On the irrationality of $\zeta_q(2)$

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For complex q, $|q|<1$, we define the quantity

$$\zeta_q(2):=\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}=\sum_{n=1}^{\infty}\sigma(n)q^n;\qquad \lim_{\substack{q\to1\\|q|<1}}(1-q)^2\zeta_q(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6},$$

where $\sigma(n)$ is the sum of divisors of the positive integer n.

Theorem 1. *When $q=1/p$, $p\in\mathbb{Z}\setminus\{0,\pm1\}$, the number $\zeta_q(2)$ is irrational and its index of irrationality satisfies the inequality $\mu(\zeta_q(2))\leqslant 4.07869374\ldots$.*

Recall that the *index of irrationality* $\mu(\alpha)$ of a number $\alpha\in\mathbb{R}\setminus\mathbb{Q}$ is defined as the least upper bound of those $\mu\in\mathbb{R}$ for which the inequality $|\alpha-a/b|\leqslant|b|^{-\mu}$ has a finite number of solutions for $a,b\in\mathbb{Z}$. (Note that $\mu(\alpha)\geqslant2$ by Dirichlet's theorem.) If $\mu(\alpha)<+\infty$, we say that α is a *Liouville number.* A theorem of Nesterenko [1] implies the transcendence of $\zeta_q(2)$ for any $q\in\mathbb{Q}$ with $0<|q|<1$, although it does not follow from general bounds for the measure of transcendence [2] that this is a Liouville number.

We shall use standard q-notation [3]:

$$(T;q)_n:=\prod_{k=1}^{n}(1-q^{k-1}T),\quad \Gamma_q(t):=\frac{(q;q)_\infty}{(q^t;q)_\infty}(1-q)^{1-t},\quad [n]_q!:=\Gamma_q(n+1)=\frac{(q;q)_n}{(1-q)^n}.$$

For each $n=0,1,2,\ldots$ we define numbers $a_j=\alpha_jn+1$, $j=1,2,3$, $b_1=\beta_1n+1$, $b_k=\beta_kn+2$, $k=2,3$, where the integer parameters (*directions*) α_j and β_1,β_k satisfy the conditions $\beta_1=0\leqslant\alpha_j\leqslant\beta_k$, $\alpha_1+\alpha_2+\alpha_3\leqslant\beta_1+\beta_2+\beta_3$. Consider the q-basic hypergeometric series [3]

$$H_n(q):=\frac{[b_2-a_2-1]_q!\,[b_3-a_3-1]_q!}{(1-q)^2[a_1-b_1]_q!}\sum_{t=0}^{\infty}R(q;t)q^t,\tag{1}$$

$$\text{where}\quad R(q;t)=\frac{\Gamma_q(t+a_1)\,\Gamma_q(t+a_2)\,\Gamma_q(t+a_3)}{\Gamma_q(t+b_1)\,\Gamma_q(t+b_2)\,\Gamma_q(t+b_3)}\cdot q^{t(b_2+b_3-a_1-a_2-a_3-1)}.$$

By decomposing $R(q;t)$ as a rational function of $T=q^t$ into a sum of partial fractions and performing the summation in (1), we arrive at the following assertion.

Lemma 1. *$H_n(q)=A_n(q)\zeta_q(2)-B_n(q)$, where $A_n(q)$ and $B_n(q)$ are rational functions of the parameter q.*

Explicit formulae for $A_n(q)$ and trivial estimates for the series on the right-hand side of (1) lead to the following result.

Lemma 2. *For any $q=1/p$, $p\in\mathbb{Z}\setminus\{0,\pm1\}$,*

$$\lim_{n\to\infty}\frac{\log|H_n(q)|}{n^2\log|p|}=0,\qquad \varlimsup_{n\to\infty}\frac{\log|A_n(q)|}{n^2\log|p|}\leqslant\beta_2\beta_3-\frac{\alpha_1^2+\alpha_2^2+\alpha_3^2}{2}=:C_1.$$

To calculate the denominators of the rational functions $A_n(q),B_n(q)$ as in [4], [5] for linear approximations to $\zeta(2)$, we apply a group $\mathfrak{G}\subset\mathfrak{S}_{10}$ of permutations of the 10-element set

$$c_{00}=(\beta_2+\beta_3)-(\alpha_1+\alpha_2+\alpha_3),\qquad c_{jk}=\begin{cases}\alpha_j-\beta_k & \text{if } k=1,\\ \beta_k-\alpha_j & \text{if } k=2,3,\end{cases}\qquad j,k=1,2,3.\tag{2}$$

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This group has 120 elements, and the quantity

$$\frac{H_n(q)}{[c_{00}n]_q!\,[c_{21}n]_q!\,[c_{22}n]_q!\,[c_{33}n]_q!\,[c_{31}n]_q!}$$

is invariant under its action. Moreover, the quantity $H_n(q)$ itself is invariant under the action of a subgroup $\mathfrak{G}_0 \subset \mathfrak{G}$ of order 10. We put

$$M := \max_{\mathfrak{g}\in\mathfrak{G}_0}\{\widetilde{M}(\mathfrak{g}\boldsymbol{c})\}, \qquad \widetilde{M}(\boldsymbol{c}) := \begin{cases} c_{00}c_{21} + c_{31}c_{33} - c_{21}c_{33} & \text{if } c_{21} \leqslant c_{31}, \\ c_{00}c_{31} + c_{21}c_{22} - c_{31}c_{22} & \text{if } c_{21} \geqslant c_{31}, \end{cases}$$

$$\omega(z) := \max_{\mathfrak{g}\in\mathfrak{G}}\{\widetilde{\omega}(\boldsymbol{c};z) - \widetilde{\omega}(\mathfrak{g}\boldsymbol{c};z)\}, \qquad \widetilde{\omega}(\boldsymbol{c};z) := \lfloor c_{00}z\rfloor + \lfloor c_{21}z\rfloor + \lfloor c_{22}z\rfloor + \lfloor c_{33}z\rfloor + \lfloor c_{31}z\rfloor,$$

where $\mathfrak{g}\boldsymbol{c}$ denotes the action of the corresponding permutation on the set (2), $\lfloor\cdot\rfloor$ denotes the integer part function, and the function $\omega(z)$ takes non-negative integer values and is 1-periodic. Also let $m_1 \geqslant m_2$ be two maximal elements standing in different places in the tuple $\boldsymbol{c}$. The *cyclotomic polynomials* $\Phi_l(x)$, and only these, occur in the decomposition of $(x;x)_n$ into irreducible factors (see, for example, [6], [7]), and the polynomial $D_n(x) := \prod_{l=1}^{n}\Phi_l(x)$ is the least common multiple of $x-1, x^2-1, \dots, x^n-1$.

Lemma 3. *Let* $\Pi_n(p) := p^{-Mn^2}\cdot D_{m_1n}(p)D_{m_2n}(p)\cdot\prod_{l=1}^{m_1n}\Phi_l(p)^{-\omega(n/l)}$, *where* $p = q^{-1} \in \mathbb{Z}\setminus\{0,\pm1\}$. *Then the coefficients of the linear form* $H_n(q)$ *satisfy the inclusions* $\Pi_n(p)A_n(q)$, $\Pi_n(p)B_n(q) \in \mathbb{Z}$.

To study the asymptotics of $\Pi_n(p)$ as $n\to\infty$ we apply the corresponding result [6] on the asymptotics of $D_n(p)$ and the q-analogue of the arithmetic scheme of Chudnovskii–Rukhadze–Hata.

Lemma 4.

$$-\lim_{n\to\infty}\frac{\log|\Pi_n(p)|}{n^2\log|p|} = M - \frac{3}{\pi^2}\left(m_1^2 + m_2^2 + \int_0^1 \omega(z)\,\mathrm{d}\psi'(z)\right) =: C_0,$$

where $\psi(z)$ *is the logarithmic derivative of the gamma-function.*

If $C_0 > 0$, then $\zeta_q(2)$ is irrational for any $q = 1/p$, $p\in\mathbb{Z}\setminus\{0,\pm1\}$, and $\mu(\zeta_q(2)) \leqslant C_1/C_0$. Taking $\alpha_1 = 5$, $\alpha_2 = 6$, $\alpha_3 = 7$, $\beta_2 = 14$, $\beta_3 = 15$, we get $C_0 = 38.00236293\dots$ and $C_1 = 155$, which yields the bound in Theorem 1.

The q-arithmetic scheme and the q-hypergeometric construction of approximating linear forms also enable us to sharpen the measures of irrationality [6], [7] for the quantities

$$\zeta_q(1) = \sum_{n=1}^{\infty}\frac{q^n}{1-q^n}, \quad \ln_q(2) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^n}{1-q^n}, \qquad |q|<1, \tag{3}$$

which are the q-analogues of the (divergent) harmonic series and $\log 2$, respectively.

Theorem 2. *For* $q = 1/p$, $p\in\mathbb{Z}\setminus\{0,\pm1\}$, *the indices of irrationality of the numbers* (3) *satisfy the inequalities* $\mu(\zeta_q(1)) \leqslant 2.49846482\dots$, $\mu(\ln_q(2)) \leqslant 3.29727451\dots$.

Bibliography

[1] Yu. V. Nesterenko, *Mat. Sb.* **187**:9 (1996), 65–96; English transl., *Sbornik*: *Math* **85** (1996), 1319–1348.

[2] Yu. V. Nesterenko, *Trudy Mat. Inst. Steklov.* **218** (1997), 299–334; English transl., *Proc. Steklov Inst. Math.* **218** (1997), 294–331.

[3] G. Gasper and M. Rachman, *Basic hypergeometric series*, Cambridge Univ. Press, Cambridge 1990; Russian transl., Mir, Moscow 1993.

[4] G. Rhin and C. Viola, *Acta Arithmetica* **77**:1 (1996), 23–56.

[5] W. Zudilin, Arithmetic of linear forms involving odd zeta values, Manuscript, 2001, submitted.

[6] P. Bundschuh and K. Väänänen, *Compositio Math.* **91** (1994), 175–199.
[7] W. Van Assche, *Ramanujan J.* **5** (2001), 295–310.

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