IRRATIONALITY MEASURES FOR q-ZETA VALUES[1]

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1. q-Introduction. As usual, quantities depending on a number q and becoming classical objects as $q \to 1$ (at least formally) are regarded as *q-analogues* or *q-extensions*. A possible way to q-extend the values of Riemann's zeta function reads as follows (here $q \in \mathbb{C}$, $|q| < 1$):

$$\zeta_q(k) = \sum_{n=0}^{\infty} \sigma_{k-1}(n) q^n = \sum_{\nu=0}^{\infty} \frac{q^\nu \rho_k(q^\nu)}{(1-q^\nu)^k}, \qquad k = 1, 2, \dots, \tag{1}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the sum of powers of divisors and the polynomials $\rho_k(x) \in \mathbb{Z}[x]$ can be determined recursively by the formulae $\rho_1 = 1$ and $\rho_{k+1} = (1+(k-1)x)\rho_k + x(1-x)\rho_k'$ for $k = 1, 2, \dots$. Then we obtain

$$\lim_{\substack{q \to 1 \\ |q|<1}} (1-q)^k \zeta_q(k) = \rho_k(1) \cdot \zeta(k), \qquad k = 2, 3, \dots. \tag{2}$$

The above q-zeta values (1) present several problems in transcendence number theory that are extensions of the corresponding problems for ordinary zeta values; we state some of these problems at the end of the talk. Here we would like to explain how some recent contributions to the arithmetic study of $\zeta(k)$, $k = 2, 3, \dots$, successfully work for q-zeta values. Namely, we mean the hypergeometric construction of linear forms (due to Nikishin, Gutnik, Nesterenko) and the arithmetic approach (due to Chudnovsky, Rukhadze, Hata) accompanied with the group-structure scheme (due to Rhin, Viola). We consider the quantities $\zeta_q(1)$ and $\zeta_q(2)$ for $q^{-1} = p \in \mathbb{Z} \setminus \{0, \pm 1\}$ and start with the following table illustrating a connection of some standard ordinary and q-objects.

ordinary objects	q-extensions, $p = 1/q \in \mathbb{Z} \setminus \{0, \pm 1\}$
numbers $n \in \mathbb{Z}$	'numbers' $[n]_p = \dfrac{p^n - 1}{p - 1} \in \mathbb{Z}[p]$
primes $l \in \{2, 3, 5, 7, \dots\} \in \mathbb{Z}$	irreducible reciprocal polynomials $\Phi_l(p) = \prod_{\substack{k=1 \\ (k,l)=1}}^{l} (p - e^{2\pi i k/l}) \in \mathbb{Z}[p]$
Euler's gamma function $\Gamma(t)$	Jackson's q-gamma function $\Gamma_q(t) = \dfrac{\prod_{\nu=1}^{\infty}(1-q^\nu)}{\prod_{\nu=1}^{\infty}(1-q^{t+\nu-1})} (1-q)^{1-t}$
the factorial $n! = \Gamma(n+1)$ $n! = \prod_{\nu=1}^{n} \nu \in \mathbb{Z}$	the q-factorial $[n]_q! = \Gamma_q(n+1)$ $[n]_p! = \prod_{\nu=1}^{n} \dfrac{p^\nu - 1}{p - 1} = p^{n(n-1)/2} [n]_q! \in \mathbb{Z}[p]$
$\operatorname{ord}_l n! = \left\lfloor \dfrac{n}{l} \right\rfloor + \left\lfloor \dfrac{n}{l^2} \right\rfloor + \cdots$	$\operatorname{ord}_{\Phi_l(p)} [n]_p! = \left\lfloor \dfrac{n}{l} \right\rfloor$, $l = 2, 3, 4, \dots$

[1] An extension of the talk given at Luminy's conference *Problèmes Diophantiens* (CIRM, 6–10 May 2002).

ordinary objects	q-extensions, $p=1/q\in\mathbb{Z}\setminus\{0,\pm1\}$
$D_n=\mathrm{l.c.m.}(1,\dots,n) = \prod_{\text{primes } l\leqslant n} l^{\lfloor\log n/\log l\rfloor}\in\mathbb{Z}$	$D_n(p)=\mathrm{l.c.m.}([1]_p,\dots,[n]_p) = \prod_{l=1}^{n}\Phi_l(p)\in\mathbb{Z}[p]$
the prime number theorem $\lim_{n\to\infty}\dfrac{\log D_n}{n}=1$	Mertens' formula $\lim_{n\to\infty}\dfrac{\log\lvert D_n(p)\rvert}{n^2\log\lvert p\rvert}=\dfrac{3}{\pi^2}$

If $\psi(x)$ is the logarithmic derivative of Euler's gamma function and $\{x\}=x-\lfloor x\rfloor$ is the fractional part of a number x, then, for each demi-interval $[u,v)\subset(0,1)$, Mertens' formula yields the limit relation

$$\lim_{n\to\infty}\frac{1}{n^2\log|p|}\sum_{l:\{n/l\}\in[u,v)}\log|\Phi_l(p)|=\frac{3}{\pi^2}\bigl(\psi'(u)-\psi'(v)\bigr)=\frac{3}{\pi^2}\int_u^v\mathrm{d}\bigl(-\psi'(x)\bigr), \tag{3}$$

which can be regarded as a q-extension of the formula

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{l:\{n/l\}\in[u,v)\\ \text{primes } l>\sqrt{Cn}}}\log l=\psi(v)-\psi(u)=\int_u^v\mathrm{d}\psi(x)$$

in the arithmetic approach.

2. Rational approximations to q-zeta values and basic transformations. Let a_0, a_1, a_2, and b be positive integers satisfying the condition $a_1+a_2\leqslant b$. Then, Heine's series

$$F(\boldsymbol{a},b)=\frac{\Gamma_q(b-a_2)}{(1-q)\Gamma_q(a_1)}\sum_{t=0}^{\infty}\frac{\Gamma_q(t+a_1)\,\Gamma_q(t+a_2)}{\Gamma_q(t+1)\,\Gamma_q(t+b)}\,q^{a_0t}$$

becomes a $\mathbb{Q}(p)$-linear form $F(\boldsymbol{a},b)=A\zeta_q(1)-B$ with the property

$$p^{-M}D_m(p)\cdot F(\boldsymbol{a},b)\in\mathbb{Z}[p]\zeta_q(1)+\mathbb{Z}[p]. \tag{4}$$

Here $M=M(\boldsymbol{a},b)$ is some (explicit) integer and m is the successive maximum of the 6-element set

$$c_{00}=a_0+a_1+a_2-b-1,\qquad c_{01}=a_0-1,\qquad c_{11}=a_1-1,\qquad c_{21}=a_2-1,$$
$$c_{12}=b-a_1-1,\qquad c_{22}=b-a_2-1.$$

Taking $H(\boldsymbol{c})=F(\boldsymbol{a},b)$ and using the stability of the quantity

$$\frac{F(a_0,a_1,a_2,b)}{\Gamma_q(a_0)\,\Gamma_q(a_2)\,\Gamma_q(b-a_2)}=\frac{H(\boldsymbol{c})}{\Pi_q(\boldsymbol{c})},\qquad\text{where}\quad \Pi_q(\boldsymbol{c})=[c_{01}]_q!\,[c_{21}]_q!\,[c_{22}]_q!=p^{-N(\boldsymbol{c})}\Pi_p(\boldsymbol{c}),$$

under the actions of

$$\tau=(c_{22}\;c_{21}\;c_{01}\;c_{11}\;c_{12}\;c_{00})\colon (a_0,a_1,a_2,b)\mapsto(a_1,b-a_1,a_0,a_0+a_2),$$
$$\sigma=(c_{11}\;c_{21})(c_{12}\;c_{22})\colon \qquad (a_0,a_1,a_2,b)\mapsto(a_0,a_2,a_1,b)$$

we arrive at the better than (4) inclusions

$$p^{-M}D_m(p)\Omega^{-1}(p)\cdot F(\boldsymbol{a},b)\in\mathbb{Z}[p]\zeta_q(1)+\mathbb{Z}[p] \tag{5}$$

with

$$\Omega(p)=\prod_{l=1}^{m}\Phi_l^{\nu_l}(p), \qquad \nu_l=\max_{\mathfrak{g}\in\langle\tau^2,\sigma\rangle}\operatorname{ord}_{\Phi_l(p)}\frac{\Pi_p(\boldsymbol{c})}{\Pi_p(\mathfrak{g}\boldsymbol{c})}. \tag{6}$$

In addition, trivial estimates for $F(\boldsymbol{a},b)$ and explicit formulae for the coefficient A imply that

$$|F(\boldsymbol{a},b)|=|p|^{O(b)}, \qquad |A|\leqslant|p|^{(a_0+a_1+a_2)b-(a_1^2+a_2^2+b^2)/2+O(b)} \tag{7}$$

with some absolute constant in $O(b)$.

The basic transform τ of order 6 was proved by Heine more than 150 years ago. The group $\mathfrak{G}=\langle\tau,\sigma\rangle$ of order 12 has no ordinary analogue since corresponding (in limit $q\to1$) Gauß's hypergeometric series is divergent. We use the group $\langle\tau^2,\sigma\rangle$ of order 6 instead of the total available group $\mathfrak{G}$ to ensure the required condition $a_1+a_2\leqslant b$. Now, choosing $a_0=a_2=8n+1$, $a_1=6n+1$, and $b=15n+1$, and taking in mind (5), (7), and (3) we conclude that the irrationality exponent of $\zeta_q(1)$ satisfies the estimate

$$\mu(\zeta_q(1))\leqslant 2.42343562\ldots$$

that can be compared with the previous result $\mu(\zeta_q(1))\leqslant 2\pi^2/(\pi^2-2)=2.50828476\ldots$ of Bundschuh and Väänänen (corresponding to the choice $a_0=a_1=a_2=n+1$ and $b=2n+2$).

Similar arguments with a simpler group $\langle\sigma\rangle$ of order 2 can be put forward to improve Van Assche's estimate $\mu(\log_q(2))\leqslant 3.36295386\ldots$ for the following q-extension of $\log(2)$:

$$\log_q(2)=\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}q^\nu}{1-q^\nu}=\sum_{\nu=1}^{\infty}\frac{q^\nu}{1+q^\nu}.$$

Namely, we are able to prove the estimate $\mu(\log_q(2))\leqslant 3.29727451\ldots$ for $q^{-1}=p\in\mathbb{Z}\setminus\{0,\pm1\}$.

In the case of $\zeta_q(2)$, we consider the positive parameters $(\boldsymbol{a},\boldsymbol{b})=(a_1,a_2,a_3,b_2,b_3)$ satisfying the conditions $a_j<b_k$, $a_1+a_2+a_3<b_2+b_3$ and the q-basic hypergeometric series

$$\begin{aligned}\widetilde{F}(\boldsymbol{a},\boldsymbol{b})&=\frac{\Gamma_q(b_2-a_2)\,\Gamma_q(b_3-a_3)}{(1-q)^2\Gamma_q(a_1)}\sum_{t=0}^{\infty}\frac{\Gamma_q(t+a_1)\,\Gamma_q(t+a_2)\,\Gamma_q(t+a_3)}{\Gamma_q(t+1)\,\Gamma_q(t+b_2)\,\Gamma_q(t+b_3)}\,q^{(b_2+b_3-a_1-a_2-a_3)t}\\&=\widetilde{A}\zeta_q(2)-\widetilde{B}.\end{aligned}$$

Then $p^{-M}D_{m_1}(p)D_{m_2}(p)\cdot\widetilde{F}(\boldsymbol{a},\boldsymbol{b})\in\mathbb{Z}[p]\zeta_q(2)+\mathbb{Z}[p]$, where m_1,m_2 are the two successive maxima of the 10-element set

$$c_{00}=(b_2+b_3)-(a_1+a_2+a_3)-1, \qquad c_{jk}=\begin{cases}a_j-1 & \text{if } k=1,\\ b_k-a_j-1 & \text{if } k=2,3,\end{cases} \qquad j=1,2,3,$$

and, in addition,

$$|\widetilde{F}(\boldsymbol{a},\boldsymbol{b})|=|p|^{O(\max\{b_2,b_3\})}, \qquad |\widetilde{A}|\leqslant|p|^{b_2b_3-(a_1^2+a_2^2+a_3^2)/2+O(\max\{b_2,b_3\})}.$$

The $\boldsymbol{c}$-permutation group $\mathfrak{G}\subset\mathfrak{S}_{10}$ generated by all permutations of a_1,a_2,a_3, the permutation of b_2,b_3, and the permutation $(c_{00}\ \ c_{22})(c_{11}\ \ c_{33})(c_{13}\ \ c_{31})$ has order 120 and is known in connection with the Rhin–Viola proof of the new irrationality measure for $\zeta(2)$. In notation $\widetilde{H}(\boldsymbol{c})=\widetilde{F}(\boldsymbol{a},\boldsymbol{b})$, the quantity

$$\frac{\widetilde{H}(\boldsymbol{c})}{[c_{00}]_q!\,[c_{21}]_q!\,[c_{22}]_q!\,[c_{33}]_q!\,[c_{31}]_q!}$$

is stable under the action of the group $\mathfrak{G}$. This stability yields the inclusions

$$p^{-M}D_{m_1}(p)D_{m_2}(p)\widetilde{\Omega}^{-1}(p)\cdot\widetilde{F}(\boldsymbol{a},\boldsymbol{b})\in\mathbb{Z}[p]\zeta_q(2)+\mathbb{Z}[p]$$

with a quantity $\widetilde{\Omega}(p)$ defined like in (6). Finally, choosing $a_1=5n+1$, $a_2=6n+1$, $a_3=7n+1$, and $b_2=14n+2$, $b_3=15n+2$ we deduce the estimate

$$\mu(\zeta_q(2))\leqslant 4.07869374\dots$$

that cannot be compared with previous results, although the transcendence of $\zeta_q(2)$ for algebraic q with $0<|q|<1$ follows from Nesterenko's theorem.

It is nice to mention that the simpler choice $a_1=a_2=a_3=n+1$, $b_2=b_3=2n+2$ also proves the irrationality of $\zeta_q(2)$ for $q^{-1}\in\mathbb{Z}\setminus\{0,\pm1\}$ and the limit $q\to1$ produces Apéry's original sequence of rational approximations to $\zeta(2)$.

We would like to stress that the both series $F(\boldsymbol{a},b)$ and $\widetilde{F}(\boldsymbol{a},\boldsymbol{b})$ possess (multiple) q-integral representations as those considered by Rhin and Viola in their arithmetic study of $\zeta(2)$ and $\zeta(3)$. Inspite of this fact, there exists no general pattern to change the variable of q-integration, hence we do not expect a multiple-integral explanation of the above construction.

3. General problems for q-zeta values. We start this part with mentioning that, for an even integer $k\geqslant2$, the series $E_k(q)=1-2k\zeta_q(k)/B_k$, where $B_k\in\mathbb{Q}$ are Bernoulli numbers, is known to be the Eisenstein series. Therefore the modular origin (with respect to the parameter $\tau=\frac{\log q}{2\pi i}$) of $E_4,E_6,E_8,\dots$ gives the algebraic independence of the functions $\zeta_q(2),\zeta_q(4),\zeta_q(6)$ over $\mathbb{Q}[q]$, while all other even q-zeta values are polynomials in $\zeta_q(4)$ and $\zeta_q(6)$. In this sence, the consequence of Nesterenko's theorem "the numbers $\zeta_q(2),\zeta_q(4),\zeta_q(6)$ are algebraically independent over $\mathbb{Q}$ for algebraic q, $0<|q|<1$" reads as a complete q-extension of the consequence of Lindemann's theorem "$\zeta(2)$ is transcendental".

The limit relations (2) as well as the expected algebraic structure of the ordinary zeta values motivate the following questions (we also regard $\zeta_q(1)$ to be an odd q-zeta value, although the corresponding ordinary harmonic series is divergent).

Conjecture 1. *The q-zeta values $\zeta_q(1),\zeta_q(2),\zeta_q(3),\dots$ as functions of q are linearly independent over $\mathbb{C}(q)$.*

Conjecture 2. *The q-functional set involving the three even q-zeta values $\zeta_q(2),\zeta_q(4),\zeta_q(6)$ and all odd q-zeta values $\zeta_q(1),\zeta_q(3),\zeta_q(5),\dots$ consists of functions that are algebraically independent over $\mathbb{C}(q)$.*

The associated diophantine problems are to prove the corresponding linear and algebraic independence over the algebraic closure of $\mathbb{Q}$ for algebraic q with $0<|q|<1$. Even irrationality and $\mathbb{Q}$-linear independence results at the point $q\in\mathbb{Q}$ with $q^{-1}\in\mathbb{Z}\setminus\{0,\pm1\}$ look interesting problems.

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