remarkable features.

Next, one can investigate other Humbert surfaces corresponding to different orders — and we did for H_5 as well. The corresponding second order equations are solved by hyperelliptic integrals with the underlying Jacobians admitting RM by $\mathbb{Z}[(1+\sqrt{5})/2]$. We can do this task for more Humbert surfaces.

To our surprise, not only genus 2 integrals can satisfy second order equations, but also we now have a program that produces examples of arbitrarily high genus. The corresponding second-order linear differential operator (conjecturally) has monodromy in O_{ζ_N}, meaning that the algorithm produces families of hyperelliptic curves with real multiplication by $\mathbb{Z}[\zeta_N+\zeta_N^{-1}]$.

A historical predecessor

Interestingly, the discussed Humbert surface H_8 (and it only!) has received a differential treatment almost 40 years ago in 1988:

As an illustration of their general construction, Sasaki and Yoshida explicitly gave a rank 4 system of partial differential equations for the Hilbert surface X_8.

We expect that system to be related to the one for $F(y, z)$ via suitable algebraic transformations of the variables. To test this we checked that it has a similar decomposition to rank 2, which it does.

A simple (looking) question

Our proof of the decomposition of $F(y,z)$ originates the following expectation of an arithmetic flavour, which may be regarded as a potential correction of Dwork's ex-conjecture.

Assume that $f(x)=\sum_{n=0}^{\infty} a_n x^n$ is in $1+x\mathbb{Z}[[x]]$ and that its twist $\sum_{n=0}^{\infty} \binom{2n}{n} a_n x^n$ by the central binomial coefficients satisfies a second order linear differential equation with regular singularities.
Must $f(x)$ be algebraic?

Conclusion

Our proofs also demonstrate that closed forms of arithmetic (globally bounded) series may be obtained from 'untwisting' the coefficients by the central binomial coefficients $\binom{2n}{n}$ (or more generally by $(\alpha)_n/n!$ for some $\alpha \in \mathbb{Q} \setminus \mathbb{Z}$).
Such untwisting may depend on a special choice of the variable, with respect to which the series are expanded. The choices required in our proofs were *ad hoc*, though hinted by the corresponding differential equations. We would therefore question whether there exists a *routine* strategy for performing the related change of variable directly from the equations, in a greater generality.
Such a methodology, if it exists, has potential to shed more light on a famous conjecture of Christol about globally bounded solutions of Picard–Fuchs differential equations.


Thank you very much
どうもありがとうございました
感謝