Mathematical Notes, vol. 71, no. 5, 2002, pp. 604–616.
Translated from Matematicheskie Zametki, vol. 71, no. 5, 2002, pp. 662–676.

Integrality of Power Expansions Related to Hypergeometric Series

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Received October 31, 2000

Abstract—In the present paper, we study the arithmetic properties of power expansions related to generalized hypergeometric differential equations and series. Defining the series $f(z), g(z)$ in powers of z so that $f(z)$ and $f(z)\log z + g(z)$ satisfy a hypergeometric equation under a special choice of parameters, we prove that the series $q(z) = ze^{g(Cz)/f(Cz)}$ in powers of z and its inversion $z(q)$ in powers of q have integer coefficients (here the constant C depends on the parameters of the hypergeometric equation). The existence of an integral expansion $z(q)$ for differential equations of second and third order is a classical result; for orders higher than 3 some partial results were recently established by Lian and Yau. In our proof we generalize the scheme of their arguments by using Dwork's p-adic technique.

Key words: *integral power expansion, hypergeometric series, linear differential equation, Calabi–Yau manifold, mirror map.*

1. INTRODUCTION

Choose an arbitrary integer $N \geq 2$; suppose that $N = p_1^{s_1} p_2^{s_2} \cdots p_l^{s_l}$ is its factorization into primes and the integers $q_1, q_2, \ldots, q_k$, $1 = q_1 < q_2 < \cdots < q_k < N$, form the complete set of remainders from the division by N which are prime to N. The number k of remainders in this set is given by the well-known formula

$$k = N\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_l}\right) \tag{1}$$

(see, for example, [1, Sec. 8, Problem 25]). The following assertion (given in our notation) is valid.

Lemma 1. *For an integer $N \geq 2$, we define the positive constant*

$$C_N := N^k \cdot \prod_{p \mid N} p^{k/(p-1)} = \left(\prod_{j=1}^{l} p_j^{s_j + 1/(p_j - 1)}\right)^k \tag{2}$$

which is an integer, since $(p-1) \mid k$ for any $p \mid N$ by (1). Then for any integer $m \geq 0$ the number

$$A(m) = A_N(m) := C_N^m \cdot \frac{(q_1/N)_m (q_2/N)_m \cdots (q_k/N)_m}{m!^k} \tag{3}$$

is a (positive) integer. Here $(x)_m = x(x+1)\cdots(x+m-1)$ for $m \geq 1$ and $(x)_m = 1$ denotes the Pochhammer symbol.

We prove this assertion in Sec. 3, but for now let us consider two simple particular cases. If $N = p^s$ is a power of a prime and hence $k = p^s - p^{s-1}$, then the choice of $C_N = p^{sp^s - (s-1)p^{s-1}}$ results in the integers

$$A(m) = \frac{(p^s m)!}{(p^{s-1}m)!\, m!^{p^s - p^{s-1}}}, \qquad m = 0, 1, 2, \ldots .$$

 0001-4346/2002/7156-0604$27.00

If $N = p_1p_2$, then $k = (p_1 - 1)(p_2 - 1)$, and the choice of $C_N = p_1^{p_1(p_2-1)}p_2^{(p_1-1)p_2}$ yields the integers

$$A(m) = \frac{(p_1p_2m)!}{(p_1m)!\,(p_2m)!\,m!^{p_1p_2-p_1-p_2}}, \qquad m = 0, 1, 2, \ldots .$$

By Lemma 1, the generalized hypergeometric series

$$f(z) = f_N(z) = {}_kF_{k-1}\left(\begin{matrix} q_1/N, q_2/N, \ldots, q_{k-1}/N, q_k/N \\ 1, \quad 1, \ldots, \quad 1 \end{matrix} \;\middle|\; C_N \cdot z\right) = \sum_{m=0}^{\infty} A(m)z^m \tag{4}$$

has integer coefficients in the expansion in powers of z.

Setting

$$\begin{aligned} D(x, m) &= \frac{d}{dx}\log(x)_m = \sum_{n=1}^{m} \frac{1}{x+n-1}, \qquad x \in (0, 1], \quad m = 0, 1, 2, \ldots, \\ D(m) = D_N(m) &:= \sum_{j=1}^{k} D\left(\frac{q_j}{N}, m\right) - kD(1, m), \qquad m = 0, 1, 2, \ldots, \end{aligned} \tag{5}$$

we also consider the power series

$$g(z) = g_N(z) = \sum_{m=0}^{\infty} A(m)D(m)z^m = \sum_{m=1}^{\infty} A(m)D(m)z^m, \tag{6}$$

whose coefficients, generally speaking, are no longer integers. Both series (5), (6) are convergent in the neighborhood of the point $z = 0$ (more exactly, for $|z| < 1/C_N$); in addition, in this neighborhood for $N > 2$ the functions $f(z)$ and $f(z)\log z + g(z)$ are linearly independent solutions of the linear homogeneous differential equation

$$\left(\left(z\frac{d}{dz}\right)^k - C_N \cdot z\left(z\frac{d}{dz} + \frac{q_1}{N}\right)\left(z\frac{d}{dz} + \frac{q_2}{N}\right)\cdots\left(z\frac{d}{dz} + \frac{q_k}{N}\right)\right)y = 0. \tag{7}$$

Equation (7) is the generalized hypergeometric equation whose solutions possess *maximum nilpotent monodromy* in the sense Morrison (see [2, Sec. 1; 3, Sec. 4.2]).

Theorem 1. *Suppose that for a given integer $N \geq 2$ the power series $f(z) = f_N(z)$ and $g(z) = g_N(z)$ are defined by formulas* (3)–(6). *Then the coefficients of the power expansion*

$$q(z) := \exp\left(\frac{f(z)\log z + g(z)}{f(z)}\right) = z \cdot \exp\left(\frac{g(z)}{f(z)}\right)$$

are integers.

In the case of a prime N, this theorem was proved by Lian and Yau [4, Theorem 5.5] using Dwork's p-adic technique [5]. In the present paper, we simplify the method used in [4] and prove more general results. However, the truly general assertion motivated by results from [5] and numerical experiments can be stated as follows.

Conjecture. *Suppose that $N_1, N_2, \ldots, N_r$ are integers, $N_j \geq 2$ for all $j = 1, \ldots, r$, and to the numerical sequences*

$$A(m) = A_{N_1}(m)A_{N_2}(m)\cdots A_{N_r}(m), \qquad m = 0, 1, 2, \ldots, \tag{8}$$

$$D(m) = D_{N_1}(m) + D_{N_2}(m) + \cdots + D_{N_r}(m), \qquad m = 0, 1, 2, \ldots, \tag{9}$$

correspond the power series

$$f(z) := \sum_{m=0}^{\infty} A(m)z^m, \qquad g(z) := \sum_{m=1}^{\infty} A(m)D(m)z^m. \tag{10}$$

Then the coefficients of the power expansion

$$q(z) := z \cdot \exp\left(\frac{g(z)}{f(z)}\right)$$

are integers.

Corollary. *For an arbitrary $N \geq 2$, define the power series*

$$f(z) := {}_{N-1}F_{N-2}\left(\begin{matrix} 1/N, 2/N, \dots, (N-2)/N, (N-1)/N \\ 1, \quad 1, \dots, \quad 1 \end{matrix} \;\middle|\; N^N \cdot z\right) = \sum_{m=0}^{\infty} \frac{(Nm)!}{m!^N} z^m, \tag{11}$$

$$g(z) := \sum_{m=1}^{\infty} \frac{(Nm)!}{m!^N} \left(\sum_{j=1}^{N-1} D\left(\frac{j}{N}, m\right) - (N-1)D(1,m)\right) z^m.$$

Then the coefficients of the power expansion

$$q(z) := z \cdot \exp\left(\frac{g(z)}{f(z)}\right)$$

are integers.

To prove the corollary, it suffices, by assumption, to choose

$$\left\{p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l} : \ 0 \leq \alpha_j \leq s_j, \ j = 1, \dots, l, \ \alpha_1 + \alpha_2 + \cdots + \alpha_l > 0\right\},$$

for the set of integers $\{N_j\}_{j=1,\dots,r}$; here $N = p_1^{s_1} p_2^{s_2} \cdots p_l^{s_l}$ is the factorization of the number N into primes.

Our contribution to the partial solution of the conjecture can be stated as follows.

Theorem 2. *Suppose that $N_1, N_2, \dots, N_r$ are integers, $N_j \geq 2$, and let any prime p dividing the product $N_1 N_2 \cdots N_r$ also divide each N_j, $j = 1, \dots, r$ (for example, $N_1 = N_2 = \cdots = N_r$). Define the power series $f(z), g(z)$ according to formulas (8)–(10). Then the coefficients of the power expansion*

$$q(z) := z \cdot \exp\left(\frac{g(z)}{f(z)}\right)$$

are integers.

Theorem 2 yields the corollary of the conjecture for any integer which is a power of a prime: $N = p^s$.

Theorem 3. *Suppose that $r \geq 1$ is an integer and $N = p^s$ for a prime p and an integer $s \geq 1$ and the power series $f(z)$ and $g(z)$ are defined by the formulas*

$$f(z) := \sum_{m=0}^{\infty} \left(\frac{(Nm)!}{m!^N}\right)^r z^m, \tag{12}$$

$$g(z) := r \sum_{m=1}^{\infty} \left(\frac{(Nm)!}{m!^N}\right)^r \left(\sum_{j=1}^{N-1} D\left(\frac{j}{N}, m\right) - (N-1)D(1,m)\right) z^m.$$

Then the coefficients of the power expansion

$$q(z) := z \cdot \exp\left(\frac{g(z)}{f(z)}\right)$$

are integers.

Proof. Choosing r copies of the set $\{p^\alpha\}_{1\le\alpha\le s}$ for the collection $\{N_j\}$ in Theorem 2, we obtain the required assertion. $\square$

The hypergeometric series $f(z)$ from (4), (10)–(12) and the linear differential equations they satisfy appear in a natural way in the geometry of Calabi–Yau manifolds. So, for example, the periods of the family of $(N-2)$-dimensional hypersurfaces

$$\mathcal{Q}_\psi := \{x_1^N + \cdots + x_N^N - N\psi x_1 \cdots x_N = 0\} \subset \mathbb{P}^{N-1}$$

satisfy, as functions of $z = (N\psi)^{-N}$, the same differential equation as the series (11) (see for example, [6, Corollary (2.3.8.1); 7, Sec. 8.3]). The inversion $z(q)$ of the series $q(z) = ze^{g(z)/f(z)}$ is an analytic function in the neighborhood of $q = 0$ and is called the *mirror map* for the corresponding family of hypersurfaces.

Lemma 2. *If the power series $q(z) = z + O(z^2)$ has integer coefficients, then its inversion $z(q) = q + O(q^2)$ possesses the same property.*

Theorems 1, 2, and Lemma 2 imply the existence of integral power expansions for a wide class of mirror maps. So, by setting $N = 8$ and $N = 10$ in Theorem 1, we obtain the integrality of mirror maps corresponding to the families of hypersurfaces

$$\mathcal{Q}_\psi^{(8)} := \{x_1^2 + x_2^8 + x_3^8 + x_4^8 + x_5^8 - 8\psi x_1x_2x_3x_4x_5 = 0\} \subset \mathbb{P}^4[4,1,1,1,1],$$
$$\mathcal{Q}_\psi^{(10)} := \{x_1^2 + x_2^5 + x_3^{10} + x_4^{10} + x_5^{10} - 10\psi x_1x_2x_3x_4x_5 = 0\} \subset \mathbb{P}^4[5,2,1,1,1],$$

respectively, in weighted projective spaces (for more details on these families, see for example, [2, Sec. 4]). Also note that for small values of N in Theorem 1 or of $N_1N_2\cdots N_r$ in Theorem 2 (namely, when the corresponding differential equation is of order 2 or 3) the fact that the series $z(q)$ is integral can be explained by the fact that the function represented by the series is modular as a function of $\tau = \frac{1}{\pi i}\log q$ (see [8, Sec. 1]).

The present paper is arranged as follows. In the following section, we reduce the proof of Theorems 1 and 2 to an arithmetic problem in p-adic analysis. In Sec. 3, we present the proof of Lemmas 1 and 2 and also some information on Dwork's p-adic technique [5]. Finally, in Sec. 4 we prove the main results of this paper.

2. p-ADIC REDUCTION OF THEOREMS 1 AND 2

Let p be a prime. Let $\operatorname{ord}_p\xi$ denote the p-adic order of the number $\xi \in \mathbb{Q}$ (i.e., the power to which p occurs in the irreducible fraction for ξ); the value of $\operatorname{ord}_p\xi$ can take any integer value and for $\xi = 0$ we assume $\operatorname{ord}_p\xi = +\infty$. The closure of the field $\mathbb{Q}$ with respect to the non-Archimedean norm $|\xi|_p = p^{-\operatorname{ord}_p\xi}$ is denoted by $\mathbb{Q}_p$; all elements ζ of the field $\mathbb{Q}_p$ (and, in particular, all elements of the original field $\mathbb{Q}$) possess the unique expansion

$$\zeta = p^r(c_0 + c_1p + c_2p^2 + \cdots + c_np^n + \cdots), \tag{13}$$

where $r = \operatorname{ord}_p\zeta \in \mathbb{Z}$ and $0 \le c_n < p$ for all $n = 0, 1, 2, \ldots$ and the convergence in (13) is understood in the sense of the norm $|\cdot|_p$ (see for example, [9, Chap. I, Sec. 4]). The expansion (13) is called the *p-adic representation* for the number ζ. The set of elements $\zeta \in \mathbb{Q}_p$ satisfying the condition $\operatorname{ord}_p\zeta \ge 0$ forms a ring $\mathbb{Z}_p$. The class of numbers $\zeta \in \mathbb{Q}_p$ for which $r \ge s$ in the expression (13) is denoted by $O(p^s)$.

Lemma 3. *Let the rational number ξ be an element of $\mathbb{Z}_p$ for any prime p. Then ξ is an integer.*

Proof. It suffices to give a proof only for $\xi \neq 0$. The factorization of the numerator and denominator of the irreducible fraction for ξ into prime factors induces the factorization into primes of the number $\xi = p_1^{s_1} p_2^{s_2} \cdots p_l^{s_l}$. In addition, $\operatorname{ord}_{p_j} \xi = s_j \geq 0$ for $j = 1, \ldots, l$, since $\xi \in \mathbb{Z}_{p_j}$ by assumption. Thus the number ξ is the product of primes raised to nonnegative integer powers, i.e., $\xi \in \mathbb{Z}$, as required. $\square$

Corollary. *To prove Theorems* 1 *and* 2, *it suffices to show that*

$$q(z) = z \cdot \exp\left(\frac{g(z)}{f(z)}\right) \in \mathbb{Z}_p[[z]]$$

for any prime p. (*Here* $\mathbb{Z}_p[[z]]$ *denotes the ring of formal power series with respect to* z *with coefficients from* $\mathbb{Z}_p$.)

3. PRELIMINARY NOTIONS

First, let us prove the auxiliary assertions from Sec. 1.

Proof of Lemma 2. For the power series

$$q(z) = \sum_{m=1}^{\infty} a_m z^m, \quad a_1 = 1, \quad a_m \in \mathbb{Z},\ m = 1, 2, \ldots, \qquad z(q) = \sum_{n=1}^{\infty} b_n q^n,$$

by assumption, the identity $q = q(z(q))$ is valid, and hence

$$q = \sum_{n=1}^{\infty} b_n q^n + a_2 \left(\sum_{n=1}^{\infty} b_n q^n\right)^2 + a_3 \left(\sum_{n=1}^{\infty} b_n q^n\right)^3 + \cdots. \tag{14}$$

Comparing the first coefficients of the powers of q in (14), we obtain $b_1 = 1$, $b_2 + a_2 = 0$, $b_3 + 2a_2 b_2 + a_3 = 0$, i.e., b_1, b_2, b_3 are integers. The subsequent proof is carried out by induction. Suppose we have proved that $b_1, \ldots, b_{n-1}$ for $n \geq 3$ are integers, and now consider the coefficient of q^n in (14). It is equal to

$$b_n + \sum_{i=2}^{n-1} a_i M_i + a_n = 0, \tag{15}$$

where M_i is the coefficient of q^n in the polynomial $(b_1 q + b_2 q^2 + \cdots + b_{n-1} q^{n-1})^i$, $i = 2, \ldots, n-1$. By the induction assumption, the numbers M_i are integers, and hence relation (15) implies that b_n is an integer. The lemma is proved. $\square$

Proof of Lemma 1. Suppose that the integer q, $0 < q < N$, is prime to N. Then in the factorization into primes the denominators of the numbers

$$N^m \cdot \frac{(q/N)_m}{m!} = \frac{q(q+N)(q+2N)\cdots(q+(m-1)N)}{m!}, \qquad m = 0, 1, 2, \ldots, \tag{16}$$

contain only prime divisors of the number N (see, for example, [10, Chap. I, Supplement]). The numerators of the numbers (16) are prime to N and the power to which the prime $p \mid N$ occurs in $m!$ is equal to

$$\operatorname{ord}_p m! = \left[\frac{m}{p}\right] + \left[\frac{m}{p^2}\right] + \left[\frac{m}{p^3}\right] + \cdots < \frac{m}{p-1} \tag{17}$$

(here $[\,\cdot\,]$ is the integral part of a number). Therefore, the constant (2) does cancel the denominators of the numbers

$$\frac{(q_1/N)_m (q_2/N)_m \cdots (q_k/N)_m}{m!^k}.$$

The lemma is proved. $\square$

Lemma 4. *For the elements of the sequence* (3)*, the following "factorial" representation is valid*:

$$A(m)=\frac{(a_1m)!\,(a_2m)!\cdots(a_\mu m)!}{(b_1m)!\,(b_2m)!\cdots(b_\eta m)!},\qquad m=0,1,2,\dots, \tag{18}$$

where

$$\{a_j\}_{j=1,\dots,\mu}=\left\{N,\frac{N}{p_{j_1}p_{j_2}},\frac{N}{p_{j_1}p_{j_2}p_{j_3}p_{j_4}},\dots\right\}_{1\le j_1<j_2<\dots\le l},$$
$$\{b_i\}_{i=1,\dots,\eta}=\left\{1,\dots,1,\frac{N}{p_{j_1}},\frac{N}{p_{j_1}p_{j_2}p_{j_3}},\dots\right\}_{1\le j_1<j_2<\dots\le l}$$

are integer sets corresponding to the given integer $N=p_1^{s_1}p_2^{s_2}\cdots p_l^{s_l}$ *and, in addition,*

$$a_1+a_2+\dots+a_\mu=b_1+b_2+\dots+b_\eta. \tag{19}$$

Proof. Using a modification of a formal logic principle (see [1, Sec. 8, Problems 21–25]), for an integer $n\ge 0$ we obtain

$$\begin{aligned}\prod_{\substack{1\le q\le N\\(q,N)=1}}\left(\frac{q}{N}+n\right)&=\prod_{1\le q\le N}\left(\frac{q}{N}+n\right)\cdot\left(\prod_{1\le j\le l}\prod_{\substack{1\le q\le N\\p_j\mid q}}\left(\frac{q}{N}+n\right)\right)^{-1}\left(\prod_{1\le j_1<j_2\le l}\prod_{\substack{1\le q\le N\\p_{j_1}p_{j_2}\mid q}}\left(\frac{q}{N}+n\right)\right)\\&\quad\times\left(\prod_{1\le j_1<j_2<j_3\le l}\prod_{\substack{1\le q\le N\\p_{j_1}p_{j_2}p_{j_3}\mid q}}\left(\frac{q}{N}+n\right)\right)^{-1}\cdots\\&=\prod_{1\le q\le N}\left(\frac{q}{N}+n\right)\cdot\left(\prod_{1\le j\le l}\prod_{1\le q\le N/p_j}\left(\frac{q}{N/p_j}+n\right)\right)^{-1}\\&\quad\times\left(\prod_{1\le j_1<j_2\le l}\prod_{1\le q\le N/(p_{j_1}p_{j_2})}\left(\frac{q}{N/(p_{j_1}p_{j_2})}+n\right)\right)\\&\quad\times\left(\prod_{1\le j_1<j_2<j_3\le l}\prod_{1\le q\le N/(p_{j_1}p_{j_2}p_{j_3})}\left(\frac{q}{N/(p_{j_1}p_{j_2}p_{j_3})}+n\right)\right)^{-1}\cdots.\end{aligned}$$

Let us multiply the obtained expression by

$$\begin{aligned}&N^{-N}\cdot\left(\prod_{1\le j\le l}\left(\frac{N}{p_j}\right)^{N/p_j}\right)^{-1}\cdot\left(\prod_{1\le j_1<j_2\le l}\left(\frac{N}{p_{j_1}p_{j_2}}\right)^{N/(p_{j_1}p_{j_2})}\right)\\&\quad\times\left(\prod_{1\le j_1<j_2<j_3\le l}\left(\frac{N}{p_{j_1}p_{j_2}p_{j_3}}\right)^{N/(p_{j_1}p_{j_2}p_{j_3})}\right)^{-1}\cdots=C_N,\end{aligned}$$

take the product over all $n=0,1,\dots,m-1$, and divide the resulting expression by $m!^k$. Then, in view of (3), we obtain the required identity (18). The lemma is proved. □

Let us fix an arbitrary prime p until the end of this section.

Dwork's lemma [9, Chap. VI, Sec. 2, Lemma 3; 11, Chap. 14, Sec. 2]. *Suppose that the function* $F(z)$ *belongs to* $1+z\mathbb{Q}[[z]]$. *Then* $F(z)\in 1+z\mathbb{Z}_p[[z]]$ *if and only if*

$$\frac{F(z^p)}{F(z)^p}\in 1+pz\mathbb{Z}_p[[z]].$$

Despite the fact that the proof of the following assertion is contained (with a misprint) in [4], we present it in this paper for the sake of completeness.

Lemma 5 [4, Corollary 6.7]. *Let* $f(z)\in z\mathbb{Q}[[z]]$. *Then* $e^{f(z)}\in 1+z\mathbb{Z}_p[[z]]$ *if and only if*

$$f(z^p)-pf(z)\in pz\mathbb{Z}_p[[z]].$$

Proof. Set $F(z)=e^{f(z)}\in 1+z\mathbb{Q}[[z]]$.

Necessity. Let $F(z)\in 1+z\mathbb{Z}_p[[z]]$. Then by Dwork's lemma

$$e^{f(z^p)-pf(z)}=\frac{F(z^p)}{F(z)^p}=1-pG(z)$$

for some power series $G(z)\in z\mathbb{Z}_p[[z]]$. Therefore,

$$f(z^p)-pf(z)=\log\bigl(1-pG(z)\bigr)=-\sum_{m=1}^{\infty}\frac{p^mG(z)^m}{m}\in pz\mathbb{Z}_p[[z]],$$

where we have used the fact that $p^m/m\in p\mathbb{Z}_p$ for all integers $m\geq 1$.

Sufficiency. Now let $f(z^p)-pf(z)=pH(z)$ for some $H(z)\in z\mathbb{Z}_p[[z]]$. Since for an integer $m\geq 1$ the power to which the prime p occurs in $m!$ is always *less* than m (see (17)), we can conclude that

$$\frac{F(z^p)}{F(z)^p}=e^{pH(z)}=1+\sum_{m=1}^{\infty}\frac{p^m}{m!}H(z)^m\in 1+pz\mathbb{Z}_p[[z]].$$

The application of Dwork's lemma yields the required inclusion $e^{f(z)}=F(z)\in 1+z\mathbb{Z}_p[[z]]$. The lemma is proved. $\square$

Lemma 6. *Suppose that the coefficients of the power series*

$$f(z)=\sum_{m=0}^{\infty}A(m)z^m\in 1+z\mathbb{Z}_p[[z]],\qquad A(m)\neq 0,\quad m=1,2,\ldots, \tag{20}$$

for all nonnegative integers u,v,n,s *such that* $0\leq u<p^s$ *and* $0\leq v<p$ *satisfy the condition*

$$\frac{A(v+up+np^{s+1})}{A(v+up)}-\frac{A(u+np^s)}{A(u)}\in p^{s+1}\mathbb{Z}_p. \tag{21}$$

Set

$$f_\nu(z)=\sum_{m=\nu}^{\infty}A(m)z^m,\qquad \nu=1,2,\ldots. \tag{22}$$

Then for any positive integer ν *one has the congruence*

$$\frac{f_\nu(z^p)}{f(z^p)}\equiv\frac{f_{\nu p}(z)}{f(z)}\pmod{\nu p\mathbb{Z}_p[[z]]}. \tag{23}$$

Proof. Let us choose an arbitrary positive integer ν and set $s = \operatorname{ord}_p \nu$. Then $\nu = np^s$ for some positive integer n and $\nu p\mathbb{Z}_p[[z]] = p^{s+1}\mathbb{Z}_p[[z]]$.

In Theorem 1.1 from [5], setting

$$A^{(r)}(m) = A(m), \quad g_r(m) = 1, \qquad r = 0, 1, 2, \dots, \quad m = 0, 1, 2, \dots,$$

by condition (21) we obtain the congruence

$$f(z) \sum_{m=kp^s}^{(k+1)p^s-1} A(m)z^{mp} \equiv f(z^p) \sum_{m=kp^{s+1}}^{(k+1)p^{s+1}-1} A(m)z^m \pmod{p^{s+1}\mathbb{Z}_p[[z]]}, \tag{24}$$

valid for all positive integers k. Summing the congruences (24) over all $k = n, n+1, n+2, \dots$, we find that

$$f(z)f_\nu(z^p) \equiv f(z^p)f_{\nu p}(z) \pmod{p^{s+1}\mathbb{Z}_p[[z]]}. \tag{25}$$

Finally, since $f(z) \in 1 + z\mathbb{Z}_p[[z]]$, and hence also $f(z^p) \in 1 + z\mathbb{Z}_p[[z]]$, we can multiply both parts of the congruence (25) by the power series $\bigl(f(z)f(z^p)\bigr)^{-1} \in 1 + z\mathbb{Z}_p[[z]]$, and this precisely yields the required congruence (23). The lemma is proved. □

Proposition 1. *Suppose that the coefficients of the power series* (20) *for all nonnegative integers* u, v, n, s *such that* $0 \le u < p^s$ *and* $0 \le v < p$ *satisfy condition* (21) *and the series* $g(z)$ *is defined by the expansion*

$$g(z) = \sum_{m=1}^{\infty} A(m)D(1, m)z^m = \sum_{m=1}^{\infty} A(m)\left(\sum_{\nu=1}^{m} \frac{1}{\nu}\right)z^m. \tag{26}$$

Then $e^{g(z)/f(z)} \in \mathbb{Z}_p[[z]]$.

Proof. Changing the summation in (26), we obtain the expansion

$$g(z) = \sum_{\nu=1}^{\infty} \frac{1}{\nu} f_\nu(z),$$

where the series $f_\nu(z)$ are defined by formulas (22). Therefore,

$$\begin{aligned} \frac{g(z^p)}{f(z^p)} - p\frac{g(z)}{f(z)} &= \sum_{\nu=1}^{\infty} \frac{1}{\nu}\frac{f_\nu(z^p)}{f(z^p)} - p\sum_{\nu=1}^{\infty} \frac{1}{\nu}\frac{f_\nu(z)}{f(z)} \\ &= \sum_{\nu=1}^{\infty} \frac{1}{\nu}\left(\frac{f_\nu(z^p)}{f(z^p)} - \frac{f_{\nu p}(z)}{f(z)}\right) - p\sum_{\substack{\nu=1 \\ p\nmid\nu}}^{\infty} \frac{1}{\nu}\frac{f_\nu(z)}{f(z)} \in pz\mathbb{Z}_p[[z]], \end{aligned} \tag{27}$$

where each summand of the first sum in (27) lies in $p\mathbb{Z}_p[[z]]$ by Lemma 6 and each summand of the second sum in $\mathbb{Z}_p[[z]]$, since $f_\nu(z) \in z\mathbb{Z}_p[[z]]$, $f(z) \in 1 + z\mathbb{Z}_p[[z]]$, and $1/\nu \in \mathbb{Z}_p$ for any integer ν prime to p. By Lemma 5, the inclusion (27) means that $e^{g(z)/f(z)} \in \mathbb{Z}_p[[z]]$. The proposition is proved. □

In what follows, we need certain properties of the *p-adic gamma-function*

$$\Gamma_p(n) = (-1)^n\gamma_p(n), \qquad \text{where} \quad \gamma_p(n) = \prod_{\substack{k=1 \\ (k,p)=1}}^{n-1} k. \tag{28}$$

Lemma 7. *For any integer $n \geq 0$, one has the identity*

$$\frac{(np)!}{n!} = p^n \gamma_p(1+np).$$

Proof. From the definition (28), we obtain

$$\gamma_p(1+np) = \frac{(np)!}{p \cdot 2p \cdot 3p \cdots np} = \frac{(np)!}{n!\, p^n},$$

and this yields the required identity. $\square$

Lemma 8 [11, Lemma 1.1]. *For all positive integers k, n, s, one has*

$$\Gamma_p(k+np^s) \equiv \Gamma_p(k) \pmod{p^s}.$$

Lemma 9. *Suppose that the sequence of integers $A(m) = A_N(m)$, $m = 0, 1, 2, \ldots$, is defined by* (3). *Then for any nonnegative integer m one has*

$$\frac{A(mp)}{A(m)} = 1 + O(p)$$

and, in particular, $\operatorname{ord}_p A(mp) = \operatorname{ord}_p A(m)$.

Proof. By Lemmas 7 and 8, for any positive integer a we have

$$\begin{aligned}\frac{(amp)!}{(am)!} &= p^{am}\gamma_p(1+amp) = p^{am}(-1)^{1+amp}\Gamma_p(1+amp) \\ &= p^{am}(-1)^{1+amp}\Gamma_p(1)\bigl(1+O(p)\bigr) = p^{am}(-1)^{amp}\bigl(1+O(p)\bigr);\end{aligned}$$

hence, using Lemma 4, we obtain the required relation

$$\frac{A(mp)}{A(m)} = p^{(a_1+\cdots+a_\mu-b_1-\cdots-b_\eta)m}(-1)^{(a_1+\cdots+a_\mu-b_1-\cdots-b_\eta)mp}\bigl(1+O(p)\bigr) = 1+O(p).$$

The lemma is proved. $\square$

4. PROOF OF THEOREMS 1 AND 2

For a prime p, on the set $\mathcal{C}_p := \{\xi \in \mathbb{Q} : \xi > 0,\ \operatorname{ord}_p \xi \geq 0\}$ we define the mapping

$$' \colon \mathcal{C}_p \to \mathcal{C}_p, \quad \xi \mapsto \xi', \tag{29}$$

by the following rule: the number $p\xi' - \xi$ is the minimal representative of the class of residues $-\xi \pmod p$ (in other words, ξ' is the minimal element in $\mathcal{C}_p$ such that $p\xi' - \xi \in \mathbb{Z}$).

Suppose that $N \geq 2$ is an integer and the numbers $q_1, q_2, \ldots, q_k$ form the complete set of remainders from the division by N which are prime to N.

Lemma 10. *If the prime p does not divide N, then the mapping* (29) *is a bijection of the set $\{q_j/N\}_{j=1,\ldots,k}$ onto itself.*

Proof. Denote by q'_j/N the image of the element q_j/N, $j=1,\dots,k$, under the mapping (29). By definition, we have

$$pq'_j \equiv q_j \pmod N, \qquad j=1,\dots,k.$$

Since p is prime to N, there exists a p' such that $pp' \equiv 1 \pmod N$. Therefore,

$$\{q'_j\}_{j=1,\dots,k} \equiv \{p'q_j\}_{j=1,\dots,k} \equiv \{q_j\}_{j=1,\dots,k} \pmod N,$$

whence, in view of the minimality of $q'_j > 0$, we obtain the required assertion. □

Proposition 2. *Suppose that the prime p is not a divisor of N under the assumptions of Theorem 1 or a divisor of $N_1N_2\cdots N_r$ under the assumptions of Theorem 2. Then the power series $q(z)$ from the statement of the corresponding theorem satisfies the inclusion $q(z) \in \mathbb{Z}_p[[z]]$.*

Proof. By Lemma 10, the set of parameters $\{q_j/N\}_{j=1,\dots,k}$ of the generalized hypergeometric series

$$\widehat{f}(z) := {}_kF_{k-1}\left(\begin{matrix} q_1/N, q_2/N, \dots, q_{k-1}/N, q_k/N \\ 1, \quad 1, \dots, \quad 1 \end{matrix} \,\middle|\, z\right) \tag{30}$$

is invariant with respect to the transformation (29). Setting

$$\widehat{g}(z) = \sum_{m=1}^{\infty} D_N(m) \frac{(q_1/N)_m (q_2/N)_m \cdots (q_k/N)_m}{m!^k} z^m, \tag{31}$$

from [5, Theorem 4.1] we obtain the congruence

$$\frac{\widehat{g}(z^p)}{\widehat{f}(z^p)} \equiv p \frac{\widehat{g}(z)}{\widehat{f}(z)} \pmod{p\mathbb{Z}_p[[z]]};$$

hence, by using Lemma 5, we find that $e^{\widehat{g}(z)/\widehat{f}(z)} \in \mathbb{Z}_p[[z]]$. Therefore, in Theorem 1 we have

$$q(z) = z \cdot \exp\left(\frac{g(z)}{f(z)}\right) = z \cdot \exp\left(\frac{\widehat{g}(C_N z)}{\widehat{f}(C_N z)}\right) \in z\mathbb{Z}_p[[C_N z]] = z\mathbb{Z}_p[[z]],$$

where we have used the fact that the constant C_N from (2) and the number p are coprime.

All the arguments with necessary modifications of the power expansions (30), (31) and the substitution of $C_{N_1}C_{N_2}\cdots C_{N_r}$ for C_N remain valid also for the series $q(z)$ in Theorem 2. Thus the proof of the proposition is complete. □

Therefore, it remains to prove the reduced versions of Theorems 1 and 2 only for the primes p dividing N or $N_1N_2\cdots N_r$, respectively.

Lemma 11. *Suppose that p is a prime divisor of the number N and the sequence of integers $A(m) = A_N(m)$, $m = 0, 1, 2, \dots$, is defined by (3). Then for all nonnegative integers u, n, s, $0 \le u < p^s$, one has*

$$\operatorname{ord}_p \frac{A(u+np^s)}{A(u)} \ge 0. \tag{32}$$

Proof. Since $p \mid N$ for all nonnegative integers v, m, $0 \le v < p$, by (3) we have

$$\begin{aligned} \frac{A(v+mp)}{A(mp)} &= D_N^v \prod_{i=1}^{v} \frac{(q_1+(i-1)N+mNp)\cdots(q_k+(i-1)N+mNp)}{i+mp} \\ &= D_N^v \frac{(q_1\cdots q_k)^v}{v!}\bigl(1+O(p)\bigr), \end{aligned}$$

where

$$D_N = N^{-k} \cdot C_N = \prod_{p|N} p^{k/(p-1)}. \tag{33}$$

Therefore,

$$\operatorname{ord}_p \frac{A(v+mp)}{A(mp)} = \operatorname{ord}_p D_N^v = \frac{kv}{p-1}. \tag{34}$$

To prove the lemma, we argue by induction. For $s=0$, we have $u=0$, so that the estimate (32) takes the form $\operatorname{ord}_p A(n) \geq 0$ and the last inequality is a consequence of Lemma 1.

Let us now prove the estimate (32) for $s \geq 1$, assuming it to be proven for smaller values of s. Let us express the number u as $u = v + u_1 p$, where $0 \leq v < p$ and $0 \leq u_1 < p^{s-1}$. Then, using (34) and Lemma 9, we obtain

$$\begin{aligned}
\operatorname{ord}_p \frac{A(u+np^s)}{A(u)} &= \operatorname{ord}_p A(v+u_1p+np^s) - \operatorname{ord}_p A(v+u_1p) \\
&= \left(\operatorname{ord}_p A(u_1p+np^s) - \frac{kv}{p-1}\right) - \left(\operatorname{ord}_p A(u_1p) - \frac{kv}{p-1}\right) \\
&= \operatorname{ord}_p A(u_1+np^{s-1}) - \operatorname{ord}_p A(u_1) = \operatorname{ord}_p \frac{A(u_1+np^{s-1})}{A(u_1)} \geq 0,
\end{aligned}$$

where the last inequality follows from the induction assumption. Thus the estimate (32) is proved for all nonnegative integers u, n, s, $0 \leq u < p^s$. $\square$

Lemma 12. *Suppose that the prime p divides N. Then the elements of the sequence* (3) *for all nonnegative integers u, v, n, s such that $0 \leq u < p^s$ and $0 \leq v < p$ satisfy condition* (21).

Proof. We have

$$\begin{aligned}
\frac{A(v+up+np^{s+1})}{A(up+np^{s+1})} &= \prod_{i=1}^{v} \frac{C_N(q_1/N+i-1+up+np^{s+1}) \cdots (q_k/N+i-1+up+np^{s+1})}{i+up+np^{s+1}} \\
&= D_N^v \prod_{i=1}^{v} \frac{(q_1+(i-1)N+uNp+nNp^{s+1}) \cdots (q_k+(i-1)N+uNp+nNp^{s+1})}{i+up+np^{s+1}} \\
&= D_N^v \prod_{i=1}^{v} \frac{(q_1+(i-1)N+uNp) \cdots (q_k+(i-1)N+uNp)}{i+up} \bigl(1+O(p^{s+1})\bigr) \\
&= \frac{A(v+up)}{A(up)} \bigl(1+O(p^{s+1})\bigr),
\end{aligned} \tag{35}$$

where the constant D_N is defined in (33).

Now we use the factorial representation (18) and the properties of the p-adic gamma-function. By Lemma 7, for any positive integer a we have

$$\begin{aligned}
\frac{(aup)!}{(au)!} &= p^{au}\gamma_p(1+aup), \\
\frac{(a(up+np^{s+1}))!}{(a(u+np^s))!} &= p^{a(u+np^s)}\gamma_p(1+aup+anp^{s+1}) \\
&= p^{a(u+np^s)}(-1)^{1+aup+anp^{s+1}}\Gamma_p(1+aup+anp^{s+1}) \\
&= p^{au+anp^s}(-1)^{1+aup+anp^{s+1}}\Gamma_p(1+aup)\bigl(1+O(p^{s+1})\bigr) \\
&= (-p)^{anp^s}\frac{(aup)!}{(au)!}\bigl(1+O(p^{s+1})\bigr),
\end{aligned} \tag{36}$$

where we have also applied Lemma 8. Now, in view of (19), using Lemma 4 and relations (36) for $a \in \{a_1, \dots, a_\mu, b_1, \dots, b_\eta\}$, we obtain

$$\frac{A(up+np^{s+1})}{A(u+np^s)} = \frac{A(up)}{A(u)}\bigl(1+O(p^{s+1})\bigr)\prod_{j=1}^{\mu}(-p)^{a_j np^s}\prod_{i=1}^{\mu}(-p)^{-b_i np^s} = \frac{A(up)}{A(u)}\bigl(1+O(p^{s+1})\bigr). \quad (37)$$

On multiplying relations (35) and (37), we find that

$$\frac{A(v+up+np^{s+1})}{A(u+np^s)} = \frac{A(v+up)}{A(u)}\bigl(1+O(p^{s+1})\bigr);$$

hence

$$\frac{A(v+up+np^{s+1})}{A(v+up)} = \frac{A(u+np^s)}{A(u)}\bigl(1+O(p^{s+1})\bigr). \quad (38)$$

By Lemma 11, the right-hand side of (38) lies in $\mathbb{Z}_p$, and hence we have the inclusion (21). The lemma is proved. □

Corollary. *Let the prime p divide each of the numbers $N_1, N_2, \dots, N_r$. Then the elements of the sequence* (8) *for all nonnegative integers u, v, n, s such that $0 \le u < p^s$ and $0 \le v < p$ satisfy condition* (21).

Proof. Relations (38) and (32) are valid for any of the sequences $A(m) = A_{N_j}(m)$, $j = 1, \dots, r$, $m = 0, 1, 2, \dots$. Hence we obtain the inclusion (21) for the elements of the sequence (8). □

Proposition 3. *Suppose that the prime p is a divisor of N under the assumptions of Theorem* 1 *or a divisor of $N_1N_2\cdots N_r$ under the assumptions of Theorem* 2. *Then for the power series $q(z)$ from the statement of the corresponding theorem the inclusion $q(z) \in \mathbb{Z}_p[[z]]$ is valid.*

Proof. For simplicity, we restrict ourselves to the proof of the proposition under the assumptions of Theorem 1. The proof follows the same outline also in the general case in which the prime p divides each N_j, $j = 1, \dots, r$.

Let us express the function (6) as the sum $g(z) = g_1(z) + g_2(z)$, where

$$g_1(z) = \sum_{m=1}^{\infty} A(m)\sum_{j=1}^{k} D\left(\frac{q_j}{N}, m\right)z^m, \qquad g_2(z) = -k\sum_{m=1}^{\infty} A(m)D(1,m)z^m,$$

and prove that $e^{g_i(z)/f(z)} \in \mathbb{Z}_p[[z]]$, $i = 1, 2$.

Note that

$$D\left(\frac{q_j}{N}, m\right) = N\sum_{n=1}^{m}\frac{1}{q_j+(n-1)N} \in p\mathbb{Z}_p, \qquad j = 1, \dots, k, \quad m = 1, 2, \dots, \quad (39)$$

since N is divisible by p and the denominator of each summand in (39) is prime to N (and hence to p). In addition, all elements of the sequence (3) lie in $\mathbb{Z}_p$, $f(z) \in 1 + z\mathbb{Z}_p[[z]]$. Therefore, $g_1(z) \in pz\mathbb{Z}_p[[z]]$ and $g_1(z)/f(z) \in pz\mathbb{Z}_p[[z]]$, so that the assumptions of Lemma 5 are satisfied for the series $g_1(z)/f(z)$. By Lemma 5, we have the inclusion $e^{g_1(z)/f(z)} \in \mathbb{Z}_p[[z]]$.

By Lemma 12, the elements of the sequence (3) after multiplication by the integer $-k$ satisfy condition (21). Therefore, from Proposition 1 we obtain the inclusion $e^{g_2(z)/f(z)} \in \mathbb{Z}_p[[z]]$.

Finally,

$$q(z) = ze^{g(z)/f(z)} = ze^{g_1(z)/f(z)}\cdot ze^{g_2(z)/f(z)} \in z\mathbb{Z}_p[[z]],$$

as was required. □

By the corollary of Lemma 3, the application of Propositions 2 and 3 proves Theorems 1 and 2.

ACKNOWLEDGMENTS

The subject matter of the present paper served as a basis for special seminars on Diophantine approximations and transcendental numbers conducted at the Department of Mechanics and Mathematics of Moscow University. The author wishes to thank all participants of these seminars for a number of valuable remarks and simplifications as a result of a detailed discussion of the proof of Theorems 1 and 2.

This research was supported in part jointly by INTAS and the Russian Foundation for Basic Research under grant no. IR-97-1904 and also by the State Scientific Grant for Young Scientists of the Russian Academy of Sciences.

REFERENCES

1. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, 3d ed., Springer-Verlag, Berlin–Göttingen–Heidelberg–New York, 1964.
2. D. R. Morrison, "Picard–Fuchs equations and mirror maps for hypersurfaces," in: *Essays on Mirror Manifolds* (S.-T. Yau, editor), International Press, Hong Kong, 1992, pp. 241–264; in: *Mirror Symmetry, I* (S.-T. Yau, editor), vol. 9, IP Stud. Adv. Math., Amer. Math. Soc., Providence, R.I., 1998, pp. 185–199.
3. V. V. Batyrev and D. van Straten, "Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties," *Comm. Math. Phys.*, **168** (1995), no. 3, 493–533.
4. B. H. Lian and S.-T. Yau, "Mirror maps, modular relations and hypergeometric series I," in: *E-print* http://arXiv.org/abs/hep-th/9507151; "Integrality of certain exponential series," in: *Lectures in Algebra and Geometry* (M.-C. Kang, editor), Proceedings of the International Conference on Algebra and Geometry, National Taiwan University (Taipei, Taiwan, December 26–30, 1995), International Press, Cambridge, MA, 1998, pp. 215–227.
5. B. Dwork, "On p-adic differential equations IV. Generalized hypergeometric functions as p-adic analytic functions in one variable," *Ann. Sci. École Norm. Sup.* (*4*), **6** (1973), no. 3, 295–315.
6. N. M. Katz, "Algebraic solutions of differential equations (p-curvature and the Hodge fibration)," *Invent. Math.*, **18** (1972), no. 1/2, 1–118.
7. D. R. Morrison, "Mathematical aspects of mirror symmetry," in: *Complex Algebraic Geometry* (J. Kollár, editor), vol. 3, Lectures of a Summer Program (Park City, UT, 1993), IAS/Park City Math. Ser., Amer. Math. Soc., Providence, R.I., 1997, pp. 267–340.
8. W. Zudilin, "Number theory casting a look at the mirror," in: *E-print* http://arXiv.org/abs/math/0008237 (preprint, 2000; submitted for publication).
9. N. Kobliz, *p-Adic Numbers, p-Adic Analysis, and Zeta-Functions*, Springer-Verlag, Heidelberg, 1977.
10. Y. André, *G-Functions and Geometry*, vol. E13, Aspects Math. (A Publication of the Max-Planck-Institut für Mathematik, Bonn), Vieweg, Braunschweig, 1989.
11. S. Lang, *Cyclotomic Fields, I, II*, Combined 2nd edition, vol. 121, Graduate Texts in Math., Springer-Verlag, New York, 1990.

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