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On the Functional Transcendence of q-Zeta Values

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For any integer $k \ge 1$, the power series

$$\zeta_q(k+1) = \sum_{n=1}^{\infty} \sigma_k(n) q^n, \qquad \sigma_k(n) = \sum_{d|n} d^k, \tag{1}$$

determines some q-extension of the value $\zeta(k+1)$ of the Riemann zeta function (see [1]). Moreover, the series in (1) is also meaningful for $k = 0$. By virtue of the trivial estimates

$$\sigma_k(n) \le n^k \sum_{d|n} 1 \le n^{k+1},$$

this series represents an analytic function inside the unit disc for each integer $k \ge 0$. The objective of this paper is to prove that the function $\zeta_q(k+1)$ is not algebraic for any $k \ge 1$. This (and even a stronger) result is well known for $\zeta_q(2), \zeta_q(4), \zeta_q(6), \dots$, because the functions $1 + c_k\zeta_q(k)$ with suitable $c_k \in \mathbb{Q}$ are Eisenstein series for each even $k \ge 2$.

Theorem. *For each $k \ge 0$, the function $\zeta_q(k+1)$ analytic on the domain $|q| < 1$ is transcendental over $\mathbb{C}(q)$.*

In essence, this result is an application of problems from [2, Division 8] (see also the original work [3, pp. 368–371]). Hereafter, by an *integral* power series we mean a series with integer coefficients (such is, e.g., the power series in (1)).

Lemma 1 [2, Division 8, Chap. 3, Sec. 4, Problem 163]**.** *If a rational function is represented by an integral power series, then the coefficients of this series, starting with some coefficient, are periodic for any modulus.*

Lemma 2 [2, Division 8, Chap. 3, Sec. 5, Problem 167]**.** *If an integer series represents an algebraic irrational function, then its radius of convergence is less than one.*

Lemma 3. *For each integer $k \ge 0$ and any $N \in \mathbb{N}$, the function $\sigma_k(n)$ of positive integer argument $n > N$ is not periodic modulo 2.*

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Proof. First, note that the function $\sigma_k(n)$ is *multiplicative*, i.e.,

$$\sigma_k(m_1 m_2) = \sigma_k(m_1)\sigma_k(m_2), \qquad m_1, m_2 \in \mathbb{Z}, \quad (m_1, m_2) = 1,$$

and if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$ is the canonical prime decomposition of the number n, then

$$\sigma_k(n) = \prod_{j=1}^{l} \bigl(1 + p_j^k + p_j^{2k} + p_j^{3k} + \dots + p_j^{\alpha_j k}\bigr) \tag{2}$$

(see, e.g., [2, Division 8, Chap. 1, Sec. 6, Problem 44]). In particular, (2) implies $\sigma_k(n^2) \equiv 1 \pmod 2$ for any $n \ge 1$ and $\sigma_k(p) \equiv 0 \pmod 2$ for any odd prime p. Therefore, if $\sigma_k(n)$ with $n > N$ is periodic modulo 2, then the period is larger than one. Suppose that, on the contrary, the sequence under consideration has period $m > 1$ modulo 2, i.e.,

$$\sigma_k(n_1) \equiv \sigma_k(n_2) \pmod 2, \qquad n_1, n_2 > N, \quad n_1 \equiv n_2 \pmod m. \tag{3}$$

Choose an odd prime $p > N$ coprime to m and an exponent $\alpha \ge 1$ such that $m^{2\alpha} > N$. According to what is said above, $\sigma_k(p) \equiv 0 \pmod 2$ and $\sigma_k(m^{2\alpha}) \equiv 1 \pmod 2$, whence

$$\sigma_k(pm^{2\alpha}) = \sigma_k(p)\sigma_k(m^{2\alpha}) \equiv 0 \pmod 2.$$

This congruence contradicts the assumption (3), because $pm^{2\alpha} \equiv m^{2\alpha} \pmod m$. This contradiction completes the proof of the lemma. □

Proof of the theorem. According to Lemmas 1 and 3, the function $\zeta_q(k+1)$ is irrational over the field $\mathbb{C}(q)$; therefore, Lemma 2 and the convergence of the series in (1) in the domain $|q| < 1$ implies the transcendence of this function over $\mathbb{C}(q)$. □

Remark. The power series in (1) can also be represented in the domain $|q| < 1$ as the Lambert series

$$\zeta_q(k+1) = \sum_{l=1}^{\infty} \frac{f(l) q^l}{1 - q^l},$$

where $f(l) = l^k$ is a multiplicative function of a positive integer argument. Similar Lambert series are not always irrational (and transcendental); for instance, the choice of the (multiplicative) Euler function $f(l) = \varphi(l)$ (the quantity of numbers $0 \le k < l$ coprime to l) leads to the rational function $q/(1-q)^2$ (see [2, Division 8, Chap. 1, Sec. 7, Problem 69]).

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