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One of the Eight Numbers $\zeta(5), \zeta(7), \dots, \zeta(17), \zeta(19)$ Is Irrational

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1. INTRODUCTION

In 1978 R. Apéry [1] produced a sequence of rational approximations proving the irrationality of the number $\zeta(3)$. Despite the apparent simplicity, attempts to generalize the Apéry result and prove the irrationality of the values $\zeta(5), \zeta(7), \dots$ of the Riemann zeta-function were futile. In 2000 T. Rivoal proposed a construction [2] which allowed us to obtain "nice" linear forms with rational coefficients in the values of the zeta-function $\zeta(s)$ at odd positive points. The generalization of this construction [3–6] leads to linear forms in 1 and in the numbers

$$\zeta(s),\ \zeta(s+2),\ \dots,\ \zeta(s+2m), \qquad \text{where } s \ge 3 \text{ is odd}, \tag{1}$$

and for a properly chosen integer $m \ge 1$ one can prove the irrationality of at least one element of the set (1). As a consequence, the assertion that one of the numbers $\zeta(5)$, $\zeta(7)$, ..., $\zeta(19)$, $\zeta(21)$ is irrational was obtained. Two different proofs of this assertion were proposed in [3] and [4]. In the present paper, we prove the following result.

Theorem 1. *At least one of the eight numbers*

$$\zeta(5),\ \zeta(7),\ \zeta(9),\ \zeta(11),\ \zeta(13),\ \zeta(15),\ \zeta(17),\ \zeta(19) \tag{2}$$

is irrational.

The main difference of the construction given below from those used in [3, 4] is in the *arithmetical* part; the *analytical part*—the asymptotics of linear forms—is fully described in [4, Sec. 2]; therefore, we restrict ourselves, where necessary, to references to the relevant assertions in [4].

2. ANALYTIC CONSTRUCTION

Choose positive integer parameters a, b, c, d, where $b \ge 3$ is odd, $a \ge bd/c$ is even, and $d \ge 2c$. For each positive integer n, consider the *odd* rational function

$$\begin{aligned} R_n(t) &= t \cdot \frac{\bigl((t \pm (cn+1)) \cdots (t \pm (cn \pm dn))\bigr)^b}{\bigl(t(t\pm 1)\cdots(t \pm cn)\bigr)^a} \cdot \frac{(2cn)!^a}{(dn)!^{2b}} \\ &= t \cdot \frac{\Gamma(t+(c+d)n+1)^b \Gamma(t-cn)^{a+b}}{\Gamma(t-(c+d)n)^b \Gamma(t+cn+1)^{a+b}} \cdot \frac{(2cn)!^a}{(dn)!^{2b}} \\ &= (-1)^{(c+d)n} \frac{\sin^b \pi t}{\pi^b} \cdot \frac{t(\pm t+(c+d)n)^b (2cn)^a}{(t+cn)^{a+b}(dn)^{2b}} \cdot \frac{\Gamma(\pm t+(c+d)n)^b \Gamma(t-cn)^{a+b} \Gamma(2cn)^a}{\Gamma(t+cn)^{a+b} \Gamma(dn)^{2b}} \end{aligned} \tag{3}$$

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and assign to it the number

$$I_n = \frac{1}{(b-1)!} \sum_{t=cn+1}^{\infty} \frac{\mathrm{d}^{b-1} R(t)}{\mathrm{d}t^{b-1}} = -\frac{1}{2\pi i} \int_{M-i\infty}^{M+i\infty} \pi^b \cot_b \pi t \cdot R_n(t)\, \mathrm{d}t,$$

where M is an arbitrary constant from the interval $cn < M < (c+d)n$ and

$$\cot_b z = \frac{(-1)^{b-1}}{(b-1)!} \frac{\mathrm{d}^{b-1} \cot z}{\mathrm{d}z^{b-1}}, \qquad b = 1, 2, \dots$$

(see [4, Lemma 2.4]). As is readily verified, the number I_n is a linear form with rational coefficients in the numbers

$$1,\ \zeta(b+2),\ \zeta(b+4),\ \dots,\ \zeta(a+b-2) \tag{4}$$

(see [4, Lemma 1.1]).

Setting $r = d/(2c)$, replacing $(dn)!^{2b}$ by $(2cn)!^{bd/c}$ and cn by n in the definition of $R_n(t)$, we obtain the linear forms I_n (see [4]) in the numbers (4) (for even a and odd b). The only difference from [4] is the appearance of the additional factor t in (3); this makes it possible to consider the numbers a, b of different parity (this device was used in [3]), which does not affect the asymptotics of I_n as $n \to \infty$. Therefore, we can use Proposition 2.3 from [4].

Lemma 1. *Suppose that for a real root* $\mu_1 \in (c+d, +\infty)$ *of the polynomial*

$$(\tau + c + d)^b (\tau - c)^{a+b} - (\tau - c - d)^b (\tau + c)^{a+b} \tag{5}$$

the following inequality is valid:

$$\mu_1 \le c + d + (2c+d)d \cdot \min\left\{ \frac{b}{8c(a+b)}, \frac{1}{12(c+d)} \right\},$$

and τ_0 *is a complex root of* (5) *in the domain* $\operatorname{Re}\tau > 0$, $\operatorname{Im}\tau > 0$ *with is the maximum possible real part* $\operatorname{Re}\tau_0$. *Further, suppose that the function*

$$\begin{aligned} f_0(\tau) &= b(c+d)\log(\tau + c + d) + b(c+d)\log(-\tau + c + d) \\ &\quad - (a+b)c\log(\tau - c) - (a+b)c\log(\tau + c) + 2ac\log(2c) - 2bd\log d \end{aligned}$$

satisfies the condition

$$\operatorname{Im} f_0(\tau_0) \not\equiv 0 \pmod{\pi\mathbb{Z}}.$$

Then

$$\varlimsup_{n\to\infty} \frac{\log|I_n|}{n} = \operatorname{Re} f_0(\tau_0).$$

3. DENOMINATORS OF THE LINEAR FORMS

To study of the arithmetical properties of the linear forms I_n, let us express the function (3) as $R_n(t) = tH_n(t)^a G_n(t)^b$, where

$$H_n(t) = \frac{(2cn)!}{t(t \pm 1) \cdots (t \pm cn)}, \qquad G_n(t) = \frac{(t \pm (cn+1)) \cdots (t \pm (cn+dn))}{(dn)!^2}.$$

The following assertion is a generalization of the arithmetical Nikishin–Rivoal scheme.

Lemma 2 [4, Lemma 4.1]. *Suppose that for a polynomial $P(t)$, $\deg P(t) < m(n+1)$, the rational (not necessarily irreducible) function*

$$R(t) = \frac{P(t)}{\big((t+s)(t+s+1)\cdots(t+s+n)\big)^m}$$

satisfies, for each $j = 0, 1, \dots, m-1$, the inclusions

$$\frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(R(t)(t+k)^m\big)\Big|_{t=-k} \in \mathbb{Z}, \qquad k = s, s+1, \dots, s+n, \tag{6}$$

where D_n is the least common multiple of the numbers $1, 2, \dots, n$. Then the inclusions (6) *are satisfied for all nonnegative integers j.*

It follows from Lemma 2 that for all nonnegative integers j we have

$$\frac{D_{2cn}^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(H_n(t)(t+k)\big)\Big|_{t=-k} \in \mathbb{Z}, \qquad k = 0, \pm1, \dots, \pm cn. \tag{7}$$

In addition, from the properties of integer-valued polynomials (see, for example, [7])

$$\frac{(t+(cn+1))\cdots(t+(cn+dn))}{(dn)!}, \qquad \frac{(t-(cn+1))\cdots(t-(cn+dn))}{(dn)!}$$

and from the Leibniz rule for differentiating a product, we obtain the inclusions

$$\frac{D_{dn}^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}G_n(t)\Big|_{t=-k} \in \mathbb{Z}, \qquad k = 0, \pm1, \pm2, \dots\,. \tag{8}$$

Lemma 3 (cf. [4, Lemma 4.5]). *For each prime p, let us define the exponents*

$$\nu_p = \min_{k=0,\pm1,\dots,\pm cn}\left\{\operatorname{ord}_p \frac{((c+d)n+k)!\,((c+d)n-k)!}{(cn+k)!\,(cn-k)!\,(dn)!^2}\right\} \tag{9}$$

and set

$$\Pi_n = \prod_{p\,:\,\sqrt{(2c+d)n}<p\le dn} p^{\nu_p}. \tag{10}$$

Then for all nonnegative integers j we have

$$\Pi_n^{-1}\cdot D_{dn}^j\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}G_n(t)\Big|_{t=-k} \in \mathbb{Z}, \qquad k = 0, \pm1, \dots, \pm cn. \tag{11}$$

Proof. Since

$$G_n(t)\big|_{t=-k} = (-1)^{dn}\frac{((c+d)n+k)!\,((c+d)n-k)!}{(cn+k)!\,(cn-k)!\,(dn)!^2}, \qquad k = 0, \pm1, \dots, \pm cn,$$

the inclusions (11) for $j = 0$ follow from (9), (10).

For any prime p, let us prove by induction on j that

$$\operatorname{ord}_p\left(\Pi_n^{-1}\cdot D_{dn}^j\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}G_n(t)\Big|_{t=-k}\right) \ge 0, \qquad k = 0, \pm1, \dots, \pm cn. \tag{12}$$

We shall prove the estimates (12) for $j+1$, assuming them to be proved for all the previous values of j. If the prime p does not divide Π_n, then $\operatorname{ord}_p \Pi_n^{-1}=0$ and relations (12) follow from (8). Therefore, we next assume that p is a divisor of Π_n, whence, in particular, $p>\sqrt{(2c+d)n}$ and $p\le dn$. Setting

$$g_n(t)=\frac{G_n'(t)}{G_n(t)}=\sum_{l=cn+1}^{cn+dn}\frac{1}{t\pm l},$$

we obtain

$$\frac{\mathrm{d}^{j+1}G_n(t)}{\mathrm{d}t^{j+1}}=\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(g_n(t)G_n(t)\big)=\sum_{m=0}^{j}\binom{m}{j}\cdot\frac{\mathrm{d}^{j-m}g_n(t)}{\mathrm{d}t^{j-m}}\cdot\frac{\mathrm{d}^m G_n(t)}{\mathrm{d}t^m}. \tag{13}$$

For $m\le j$, we have

$$\operatorname{ord}_p\left(\frac{\mathrm{d}^{j-m}g_n(t)}{\mathrm{d}t^{j-m}}\bigg|_{t=-k}\right)=\operatorname{ord}_p\left(\sum_{l=cn+1}^{cn+dn}\frac{1}{(l\pm k)^{j-m+1}}\right)\ge-(j-m+1),\qquad k=0,\pm1,\dots,\pm cn, \tag{14}$$

since $p>\sqrt{(2c+d)n}$ and $|l\pm k|\le(2c+d)n$ for all the denominators in (14); we also have

$$\operatorname{ord}_p D_{dn}^{j-m+1}\ge j-m+1, \tag{15}$$

since $p\le dn$; and, finally,

$$\operatorname{ord}_p\left(\Pi_n^{-1}\cdot D_{dn}^m\cdot\frac{\mathrm{d}^m G_n(t)}{\mathrm{d}t^m}\bigg|_{t=-k}\right)\ge 0,\qquad k=0,\pm1,\dots,\pm cn, \tag{16}$$

by the induction assumption. Substituting $t=-k$ into (13) and using the estimates (14)–(16), we find that the estimates (12) are satisfied for $j+1$. Thus the induction step is justified. The lemma is proved. □

Set $a_0=\lceil bd/c\rceil\ge bd/c$. From Lemma 3, the inclusions (7), and the Leibniz rule, in view of $d\ge 2c$, for all nonnegative integers j we obtain

$$\Pi_n^{-b}\cdot D_{dn}^j\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(G_n(t)^b\cdot H_n(t)^{a_0}(t+k)^{a_0}\big)\bigg|_{t=-k}\in\mathbb{Z},\qquad k=0,\pm1,\dots,\pm cn.$$

Hence, in particular, for $j=0,1,\dots,a_0-1$ we have

$$a_0!\,\Pi_n^{-b}\cdot\left(\frac{D_{dn}}{D_{2cn}}\right)^{a_0-1}\cdot\frac{D_{2cn}^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(G_n(t)^b\cdot H_n(t)^{a_0}(t+k)^{a_0}\big)\bigg|_{t=-k}\in\mathbb{Z},\qquad k=0,\pm1,\dots,\pm cn. \tag{17}$$

By Lemma 2, the inclusions (17) are valid for all nonnegative integers j, since

$$\deg G_n(t)^b=2bdn<a_0(2cn+1)=\deg H_n(t)^{a_0}.$$

Therefore, by standard arguments (see, for example, [4, Lemma 1.4] or [3, Lemma 2]), we obtain the following assertion.

Lemma 4. *The numbers*

$$\tilde{I}_n = a_0!\,\Pi_n^{-b} \cdot D_{dn}^{a_0-1} \cdot D_{2cn}^{a+b-a_0} \cdot I_n \tag{18}$$

are linear forms in the numbers (4) *with integer coefficients.*

The asymptotics of D_{dn} and D_{2cn} in (18) as $n \to \infty$ is determined by the asymptotic law of distribution of the primes:

$$\lim_{n\to\infty} \frac{\log D_n}{n} = 1. \tag{19}$$

To calculate the asymptotic behavior of Π_n as $n \to \infty$, we should note that, by (9), for the primes $p > \sqrt{(2b+c)n}$ we have

$$\nu_p \ge \varphi\biggl(\frac{n}{p}\biggr) = \varphi\biggl(\frac{n}{p} - \biggl\lfloor \frac{n}{p} \biggr\rfloor\biggr), \tag{20}$$

where

$$\varphi(x) = \min_{y\in\mathbb{R}} \bigl\{ \lfloor (c+d)x + y \rfloor + \lfloor (c+d)x - y \rfloor - \lfloor cx + y \rfloor - \lfloor cx - y \rfloor - 2\lfloor dx \rfloor \bigr\}. \tag{21}$$

The function under the sign of the minimum in (21) is periodic (with period 1) in each argument; therefore, the minimum can be taken only for $y \in [0, 1)$. Using the arguments of Chudnovskii and Hata, by (10), (20), (19), we finally obtain

$$\varpi = \lim_{n\to\infty} \frac{\log \Pi_n}{n} \ge \int_0^1 \varphi(x)\,\mathrm{d}\psi(x) - \int_0^{1/d} \varphi(x)\,\frac{\mathrm{d}x}{x^2} \tag{22}$$

(see [4, Lemma 4.4 and the proof of Lemma 4.5]), where $\psi(x)$ is the logarithmic derivative of the gamma-function, while the subtraction on the right-hand side of (22) "eliminates" the primes $p > dn$.

4. PROOF OF THEOREM 1

To obtain the result announced in the title, we assume $a = 18$, $b = 3$, $c = 3$, and $d = 7$. Then $a_0 = \lceil bd/c \rceil = 7$ and

$$\mu_1 \approx 10.305445, \qquad \tau_0 \approx 9.856603 + 0.197639i, \qquad f_0(\tau_0) \approx -123.071169 - 30.779083i.$$

In this case, for $x \in [0, 1)$ the function (21) is of the form

$$\varphi(x) = \begin{cases} 1 & \text{if } x \in \bigl[\frac{1}{10}, \frac{1}{7}\bigr) \cup \bigl[\frac{1}{4}, \frac{2}{7}\bigr) \cup \bigl[\frac{2}{5}, \frac{3}{7}\bigr) \cup \bigl[\frac{11}{20}, \frac{4}{7}\bigr) \cup \bigl[\frac{13}{20}, \frac{2}{3}\bigr) \\ & \qquad \cup \bigl[\frac{7}{10}, \frac{5}{7}\bigr) \cup \bigl[\frac{4}{5}, \frac{5}{6}\bigr) \cup \bigl[\frac{17}{20}, \frac{6}{7}\bigr) \cup \bigl[\frac{19}{20}, 1\bigr), \\ 0 & \text{for other values of } x \in [0, 1); \end{cases}$$

therefore, by (22), we have $\varpi \ge 1.150969$. Finally, for the linear forms (18) with integer coefficients in 1 and in the numbers (2), by Lemmas 1 and 4, we have

$$\overline{\lim_{n\to\infty}} \frac{\log |\tilde{I}_n|}{n} = -b\varpi + (a_0 - 1)d + 2(a + b - a_0)c + \operatorname{Re} f_0(\tau_0) < -0.524077,$$

and hence there is an irrational number among the numbers (2).

5. OTHER RESULTS ON IRRATIONALITY

The construction described in this paper also allows us to prove the irrationality of one of the numbers in the set (1) taken for $s = 7$ and $s = 9$ with a lesser value of m than that in Theorem 1 from [4–6]. Indeed, setting

$$a = 32,\ b = 5,\ c = 5,\ d = 12 \qquad \text{and} \qquad a = 46,\ b = 7,\ c = 1,\ d = 2,$$

we obtain the following results.

Theorem 2. *At least one of the fifteen numbers*

$$\zeta(7),\ \zeta(9),\ \zeta(11),\ \dots,\ \zeta(33),\ \zeta(35)$$

is irrational.

Theorem 3. *At least one of the (twenty two) numbers*

$$\zeta(9),\ \zeta(11),\ \zeta(13),\ \dots,\ \zeta(49),\ \zeta(51)$$

is irrational.

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