On the irrationality of the values of the zeta function at odd integer points

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Only a few results are currently known on the arithmetic nature of the values of the Riemann zeta function $\zeta(s)$ for odd $s > 1$. The first of these is the irrationality of $\zeta(3)$, which was proved by Apéry [1] in 1978; the most recent is the result of Rivoal [2], who established in 2000 the following asymptotic estimate for the dimensions $\delta(a)$ of the spaces spanned over $\mathbb{Q}$ by the numbers $1, \zeta(3), \zeta(5), \dots, \zeta(a-2), \zeta(a)$ for odd a:

$$\delta(a) \ge \frac{\log a}{1+\log 2}\bigl(1+o(1)\bigr) \qquad \text{as } a \to \infty. \tag{1}$$

In particular, it follows from (1) that infinitely many of the values $\zeta(3), \zeta(5), \dots$ are irrational. We generalize Rivoal's construction from [2] and prove the following results.

Theorem 1. *Each of the sets*

$$\begin{gathered} \{\zeta(5),\ \zeta(7),\ \zeta(9),\ \zeta(11),\ \zeta(13),\ \zeta(15),\ \zeta(17),\ \zeta(19),\ \zeta(21)\}, \\ \{\zeta(7),\ \zeta(9),\ \zeta(11),\ \dots,\ \zeta(35),\ \zeta(37)\}, \\ \{\zeta(9),\ \zeta(11),\ \zeta(13),\ \dots,\ \zeta(51),\ \zeta(53)\} \end{gathered} \tag{2}$$

contains at least one irrational number.

Theorem 2. *There exists an odd integer $a \le 145$ such that $1, \zeta(3)$, and $\zeta(a)$ are linearly independent over $\mathbb{Q}$.*

Theorem 2 strengthens the corresponding result in [3]: $a \le 145$ instead of $a \le 169$.

Theorem 3. *for every odd $a \ge 3$, the following absolute estimate holds*:

$$\delta(a) > 0.395 \log a > \frac{2}{3} \cdot \frac{\log a}{1+\log 2}.$$

We fix positive odd parameters a, b, and c such that $a > b(c-1)$ and $c \ge 3$, and for each positive n we consider the rational function

$$\begin{aligned} R(t) = R_n(t) &:= \frac{\bigl((t\pm(n+1))\cdots(t\pm cn)\bigr)^b}{\bigl(t(t\pm1)\cdots(t\pm n)\bigr)^a} \cdot (2n)!^{a+b-bc} \\ &= (-1)^n \left(\frac{\Gamma(\pm t+cn+1)}{\Gamma(\pm t+n+1)}\right)^b \cdot \left(\frac{\Gamma(t)\Gamma(1-t)}{\Gamma(\pm t+n+1)}\right)^a \cdot (2n)!^{a+b-bc}, \end{aligned} \tag{3}$$

where the symbol $\pm$ means that the $+$ and $-$ signs both occur in the relevant product. On representing (3) as a sum of partial fractions, using the fact that it is odd, and recalling its behaviour as $t \to \infty$, we can conclude that

$$I = I_n := \sum_{t=n+1}^{\infty} \frac{1}{(b-1)!} \frac{\mathrm{d}^{b-1}R(t)}{\mathrm{d}t^{b-1}} = \sum_{\substack{s \text{ odd} \\ b<s<a+b}} A_s\zeta(s) - A_0, \tag{4}$$

where the coefficients $A_s = A_{s,n}$ of the linear form I are rational numbers. (When $b = 1$ and $c = 2r+1$, we get the same linear forms (4) as in [2].) We denote by D_n the least common multiple of $1, 2, \dots, n$; as is well known,

$$\lim_{n\to\infty} \frac{\log D_n}{n} = 1.$$

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Lemma 1. *For every odd integer $c \geq 3$ there exists a sequence of integers $\Pi_n = \Pi_n^{(c)} \geq 1$, $n = 1, 2, \dots$, such that the numbers $\Pi_n^{-b} D_{2n}^{a+b-1} A_{s,n}$ are integers and the following relation holds*:

$$\varpi_c := \lim_{n\to\infty} \frac{\log \Pi_n^{(c)}}{n} = -\sum_{l=1}^{(c-1)/2} \left(2\psi\left(\frac{2l}{c-1}\right) + 2\psi\left(\frac{2l}{c}\right) + \frac{2c-1}{l}\right) + 2(c-1)(1-\gamma), \tag{5}$$

in which $\gamma \approx 0.57712$ is Euler's constant and $\psi(x)$ is the logarithmic derivative of the gamma function. (*As $c \to \infty$, the value of the quantity ϖ_c in* (5) *is of order* $2c(1-\gamma) + O(\log c)$.)

Lemma 1 strengthens the corresponding estimates for the denominators of the linear forms (4), at the expense of the appearance of the multipliers Π_n^{-b}, even in the case $b = 1$ considered in [2]. This is pivotal in the deduction of Theorems 2 and 3.

The proof of the next assertion rests on the representation of the forms (4) as contour integrals and on application of the saddle-point method (cf. [4] and [5]). Additional restrictions are imposed on the parameters a, b, and c in the case where $b > 1$; they turn out to hold automatically in application to Theorem 1.

Lemma 2. *The following limit relation holds for the linear forms* (4):

$$\varkappa := \varlimsup_{n\to\infty} \frac{\log |I_n|}{n} = \log \frac{2^{2(a+b-bc)} |\tau_0 + c|^{bc} |\tau_0 - c|^{bc}}{|\tau_0 + 1|^{a+b} |\tau_0 - 1|^{a+b}}, \tag{6}$$

where τ_0 is the real root of the polynomial $(\tau + c)^b(\tau - 1)^{a+b} - (\tau - c)^b(\tau + 1)^{a+b}$ in the interval $(c, +\infty)$ when $b = 1$, and is one of the pair of complex conjugate roots with maximum possible value of $\operatorname{Re}\tau_0$ when $b > 1$. (*For $b = 1$, the upper limit in* (6) *can be replaced by the ordinary limit, and then the value of $\varkappa$ in* (6) *does not exceed* $(2a - c + 3)\log 2 - 2(a - c + 1)\log c$.)

By Lemmas 1 and 2, there is a least one irrational number among the values $\zeta(s)$ for odd s such that $b < s < a + b$, provided that $-b\varpi_c + 2(a + b - 1) + \varkappa < 0$. Theorem 1 now follows by taking the three sets of values $a = 19$, $b = 3$, $c = 3$; $a = 33$, $b = 5$, $c = 3$; $a = 47$, $b = 7$, $c = 3$ respectively for the alternatives in (2).

Lemma 3. *The following estimate holds for the non-zero coefficients $A_s = A_{s,n}$ of the linear forms* (4):

$$\varlimsup_{n\to\infty} \frac{\log |A_{s,n}|}{n} \leq 2bc \log c + 2(a + b - bc) \log 2.$$

Theorems 2 and 3 are deduced from Lemmas 1–3 using the linear independence criterion to be found in [6] in the same way as it was used in [2]. For the proof of Theorem 2 (where $a = 145$ and $b = 1$) we choose $c = 21$.

Bibliography

[1] R. Apéry, *Astérisque.* **61** (1979), 11–13.
[2] T. Rivoal, *C. R. Acad. Sci. Paris. Sér. I Math.* **331** (2000), 267–270.
[3] T. Rivoal, Rapport de recherche SDAD, no. 2000-9, Univ. Caen, 2000.
[4] Yu. V. Nesterenko, *Mat. Zametki* **59** (1996), 865–880; English transl., *Math. Notes* **59** (1996), 625–636.
[5] T. G. Khessami Pilerud, *Arithmetic properties of the values of hypergeometric functions*, Candidate's Dissertation, Moscow State University, Moscow 1999. (Russian)
[6] Yu. V. Nesterenko, *Vestnik Moskov. Univ. Ser. I Math. Mekh.* **1985**:1, 46–54; English transl., *Moscow Univ. Math. Bull.* **40**:1 (1985), 69–74.

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