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BRIEF COMMUNICATIONS

On the Measure of Linear and Algebraic Independence for Values of Entire Hypergeometric Functions

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§1. Introduction

Let t and l be positive integers and let

$$\lambda_1, \dots, \lambda_{t+l}\,; \beta_1, \dots, \beta_l \in \mathbb{Q} \setminus \{-1, -2, \dots\}, \qquad \lambda_{t+l} = 0. \tag{1}$$

In the note, estimates for the measure of linear and algebraic independence of the values of generalized hypergeometric functions $f(z), f'(z), \dots, f^{(m-1)}(z)$, $m = t+l$, at a rational point $\alpha \neq 0$ are established, where

$$f(z) = \sum_{\nu=0}^{\infty} \frac{(\beta_1)_\nu \cdots (\beta_l)_\nu}{(\lambda_1+1)_\nu \cdots (\lambda_{t+l}+1)_\nu} \left(\frac{z}{t}\right)^{t\nu}, \tag{2}$$

$$(\beta)_0 = 1, \qquad (\beta)_\nu = \beta(\beta+1)\cdots(\beta+\nu-1), \quad \nu = 1, 2, \dots.$$

These estimates follow from general theorems in [1] (where the history of the problem can also be found) together with new results of the theory of differential Galois groups [2].

By the *measure of algebraic independence* of the reals $\xi_1, \dots, \xi_m$, we mean the behavior of the quantity

$$|P(\xi_1, \dots, \xi_m)|, \qquad P(y_1, \dots, y_m) \in \mathbb{Z}[y_1, \dots, y_m], \tag{3}$$

in dependence on the following quantities: the modulus of the product $\Pi(P)$ of all nonzero coefficients of the polynomial P, the height $H(P)$ (the maximum of the moduli of the coefficients), and the degree $\deg P$ of the polynomial. In the case of $\deg P = 1$, the characteristic (3) is called the *measure of linear independence* of the reals $\xi_1, \dots, \xi_m$.

§2. Galois group of a generalized hypergeometric equation

Let the parameters (1) of the function (2) satisfy the following conditions:

1) $\lambda_i - \beta_j \notin \mathbb{Z}$ for all $i = 1, \dots, t+l$ and $j = 1, \dots, l$;
2) there is no common divisor $d > 1$ of the numbers t and l such that $(\lambda_1 + 1/d, \dots, \lambda_{t+l} + 1/d) \sim (\lambda_1, \dots, \lambda_{t+l})$, $(\beta_1 + 1/d, \dots, \beta_l + 1/d) \sim (\beta_1, \dots, \beta_l)$.

(Here the notation $(\beta'_1, \dots, \beta'_m) \sim (\beta_1, \dots, \beta_m)$ means that for some nonidentity permutation $\sigma : \{1, \dots, m\} \to \{1, \dots, m\}$ we have $\beta'_j - \beta_{\sigma(j)} \in \mathbb{Z}$ for any $j \in \{1, \dots, m\}$.)

Condition 1) is usually called the *linear irreducibility condition* and Condition 2) the *Kummer irreducibility condition.*

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Let G be the Galois group of the linear homogeneous differential equation

$$\Big(\Big(z\frac{d}{dz}+t\lambda_1\Big)\cdots\Big(z\frac{d}{dz}+t\lambda_{t+l}\Big)-z^t\Big(z\frac{d}{dz}+t\beta_1\Big)\cdots\Big(z\frac{d}{dz}+t\beta_l\Big)\Big)y=0, \tag{4}$$

which is satisfied by function (2).

Proposition 1. *If $t=1$ and if conditions* 1) *and* 2) *hold, then $G=\mathrm{GL}_m$.*

Proposition 2. *If t is odd and if conditions* 1) *and* 2) *hold, then $\mathrm{SL}_m\subset G\subset\mathbb{C}^*\times\mathrm{SL}_m=\mathrm{GL}_m$.*

In the case of even t, the following additional condition of *quadratic irreducibility* is needed for the group G to contain SL_m:

3) there is no real τ such that $(\lambda_1+\tau,\dots,\lambda_{t+l}+\tau)\sim(-\lambda_1,\dots,-\lambda_{t+l})$, $(\beta_1+\tau,\dots,\beta_l+\tau)\sim(-\beta_1,\dots,-\beta_l)$.

Proposition 3. *If t is even and if conditions* 1)–3) *hold, then $\mathrm{SL}_m\subset G\subset\mathbb{C}^*\times\mathrm{SL}_m=\mathrm{GL}_m$.*

The proofs of Propositions 1–3 can be found in [2].

Now let us write out the definition in [1] for a linear differential equation. Let $\psi_1(z),\dots,\psi_m(z)$ be a fundamental system of solutions of a linear homogeneous differential equation

$$y^{(m)}+A_1(z)y^{(m-1)}+\dots+A_{m-1}(z)y'+A_m(z)y=0,\qquad A_j\in\mathbb{C}(z),\quad j=1,\dots,m. \tag{5}$$

We say that Eq. (5) of order m *belongs to the class* $\mathbf{W}^0$ if the functions

$$\psi_j^{(l-1)}(z),\qquad j,l=1,\dots,m, \tag{6}$$

are homogeneously algebraically independent over $\mathbb{C}(z)$.

Theorem 1. *Let conditions* 1) *and* 2) *hold in the case of odd t or conditions* 1)–3) *hold in the case of even t. Then the linear homogeneous differential equation* (4) *belongs to the class $\mathbf{W}^0$.*

Proof. Let $\psi_1(z),\dots,\psi_m(z)$ be a fundamental system of solutions of Eq. (4) and let G be the Galois group of this equation. The condition $G=\mathrm{GL}_m$ is equivalent to the condition that the functions (6) be algebraically independent. However, if $G\neq\mathrm{GL}_m$ and $G\supset\mathrm{SL}_m$, then there exists exactly one algebraic relation among the functions (6). For the generalized hypergeometric equation (4), this relation is known: the Wronskian of a fundamental system of solutions is a rational function, in other words,

$$\det\big(\psi_j^{(l-1)}(z)\big)_{j,l=1,\dots,m}=A(z)\in\mathbb{C}(z).$$

We can readily see that this single algebraic relation is not homogeneous. Therefore, for the case in which $\mathrm{SL}_m\subset G\subset\mathrm{GL}_m$ (and this follows from Propositions 1–3), equation (4) belongs to the class $\mathbf{W}^0$.

This completes the proof of the theorem. □

§3. Estimates for the measures

Now we state the main result of the present note.

Theorem 2. *Let the parameters* (1) *of the function* (2) *satisfy conditions* 1) *and* 2) *for t odd and conditions* 1)–3) *for t even. Let a rational point $\alpha\neq0$ and a positive integer d be given. Then there exist positive constants $\gamma=\gamma\big(f(z),\alpha,d\big)$ and $C=C\big(f(z),\alpha,d\big)$ such that for any homogeneous polynomial $P\in\mathbb{Z}[y_1,\dots,y_m]$ of degree d we have the inequality*

$$\big|P\big(f(\alpha),f'(\alpha),\dots,f^{(m-1)}(\alpha)\big)\big|>C\Pi^{-1}H^{1-\gamma(\log\log H)^{-1/(m^2-m+2)}},$$

where $\Pi=\Pi(P)$ and $H=H(P)\geq3$.

PROOF. Since, by Theorem 1, Eq. (4) belongs to the class $\mathbf{W}_0$, we can apply [1, Theorem I]. This gives the desired inequality. □

Theorem 2 immediately implies the following result on the measure of linear independence.

THEOREM 3. *Let the parameters* (1) *of the function* (2) *satisfy conditions* 1) *and* 2) *for* t *odd and conditions* 1)–3) *for* t *even, and let* $\alpha \neq 0$ *be a rational point. Then there exist positive constants* $\gamma = \gamma\big(f(z), \alpha\big)$ *and* $C = C\big(f(z), \alpha\big)$ *such that*

$$|h_1 f(\alpha) + h_2 f'(\alpha) + \cdots + h_m f^{(m-1)}(\alpha)| > C(H_1 \cdots H_m)^{-1} H^{1-\gamma(\log\log H)^{-1/(m^2-m+2)}},$$
$$h_i \in \mathbb{Z}, \quad H_i = \max\{1, |h_i|\}, \quad i = 1, \dots, m, \qquad H = \max_{1 \le i \le m}\{H_i\} \ge 3.$$

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