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The group structure for $\zeta(3)$

by

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1. Introduction. In his proof of the irrationality of $\zeta(3)$, Apéry [1] gave sequences of rational approximations to $\zeta(2) = \pi^2/6$ and to $\zeta(3)$ yielding the irrationality measures $\mu(\zeta(2)) < 11.85078\ldots$ and $\mu(\zeta(3)) < 13.41782\ldots$ Several improvements on such irrationality measures were subsequently given, and we refer to the introductions of the papers [3] and [4] for an account of these results. As usual, we denote here by $\mu(\alpha)$ the *least irrationality measure* of an irrational number α, i.e., the least exponent λ such that for any $\varepsilon > 0$ there exists a constant $q_0 = q_0(\varepsilon) > 0$ for which

$$\left|\alpha - \frac{p}{q}\right| > q^{-\lambda-\varepsilon}$$

for all integers p and q with $q > q_0$.

In the arithmetical study we made in [4] of a family $\mathcal{F}$ of double integrals lying in $\mathbb{Q} + \mathbb{Z}\zeta(2)$, we introduced a new algebraic method which enabled us to prove the best irrationality measure of $\zeta(2)$ obtained so far, namely

$$\mu(\zeta(2)) < 5.441243. \tag{1.1}$$

Roughly, our algebraic method in [4] was based on the study of the structure of a suitable permutation group acting on ten parameters related to the exponents of the five factors appearing in an integral of $\mathcal{F}$. Such a permutation group arose on the one hand from the action on the double integrals of the birational transformation

$$\tau : \begin{cases} \xi = \dfrac{1-x}{1-xy} \\ \eta = 1 - xy, \end{cases} \tag{1.2}$$

and on the other hand from an integral transformation which we called "hypergeometric", based on Euler's integral representation of Gauss's hypergeometric function. With the latter tool one transforms any integral of

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$\mathcal{F}$ into an integral of $\mathcal{F}$ multiplied by a quotient of factorials, and the p-adic valuation of such factorials yields arithmetic information on the rational part of the integral considered.

In the present paper we extend the algebraic method of [4] to a family of triple integrals. In spite of the analogies with [4], this extension is far from obvious, and requires new ideas. For instance, the double integrals of rational functions studied in [4] contain 5 factors, the birational transformation (1.2) has period 5, and the permutation group introduced in [4], arising from (1.2) and from the hypergeometric transformation, is isomorphic to the symmetric group $\mathfrak{S}_5$ of permutations of 5 elements. In this paper the situation is much subtler, since no number has the role played by 5 in [4]. In fact, we consider triple integrals of rational functions containing 7 factors, but, in order to ensure that the integrals have the required arithmetic properties, only 6 exponents of such factors are independent (see also Remark 2.2 below). Moreover, here the relevant birational transformation, analogous to (1.2), is

$$\vartheta : \begin{cases} X = (1-y)z \\ Y = \dfrac{(1-x)(1-z)}{1-(1-xy)z} \\ Z = \dfrac{y}{1-(1-y)z}, \end{cases} \tag{1.3}$$

and has period 8, and the permutation group $\boldsymbol{\Phi}$, arising from (1.3) and from the hypergeometric transformation, can be naturally embedded in the alternating group $\mathfrak{A}_{10}$ of the even permutations of 10 elements.

The algebraic structure underlying the triple integrals considered turns out to be so rich that, rather surprisingly, it enables us to prove in Section 5 the irrationality measure of $\zeta(3)$:

$$\mu(\zeta(3)) < 5.513891, \tag{1.4}$$

numerically close to the irrationality measure (1.1) of $\zeta(2)$. Our result (1.4) considerably improves upon the best previously known inequality $\mu(\zeta(3)) < 7.377956\ldots$, recently obtained by Hata [3].

As in our paper [4], a characteristic feature in our treatment of triple integrals consists in getting rid of the Legendre or Legendre-type polynomials and of the related partial integration method traditionally used in this context (see, e.g., [2] and [3]). This increases the flexibility of our method, and allows us to remove any unnecessary constraints on the exponents of the factors appearing in the integrals considered.

Finally, we point out that the irrationality measure (1.4) of $\zeta(3)$ proved in the present paper, as well as the irrationality measure (1.1) of $\zeta(2)$ proved in [4], is effective.

We are indebted to R. Dvornicich for helpful suggestions about the discussion, made in Section 4, concerning the embedding in $\mathfrak{A}_{10}$ of the hypergeometric permutation group $\boldsymbol{\Phi}$.

2. A family of integrals lying in $\mathbb{Q} + 2\mathbb{Z}\zeta(3)$. Let h, j, k, l, q, r, s be integers. We consider the integral

$$\int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}} \, \frac{dx\,dy\,dz}{1-(1-xy)z}. \tag{2.1}$$

In the special case where $h = j = k = l = q = r = s > 0$, the integral (2.1) was introduced by Beukers in [2].

The conditions for the integral (2.1) to be finite are most easily found by applying the change of variables

$$\begin{cases} x = 1 - \xi\eta \\ y = \dfrac{1-\eta}{1-\xi\eta} \end{cases}$$

considered in [4]. This transforms (2.1) into an integral containing the factors $1-\xi\eta$ and $1-\eta z$ in place of $1-(1-xy)z$, and then an elementary discussion shows that this integral is finite if and only if h, j, k, l, q, r, $s \geq 0$ and $h \leq k + r$, which we assume. We define $m = k + r - h$, so that $m \geq 0$ and

$$h + m = k + r. \tag{2.2}$$

We make the following further assumption:

$$j + q = l + s, \tag{2.3}$$

which ensures that (2.1) is changed into an integral of the same type by the transformation ϑ below. Using this property we will prove that, under the assumption (2.3), the integral (2.1) lies in $\mathbb{Q} + 2\mathbb{Z}\zeta(3)$. More precisely, in this section we show that for any non-negative integers h, j, k, l, m, q, r, s satisfying (2.2) and (2.3), the integral (2.1) equals $a + 2b\zeta(3)$, with $a \in \mathbb{Q}$ and $b \in \mathbb{Z}$, and we find three non-negative integers M, N and Q, as small as possible, such that

$$d_M d_N d_Q a \in \mathbb{Z}.$$

Here and in what follows we let $d_0 = 1$ and $d_n = \text{l.c.m.}\{1, \ldots, n\}$ for $n \geq 1$.

We employ the birational transformation

$$\vartheta : (x, y, z) \mapsto (X, Y, Z)$$

defined by the equations

$$\vartheta : \begin{cases} X = (1-y)z \\ Y = \dfrac{(1-x)(1-z)}{1-(1-xy)z} \\ Z = \dfrac{y}{1-(1-y)z}. \end{cases}$$

It is easy to check that:

$$\vartheta \text{ has period } 8, \tag{2.4}$$
$$\vartheta \text{ maps the open unit cube } (0,1)^3 \text{ onto itself,}$$

and that under the action of ϑ we have

$$\frac{X(1-X)Y(1-Y)Z(1-Z)}{1-(1-XY)Z} = \frac{x(1-x)y(1-y)z(1-z)}{1-(1-xy)z}$$

and

$$\frac{dX\,dY\,dZ}{1-(1-XY)Z} = -\frac{dx\,dy\,dz}{1-(1-xy)z}. \tag{2.5}$$

Denote the above integral (2.1) by

$$I(h,j,k,l,m,q,r,s).$$

If we apply the transformation ϑ to it, i.e., if we make in (2.1) the change of variables

$$\vartheta^{-1} : \begin{cases} x = \dfrac{(1-Y)(1-Z)}{1-(1-XY)Z} \\ y = (1-X)Z \\ z = \dfrac{X}{1-(1-X)Z}, \end{cases} \tag{2.6}$$

and then replace X, Y, Z with x, y, z respectively, by virtue of (2.2), (2.3), (2.4) and (2.5) we obtain the integral $I(j,k,l,m,q,r,s,h)$. Hence with the action of ϑ on $I(h,j,k,l,m,q,r,s)$ we associate the cyclic permutation

$$\boldsymbol{\vartheta} = (h \;\; j \;\; k \;\; l \;\; m \;\; q \;\; r \;\; s).$$

Similarly, if we apply to $I(h,j,k,l,m,q,r,s)$ the transformation

$$\sigma : \begin{cases} X = y \\ Y = x \\ Z = z, \end{cases}$$

i.e., if we interchange the variables x, y in (2.1), we get $I(k,j,h,s,r,q,m,l)$ by (2.2). Hence with the action of σ on $I(h,j,k,l,m,q,r,s)$ we associate the permutation of h, j, k, l, m, q, r, s defined by

$$\boldsymbol{\sigma} = (h \;\; k)(l \;\; s)(m \;\; r).$$

Clearly $\boldsymbol{\sigma}$ transforms a regular octagon of vertices h, j, k, l, m, q, r, s, written in this order, by the symmetry about the diagonal jq. Thus the permutation group $\boldsymbol{\Theta} = \langle \boldsymbol{\vartheta}, \boldsymbol{\sigma} \rangle$ generated by $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ is isomorphic to the dihedral group $\mathfrak{D}_8$ of order 16, and the value of $I(h,j,k,l,m,q,r,s)$ is invariant under the action of the group $\boldsymbol{\Theta}$.

With the integers

$$h,\ j,\ k,\ l,\ m,\ q,\ r,\ s \tag{2.7}$$

we associate the following eight auxiliary integers, each of which, by (2.2) or (2.3), can be written in two different ways:

$$\begin{aligned} h+l-j&=h+q-s,\\ j+m-k&=j+r-h,\\ k+q-l&=k+s-j,\\ l+r-m&=l+h-k,\\ m+s-q&=m+j-l,\\ q+h-r&=q+k-m,\\ r+j-s&=r+l-q,\\ s+k-h&=s+m-r. \end{aligned} \tag{2.8}$$

We extend the actions of the permutations $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ on any linear combination of the integers (2.7) by linearity. Thus $\boldsymbol{\vartheta}(h+l-j)=\boldsymbol{\vartheta}(h)+\boldsymbol{\vartheta}(l)-\boldsymbol{\vartheta}(j)=j+m-k$, $\boldsymbol{\sigma}(h+l-j)=k+s-j$, etc. Moreover, using (2.2) and (2.3), we get $\boldsymbol{\vartheta}(h+m)=\boldsymbol{\vartheta}(h)+\boldsymbol{\vartheta}(m)=j+q=l+s=\boldsymbol{\vartheta}(k)+\boldsymbol{\vartheta}(r)=\boldsymbol{\vartheta}(k+r)$, $\boldsymbol{\vartheta}(j+q)=k+r=h+m=\boldsymbol{\vartheta}(l+s)$, and similarly $\boldsymbol{\sigma}(h+m)=\boldsymbol{\sigma}(k+r)$, $\boldsymbol{\sigma}(j+q)=\boldsymbol{\sigma}(l+s)$. Hence $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ permute the integers (2.8).

We use the notation max, max′, max″, … to denote the successive maxima in a finite set or sequence of real numbers. More precisely, let $\mathcal{A}=(a_1,\dots,a_n)$ be any finite sequence of real numbers with $n\ge 3$. If $i_1,\dots,i_n$ is a reordering of $1,\dots,n$ such that

$$a_{i_1}\ge a_{i_2}\ge a_{i_3}\ge\dots\ge a_{i_n},$$

we let

$$\max\mathcal{A}=a_{i_1},\qquad \max{}'\mathcal{A}=a_{i_2},\qquad \max{}''\mathcal{A}=a_{i_3}.$$

THEOREM 2.1. *Let h, j, k, l, m, q, r, s be non-negative integers satisfying $h+m=k+r$ and $j+q=l+s$. Let $\mathcal{S}$ denote the sequence of the integers* (2.8):

$$\begin{aligned}\mathcal{S}=(h+l-j,\ j+m-k,\ k+q-l,\ l+r-m,&\\ m+s-q,\ q+h-r,\ r+j-s,\ s+k-h),&\end{aligned}$$

and let

$$M=\max\mathcal{S},\qquad N=\max{}'\mathcal{S},\qquad Q=\max{}''\mathcal{S}.$$

Then the integral

$$I = I(h,j,k,l,m,q,r,s) = \int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}} \frac{dx\,dy\,dz}{1-(1-xy)z}$$

satisfies

$$I = a + 2b\zeta(3), \tag{2.9}$$

with $d_M d_N d_Q a \in \mathbb{Z}$ *and* $b \in \mathbb{Z}$.

REMARK 2.1. By (2.2) we have $(h+l-j)+(j+m-k) = l+r \geq 0$, whence $\max\{h+l-j,\ j+m-k\} \geq 0$. Similarly, using (2.2) or (2.3), we see that at least one of any two consecutive integers in the list (2.8) is non-negative, whence at least four among (2.8) are non-negative. In particular we get $M \geq N \geq Q \geq 0$.

Proof of Theorem 2.1. If $q+h-r < 0$, then I is the integral of a polynomial in x, y, z with integer coefficients and partial degrees $r+l-q-1$, $m+s-q-1$, $j+r-h-1$. Therefore

$$d_{r+l-q}\, d_{m+s-q}\, d_{j+r-h}\, I \in \mathbb{Z}.$$

Since $r+l-q$, $m+s-q$ and $j+r-h$ occur in distinct places in the list (2.8), we get $d_M d_N d_Q I \in \mathbb{Z}$, so that (2.9) holds with $b = 0$. If $q+h-r \geq 0$ but the least of the integers (2.8) is < 0, then I is changed by a suitable power of the permutation $\boldsymbol{\vartheta}$ into an integral of the preceding type, and again we obtain $d_M d_N d_Q I \in \mathbb{Z}$, since M, N and Q are invariant under the actions of $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$.

If the integers (2.8) are all ≥ 0 and if $lmqr > 0$, we use the linear decomposition of I given by the identity $(1-x)(1-z) = 1-x-(1-x)z$. We have

$$\begin{aligned} I &= I(h,j,k,l,m,q,r,s) \\ &= \int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^{l-1} y^k(1-y)^s z^j(1-z)^{q-1}}{(1-(1-xy)z)^{q+h-r}} \\ &\qquad\qquad \times (1-x-(1-x)z) \frac{dx\,dy\,dz}{1-(1-xy)z} \\ &= I(h,j,k,l-1,m-1,q-1,r-1,s) \\ &\quad - I(h+1,j,k,l-1,m-1,q-1,r,s) \\ &\quad - I(h,j+1,k,l,m-1,q-1,r-1,s), \end{aligned} \tag{2.10}$$

and each of the three integrals thus obtained satisfies (2.2) and (2.3). Furthermore, for each of these three integrals at least two of the integers (2.8) are less than the corresponding integers for I and none is greater, so that the integers M, N and Q associated with any one of the three integrals do not exceed, respectively, the integers M, N and Q for I. Thus if Theorem 2.1 holds for the three integrals, it also holds for I. Moreover, the sum of the integers (2.8) for each of the three integrals is strictly less than the sum of (2.8) for I. Therefore, if we iterate the linear decomposition (2.10) sufficiently many times, in finitely many steps we express I as a linear combination with integer coefficients of integrals, for each of which either the least of the integers (2.8) is < 0, or $lmqr = 0$.

If, for such an integral, $lmqr = 0$ but the greatest of the integers $mqrs$, $qrsh$, $rshj$, $shjk$, $hjkl$, $jklm$, $klmq$ is > 0, we apply to that integral first a suitable power of the permutation $\boldsymbol{\vartheta}$, and then the linear decomposition (2.10). Iterating this process, we decompose I into a linear combination with integer coefficients of finitely many integrals, for each of which either the least of (2.8) is < 0, or $lmqr = mqrs = qrsh = rshj = shjk = hjkl = jklm = klmq = 0$. In the latter case, we may assume (up to applying a suitable power of $\boldsymbol{\vartheta}$) that $j = 0$. We distinguish two cases.

First case: $j = q = 0$. By (2.3) we have $l = s = 0$. If $h \neq k$, the least of the integers (2.8) is < 0, since either $s + k - h = k - h$ or $l + h - k = h - k$ is < 0. If $h = k$, by (2.2) we have $m = r$. If $h = k \neq m = r$, either $j + m - k = m - k$ or $q + k - m = k - m$ is < 0, and again the least of (2.8) is < 0. If $h = k = m = r$, the integral is

$$\int_0^1\!\!\int_0^1\!\!\int_0^1 x^h y^h \frac{dx\,dy\,dz}{1-(1-xy)z} = \int_0^1\!\!\int_0^1 \frac{-\log(xy)}{1-xy}\, x^h y^h\, dx\,dy = -2\sum_{\nu=1}^{h} \nu^{-3} + 2\,\zeta(3), \tag{2.11}$$

by Lemma 1 of [2]. Hence, for this integral, (2.9) holds with $d_h^3 a \in \mathbb{Z}$ and $b = 1$. Since, for the integral (2.11), four of the integers (2.8) are equal to h and the other four vanish, we get $d_M d_N d_Q a = d_h^3 a \in \mathbb{Z}$.

Second case: $j = 0$, $q > 0$. Then $lmr = mrs = rsh = klm = 0$. If we had $mr > 0$ we should obtain $l = s = 0$, whence, by (2.3), $q = l + s - j = 0$, contradicting the assumption $q > 0$. Therefore $mr = 0$. The permutation $\boldsymbol{\sigma}$ interchanges the cases $m = 0$ and $r = 0$, so we may assume, e.g., $r = 0$. If $s > 0$ we get $r + j - s = -s < 0$, and the least of (2.8) is < 0. Thus we may assume $s = 0$. Then, by (2.3), we have $l = q > 0$. If $k = 0$, since $r = 0$, the permutation $\boldsymbol{\vartheta}$ changes the integral into one having $j = q = 0$, and this has been treated in the first case above. Hence we may assume $k > 0$. Since $klm = 0$ and $kl > 0$, we get $m = 0$, whence $j + m - k = -k < 0$, and the least of (2.8) is < 0. ∎

REMARK 2.2. If h, j, k, l, q, r, s are non-negative integers satisfying $h \le k + r$ and, in place of (2.3), $j + q < l + s$, we still have

$$I=\int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}}\,\frac{dx\,dy\,dz}{1-(1-xy)z}=a+2b\zeta(3),$$

with $a\in\mathbb{Q}$ and $b\in\mathbb{Z}$. For, at least one of l and s (say, l) is >0, whence

$$\begin{aligned}I&=\int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^{l-1} y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}}\,(1-x)\,\frac{dx\,dy\,dz}{1-(1-xy)z}\\&=\int_0^1\int_0^1\int_0^1 \frac{x^h(1-x)^{l-1} y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}}\,\frac{dx\,dy\,dz}{1-(1-xy)z}\\&\quad-\int_0^1\int_0^1\int_0^1 \frac{x^{h+1}(1-x)^{l-1} y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+(h+1)-(r+1)}}\,\frac{dx\,dy\,dz}{1-(1-xy)z},\end{aligned}$$

and in each of the last two integrals j, q, s and $k+r-h$ are unchanged, while l is replaced by $l-1$. Iterating this process $l+s-j-q$ times, we express I as a linear combination with integer coefficients of integrals satisfying condition (2.3), and hence of the type $a+2b\zeta(3)$ with $a\in\mathbb{Q}$ and $b\in\mathbb{Z}$ by Theorem 2.1.

On the other hand, an elementary calculation shows that

$$\int_0^1\int_0^1\int_0^1 z\,\frac{dx\,dy\,dz}{1-(1-xy)z}=\zeta(2).$$

Thus if $j+q>l+s$, in general we expect the integral I to be a linear combination of 1, $\zeta(2)$ and $\zeta(3)$ with rational coefficients.

3. A triple contour integral. With the next lemma we show that the value of the triple contour integral which can be naturally associated with $I(h,j,k,l,m,q,r,s)$ is also invariant under the action of the permutation group $\boldsymbol{\Theta}=\langle\boldsymbol{\vartheta},\boldsymbol{\sigma}\rangle$.

LEMMA 3.1. *Let h, j, k, l, m, q, r, s be non-negative integers satisfying $h+m=k+r$ and $j+q=l+s$. Let, for any $\varrho_1,\varrho_2,\varrho_3>0$,*

$$\widetilde{I}(h,j,k,l,m,q,r,s)=\frac{1}{(2\pi i)^3}\int_C\int_{C_x}\int_{C_{x,y}} \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}}\,\frac{dx\,dy\,dz}{1-(1-xy)z}, \tag{3.1}$$

where $C=\{x\in\mathbb{C}:|x|=\varrho_1\}$, $C_x=\{y\in\mathbb{C}:|y-1/x|=\varrho_2\}$ and $C_{x,y}=\{z\in\mathbb{C}:|z-(1-xy)^{-1}|=\varrho_3\}$. Then

$$\widetilde{I}(h,j,k,l,m,q,r,s)=\widetilde{I}(j,k,l,m,q,r,s,h)=\widetilde{I}(k,j,h,s,r,q,m,l).$$

Proof. If $\varrho_2>1/\varrho_1$, the contour C_x in (3.1) can be replaced by $|y|=\varrho_2$, since this encloses the point $1/x$. Hence for any $\varrho_1,\varrho_2,\varrho_3>0$ such that

$\varrho_1\varrho_2 > 1$ the integration in (3.1) can be made over $|x| = \varrho_1$, $|y| = \varrho_2$, $|z-(1-xy)^{-1}| = \varrho_3$. On interchanging the variables x, y we get

$$\widetilde{I}(h,j,k,l,m,q,r,s) = \widetilde{I}(k,j,h,s,r,q,m,l).$$

If in (3.1) we take $\varrho_3 > 1/(\varrho_1\varrho_2)$, the contour $C_{x,y}$ can be replaced by $|z| = \varrho_3$, which encloses the point $(1-xy)^{-1}$ since $|1-xy| = \varrho_1\varrho_2$. Thus, if $\varrho_1\varrho_2\varrho_3 > 1$, the integration in (3.1) can be made over $|x| = \varrho_1$, $|y - 1/x| = \varrho_2$, $|z| = \varrho_3$, or over

$$|y| = \varrho_1, \qquad |z| = \varrho_2, \qquad |x - 1/y| = \varrho_3, \tag{3.2}$$

and if $\varrho_1\varrho_2 > 1$, $\varrho_1\varrho_2\varrho_3 > 1$, it can be made over

$$|x| = \varrho_1, \qquad |y| = \varrho_2, \qquad |z| = \varrho_3. \tag{3.3}$$

From (3.2) we get

$$\left|\frac{1}{y} - \frac{z-1}{yz}\right| = \frac{1}{|yz|} = \frac{1}{\varrho_1\varrho_2} < \varrho_3,$$

whence the contour $|x - 1/y| = \varrho_3$ encloses $(z-1)/(yz)$. It follows that, for any $\varrho_1, \varrho_2, \varrho_3 > 0$,

$$\begin{aligned}\widetilde{I}&(h,j,k,l,m,q,r,s) \\ &= \frac{1}{(2\pi i)^3} \int_{C'}\int_{C''}\int_{C'''_{y,z}} \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}} \frac{dx\,dy\,dz}{1-(1-xy)z},\end{aligned} \tag{3.4}$$

where $C' = \{y \in \mathbb{C} : |y| = \varrho_1\}$, $C'' = \{z \in \mathbb{C} : |z| = \varrho_2\}$ and $C'''_{y,z} = \{x \in \mathbb{C} : |x - (z-1)/(yz)| = \varrho_3\}$.

Clearly the contours C' and C'' in (3.4) can be replaced by $|y-1| = \varrho_1$ and $|z-(1-y)^{-1}| = \varrho_2$ respectively, provided that $\varrho_1 > 1$ and $\varrho_1\varrho_2 > 1$. Also, for any fixed complex number $\alpha \neq 0$, 1, ∞, the function $x = (1-\xi)/(1-\alpha\xi)$ of the complex variable ξ transforms any circumference enclosing $1/\alpha$ into a circumference enclosing $1/\alpha$. Choosing $\varrho_2 > 1 + 1/\varrho_1$ we have $z \neq 1$, and we can take $\alpha = (yz)/(z-1)$. Thus the contour $C'''_{y,z}$ in (3.4) can be replaced by the circumference

$$|\xi| = \left|\frac{1-x}{1 - \dfrac{yz}{z-1}x}\right| = \left|\frac{(1-x)(1-z)}{1-z+xyz}\right| = \varrho_3,$$

provided that $\varrho_3 > |(z-1)/(yz)|$. Since

$$\left|\frac{z-1}{yz}\right| = \frac{1}{|y|}\left|1 - \frac{1}{z}\right| \le \frac{1}{\varrho_1 - 1}\left(1 + \frac{\varrho_1}{\varrho_1\varrho_2 - 1}\right),$$

we see that in (3.4) the integration can be made over

$$(3.5)\qquad |y-1|=\varrho_1,\qquad \left|z-\frac{1}{1-y}\right|=\varrho_2,\qquad \left|\frac{(1-x)(1-z)}{1-z+xyz}\right|=\varrho_3,$$

for any ϱ_1, ϱ_2, ϱ_3 satisfying

$$\varrho_1>1,\qquad \varrho_2>1+\frac{1}{\varrho_1},\qquad \varrho_3>\frac{1}{\varrho_1-1}\left(1+\frac{\varrho_1}{\varrho_1\varrho_2-1}\right).$$

If in the integral over (3.5) we make the change of variables (2.6), we get, by (2.5),

$$(3.6)\qquad \widetilde{I}(h,j,k,l,m,q,r,s) = \frac{1}{(2\pi i)^3}\iiint\limits_V \frac{X^j(1-X)^mY^l(1-Y)^hZ^k(1-Z)^r}{(1-(1-XY)Z)^{r+j-s}}\,\frac{dX\,dY\,dZ}{1-(1-XY)Z},$$

where V (with a suitable orientation) is defined by

$$\begin{cases} X=(1-y)z\\ Y=\dfrac{(1-x)(1-z)}{1-(1-xy)z}\\ Z=\dfrac{y}{1-(1-y)z}, \end{cases}$$

with x, y, z satisfying (3.5). But (3.5) are equivalent to

$$|X-1|=\varrho_1\varrho_2,\qquad \left|Z-\frac{1}{1-X}\right|=\frac{1}{\varrho_2},\qquad |Y|=\varrho_3.$$

Therefore, in (3.6), V can be taken to be

$$|X|=\varrho_1',\qquad |Y|=\varrho_2',\qquad |Z|=\varrho_3',$$

for sufficiently large ϱ_1', ϱ_2', ϱ_3'. Comparing this with (3.3), we conclude that

$$\widetilde{I}(h,j,k,l,m,q,r,s)=\widetilde{I}(j,k,l,m,q,r,s,h).\ \blacksquare$$

We can now prove that the integer b in (2.9) is equal to the integral (3.1).

Theorem 3.1. *Under the assumptions of Theorem* 2.1, *the integer b in* (2.9) *is given by*

$$b=\frac{1}{(2\pi i)^3}\int\limits_C\int\limits_{C_x}\int\limits_{C_{x,y}} \frac{x^h(1-x)^ly^k(1-y)^sz^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}}\,\frac{dx\,dy\,dz}{1-(1-xy)z},$$

where C, C_x and $C_{x,y}$ are as in Lemma 3.1.

Proof. We apply to the integral $\widetilde{I}=\widetilde{I}(h,j,k,l,m,q,r,s)$ defined by (3.1), step by step, the same linear decomposition (2.10) and the same permutations lying in $\boldsymbol{\Theta}=\langle\boldsymbol{\vartheta},\boldsymbol{\sigma}\rangle$ used for I in the proof of Theorem 2.1, and we can do this by Lemma 3.1. $\widetilde{I}$ vanishes if $q+h-r<0$, and therefore, by

Lemma 3.1, it also vanishes if the least of the integers (2.8) is < 0. Thus we have, for a suitable integer $T \geq 0$,

$$I = \sum_{t=1}^{T} \beta_t I^{(t)} + (\text{rational number}) \tag{3.7}$$

and

$$\widetilde{I} = \sum_{t=1}^{T} \beta_t \widetilde{I}^{(t)}, \tag{3.8}$$

with the same $\beta_t \in \mathbb{Z}$ in (3.7) and (3.8), where

$$I^{(t)} = \int_0^1\int_0^1\int_0^1 x^{h_t} y^{h_t} \frac{dx\,dy\,dz}{1-(1-xy)z} = -2\sum_{\nu=1}^{h_t} \nu^{-3} + 2\,\zeta(3) \tag{3.9}$$

(see (2.11)), and

$$\begin{aligned}
\widetilde{I}^{(t)} &= \frac{1}{(2\pi i)^3} \int_C \int_{C_x} \int_{C_{x,y}} x^{h_t} y^{h_t} \frac{dx\,dy\,dz}{1-(1-xy)z} \\
&= -\frac{1}{(2\pi i)^2} \int_C \int_{C_x} \frac{x^{h_t} y^{h_t}}{1-xy} \biggl(\frac{1}{2\pi i} \int_{C_{x,y}} \frac{dz}{z-(1-xy)^{-1}} \biggr) dx\,dy \\
&= \frac{1}{2\pi i} \int_C x^{h_t-1} \biggl(\frac{1}{2\pi i} \int_{C_x} \frac{y^{h_t}}{y-1/x}\,dy \biggr) dx \\
&= \frac{1}{2\pi i} \int_C \frac{dx}{x} = 1.
\end{aligned}$$

Therefore, by (3.8),

$$\widetilde{I} = \sum_{t=1}^{T} \beta_t. \tag{3.10}$$

From (2.9), (3.7) and (3.9) we obtain

$$a + 2b\zeta(3) = I = \Bigl(\sum_{t=1}^{T} \beta_t\Bigr) 2\,\zeta(3) + (\text{rational number}),$$

whence, by the irrationality of $\zeta(3)$ and by (3.10),

$$b = \sum_{t=1}^{T} \beta_t = \widetilde{I}. \ \blacksquare$$

4. The hypergeometric permutation group. Here we use the hypergeometric integral transformation, based upon the Euler integral representation of Gauss's hypergeometric function and the invariance of this function under the interchange of the two parameters appearing in the numerator of the hypergeometric series. This can be done in the integral (2.1) with respect to any of the variables x, y, z.

Besides the assumptions (2.2) and (2.3), we henceforth assume the non-negative integers (2.7) to be such that (2.8) are also non-negative. Following the method of [4], pp. 36–37, we get

$$\begin{aligned}
&\int_0^1 \frac{x^h(1-x)^l}{(1-(1-xy)z)^{q+h-r+1}}\,dx \\
&\quad = (1-z)^{r-q-h-1}\int_0^1 \frac{x^h(1-x)^l}{\left(1+\dfrac{yz}{1-z}\,x\right)^{q+h-r+1}}\,dx \\
&\quad = (1-z)^{r-q-h-1}\,\frac{h!\,l!}{(q+h-r)!(r+l-q)!}\int_0^1 \frac{x^{q+h-r}(1-x)^{r+l-q}}{\left(1+\dfrac{yz}{1-z}\,x\right)^{h+1}}\,dx \\
&\quad = (1-z)^{r-q}\,\frac{h!\,l!}{(q+h-r)!(r+l-q)!}\int_0^1 \frac{x^{q+h-r}(1-x)^{r+l-q}}{(1-(1-xy)z)^{h+1}}\,dx.
\end{aligned}$$

Multiplying by $y^k(1-y)^s z^j(1-z)^q$ and integrating in $0 \le y \le 1$, $0 \le z \le 1$, we obtain

$$\begin{aligned}
(4.1)\qquad & I(h,j,k,l,m,q,r,s) \\
&\quad = \frac{h!\,l!}{(q+h-r)!(r+l-q)!}\,I(q+h-r,j,k,r+l-q,m,r,q,s).
\end{aligned}$$

Therefore

$$\frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!} = \frac{I(q+h-r,j,k,r+l-q,m,r,q,s)}{(q+h-r)!j!k!(r+l-q)!m!r!q!s!}.$$

Let φ be the hypergeometric transformation changing

$$\frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!}$$

into

$$\frac{I(q+h-r,j,k,r+l-q,m,r,q,s)}{(q+h-r)!j!k!(r+l-q)!m!r!q!s!}.$$

We associate with φ the permutation $\boldsymbol{\varphi}$ mapping the integers (2.7) respectively to $q+h-r$, j, k, $r+l-q$, m, r, q, s, and extended to any linear combination of (2.7) by linearity. Note that, by (2.2) and (2.3), $\boldsymbol{\varphi}(h+m) = \boldsymbol{\varphi}(h)+\boldsymbol{\varphi}(m) = q+h-r+m = k+q = \boldsymbol{\varphi}(k)+\boldsymbol{\varphi}(r) = \boldsymbol{\varphi}(k+r)$ and $\boldsymbol{\varphi}(j+q) = j+r = r+l-q+s = \boldsymbol{\varphi}(l+s)$. Thus $\boldsymbol{\varphi}$ is the permutation of the set of the 16 integers (2.7) and (2.8) given by the following product of transpositions:

$$\boldsymbol{\varphi} = (h \;\; q+h-r)(l \;\; r+l-q)(q \;\; r)(m+s-q \;\; s+m-r).$$

Similarly, we can apply the hypergeometric transformation with respect to z. We have

$$\int_0^1 \frac{z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r+1}}\,dz = \frac{j!\,q!}{(q+h-r)!(j+r-h)!}\int_0^1 \frac{z^{q+h-r}(1-z)^{j+r-h}}{(1-(1-xy)z)^{j+1}}\,dz.$$

Multiplying by $x^h(1-x)^l y^k(1-y)^s$ and integrating, we get

$$I(h,j,k,l,m,q,r,s) = \frac{j!\,q!}{(q+h-r)!(j+r-h)!}\, I(h,q+h-r,k,l,m,j+r-h,r,s),$$

whence

$$\frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!} = \frac{I(h,q+h-r,k,l,m,j+r-h,r,s)}{h!(q+h-r)!k!l!m!(j+r-h)!r!s!}.$$

Let χ denote this hypergeometric transformation. We associate with χ the permutation $\boldsymbol{\chi}$ mapping the integers (2.7) respectively to h, $q+h-r$, k, l, m, $j+r-h$, r, s, and extended to any linear combination of (2.7) by linearity. Again we have $\boldsymbol{\chi}(h+m) = \boldsymbol{\chi}(k+r)$ and $\boldsymbol{\chi}(j+q) = \boldsymbol{\chi}(l+s)$, and $\boldsymbol{\chi}$ is the permutation of the set of the 16 integers (2.7) and (2.8) given by

$$\boldsymbol{\chi} = (j \;\; q+h-r)(q \;\; j+r-h)(h+l-j \;\; r+l-q)(k+q-l \;\; m+j-l).$$

It is easy to see that $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ can be viewed as permutations of ten integers only, i.e., of the sums

$$(4.2) \qquad h+l,\; j+m,\; k+q,\; l+r,\; m+s,\; q+h,\; r+j,\; s+k,\; j+q,\; k+r.$$

To prove this, it suffices to show that $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ permute the sums (4.2), and that if a permutation $\boldsymbol{\varrho}$ lying in the group

$$\boldsymbol{\Phi} = \langle \boldsymbol{\varphi}, \boldsymbol{\chi}, \boldsymbol{\vartheta}, \boldsymbol{\sigma} \rangle$$

generated by $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ acts identically on the sums (4.2), it acts identically also on h, j, k, l, m, q, r, s. Note that, by (2.2) and (2.3), the last two sums in (4.2) can be written as $l+s$ and $h+m$ respectively. Then, as is easy to check, the actions of $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ on the ten sums (4.2) are the following:

$$\begin{aligned}
\boldsymbol{\varphi} &= (k+q \;\; k+r)(r+j \;\; j+q),\\
\boldsymbol{\chi} &= (j+m \;\; k+q)(q+h \;\; r+j),\\
\boldsymbol{\vartheta} &= (h+l \;\; j+m \;\; k+q \;\; l+r \;\; m+s \;\; q+h \;\; r+j \;\; s+k)(j+q \;\; k+r),\\
\boldsymbol{\sigma} &= (h+l \;\; s+k)(j+m \;\; r+j)(k+q \;\; q+h)(l+r \;\; m+s).
\end{aligned}$$

Moreover, if $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ acts identically on the sums (4.2) we get, by linearity,

$$\begin{aligned}
2\boldsymbol{\varrho}(h) = \boldsymbol{\varrho}(2h) &= \boldsymbol{\varrho}((h+l)-(l+r)+(r+k)-(k+q)+(q+h))\\
&= \boldsymbol{\varrho}(h+l)-\boldsymbol{\varrho}(l+r)+\boldsymbol{\varrho}(r+k)-\boldsymbol{\varrho}(k+q)+\boldsymbol{\varrho}(q+h)\\
&= (h+l)-(l+r)+(r+k)-(k+q)+(q+h)\\
&= 2h,
\end{aligned}$$

and similarly

$$2\boldsymbol{\varrho}(j) = \boldsymbol{\varrho}(2j) = \boldsymbol{\varrho}(j+m)-\boldsymbol{\varrho}(m+s)+\boldsymbol{\varrho}(s+l)-\boldsymbol{\varrho}(l+r)+\boldsymbol{\varrho}(r+j) = 2j,$$

and so on. Hence $\boldsymbol{\varrho}$ acts identically on h, j, k, l, m, q, r, s.

For brevity, denote the sums (4.2) by $u_1, \ldots, u_{10}$ respectively. Plainly

$$\begin{aligned}
\boldsymbol{\varphi} &= (u_3 \;\; u_{10})(u_7 \;\; u_9),\\
\boldsymbol{\chi} &= (u_2 \;\; u_3)(u_6 \;\; u_7),\\
\boldsymbol{\vartheta} &= (u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; u_5 \;\; u_6 \;\; u_7 \;\; u_8)(u_9 \;\; u_{10}),\\
\boldsymbol{\sigma} &= (u_1 \;\; u_8)(u_2 \;\; u_7)(u_3 \;\; u_6)(u_4 \;\; u_5)
\end{aligned}$$

are even permutations of the set

$$U = \{u_1, \ldots, u_{10}\}.$$

Thus we have a natural embedding of the group $\boldsymbol{\Phi} = \langle \boldsymbol{\varphi}, \boldsymbol{\chi}, \boldsymbol{\vartheta}, \boldsymbol{\sigma} \rangle$ in the alternating group $\mathfrak{A}_{10}$ of the even permutations of U. Such an embedding yields information on the structure of $\boldsymbol{\Phi}$, and in particular allows one to determine the order $|\boldsymbol{\Phi}|$.

The group $\langle \boldsymbol{\varphi}, \boldsymbol{\vartheta} \rangle$ is clearly transitive over U, and therefore so is $\boldsymbol{\Phi}$. Moreover, let

$$\mathcal{P} = \{\{u_1, u_5\}, \{u_2, u_6\}, \{u_3, u_7\}, \{u_4, u_8\}, \{u_9, u_{10}\}\}$$

be the set of the five unordered pairs indicated. $\mathcal{P}$ is a partition of U, and shows that the group $\boldsymbol{\Phi}$ is imprimitive over U, the elements of $\mathcal{P}$ being blocks of imprimitivity, since each of $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ carries every element of $\mathcal{P}$ onto an element of $\mathcal{P}$. Precisely, let $\boldsymbol{\varphi}^*$, $\boldsymbol{\chi}^*$, $\boldsymbol{\vartheta}^*$ and $\boldsymbol{\sigma}^*$ be the permutations of $\mathcal{P}$

defined by

$$\begin{aligned}
\boldsymbol{\varphi}^* &= (\{u_3, u_7\}\ \{u_9, u_{10}\}),\\
\boldsymbol{\chi}^* &= (\{u_2, u_6\}\ \{u_3, u_7\}),\\
\boldsymbol{\vartheta}^* &= (\{u_1, u_5\}\ \{u_2, u_6\}\ \{u_3, u_7\}\ \{u_4, u_8\}),\\
\boldsymbol{\sigma}^* &= (\{u_1, u_5\}\ \{u_4, u_8\})(\{u_2, u_6\}\ \{u_3, u_7\}).
\end{aligned}$$

Clearly $\boldsymbol{\varphi}^*$, $\boldsymbol{\chi}^*$, $\boldsymbol{\vartheta}^*$ and $\boldsymbol{\sigma}^*$ are the permutations of $\mathcal{P}$ induced by $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ respectively, and therefore the mapping $\boldsymbol{\varphi} \mapsto \boldsymbol{\varphi}^*$, $\boldsymbol{\chi} \mapsto \boldsymbol{\chi}^*$, $\boldsymbol{\vartheta} \mapsto \boldsymbol{\vartheta}^*$ and $\boldsymbol{\sigma} \mapsto \boldsymbol{\sigma}^*$ extends to a homomorphism $\boldsymbol{\Phi} \xrightarrow{*} \mathfrak{S}_5$ of the group $\boldsymbol{\Phi}$ into the symmetric group $\mathfrak{S}_5$ of all the permutations of $\mathcal{P}$. Note that the product $\boldsymbol{\vartheta}^*\boldsymbol{\varphi}^*$ (i.e., the permutation of $\mathcal{P}$ obtained by applying first $\boldsymbol{\varphi}^*$ and then $\boldsymbol{\vartheta}^*$) is the 5-cycle

$$(\{u_1, u_5\}\ \{u_2, u_6\}\ \{u_3, u_7\}\ \{u_9, u_{10}\}\ \{u_4, u_8\}).$$

Since the symmetric group of the 5! permutations of five elements is generated by a 5-cycle and a transposition, the permutations $\boldsymbol{\vartheta}^*\boldsymbol{\varphi}^*$ and $\boldsymbol{\varphi}^*$ generate $\mathfrak{S}_5$. Therefore $\mathfrak{S}_5 = \langle \boldsymbol{\varphi}^*, \boldsymbol{\vartheta}^* \rangle$. It follows that the above homomorphism $\boldsymbol{\Phi} \xrightarrow{*} \mathfrak{S}_5$ is surjective.

Let K denote the kernel of this homomorphism, so that we have an exact sequence of multiplicative groups:

$$1 \to K \hookrightarrow \boldsymbol{\Phi} \xrightarrow{*} \mathfrak{S}_5 \to 1.$$

Clearly a permutation $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ lies in K if and only if it maps each element of any pair lying in $\mathcal{P}$ to an element of the same pair, and hence if and only if for any pair lying in $\mathcal{P}$ either $\boldsymbol{\varrho}$ acts identically on the elements of such a pair, or $\boldsymbol{\varrho}$ interchanges them. Also, since $K \subset \boldsymbol{\Phi} \subset \mathfrak{A}_{10}$, every element of K is an even permutation of the set U. Therefore we get $K = H \cap \boldsymbol{\Phi}$, where H denotes the subgroup of $\mathfrak{A}_{10}$ consisting of the $\binom{5}{0} + \binom{5}{2} + \binom{5}{4} = 16$ permutations of U that interchange the elements in an even number of pairs lying in $\mathcal{P}$ and act identically on the remaining elements of U.

It is easy to see that H is isomorphic to the additive group $(\mathbb{Z}/2\mathbb{Z})^4$, and contains four independent generators expressible as products of permutations each of which equals either $\boldsymbol{\varphi}$ or $\boldsymbol{\vartheta}$. For instance, H is plainly generated by the four permutations

$$\begin{aligned}
(u_1\ \ u_5)(u_2\ \ u_6)(u_3\ \ u_7)(u_4\ \ u_8) &= \boldsymbol{\vartheta}^4,\\
(u_1\ \ u_5)(u_2\ \ u_6)(u_4\ \ u_8)(u_9\ \ u_{10}) &= \boldsymbol{\varphi}\boldsymbol{\vartheta}^4\boldsymbol{\varphi},\\
(u_1\ \ u_5)(u_2\ \ u_6)(u_3\ \ u_7)(u_9\ \ u_{10}) &= \boldsymbol{\vartheta}\boldsymbol{\varphi}\boldsymbol{\vartheta}^4\boldsymbol{\varphi}\boldsymbol{\vartheta}^7,\\
(u_1\ \ u_5)(u_3\ \ u_7)(u_4\ \ u_8)(u_9\ \ u_{10}) &= \boldsymbol{\vartheta}^3\boldsymbol{\varphi}\boldsymbol{\vartheta}^4\boldsymbol{\varphi}\boldsymbol{\vartheta}^5.
\end{aligned}$$

It follows that $H \subset \langle \boldsymbol{\varphi}, \boldsymbol{\vartheta} \rangle \subset \boldsymbol{\Phi}$, whence $K = H \cap \boldsymbol{\Phi} = H$. Therefore

$$|\boldsymbol{\Phi}| = |K| \cdot |\mathfrak{S}_5| = 16 \cdot 120 = 1920.$$

Moreover, if we denote by K' the kernel of the surjective homomorphism

$$\langle \boldsymbol{\varphi}, \boldsymbol{\vartheta} \rangle \xrightarrow{*} \langle \boldsymbol{\varphi}^*, \boldsymbol{\vartheta}^* \rangle = \mathfrak{S}_5,$$

the above argument shows that $K' = H \cap \langle \boldsymbol{\varphi}, \boldsymbol{\vartheta} \rangle = H = K$. Thus $|\langle \boldsymbol{\varphi}, \boldsymbol{\vartheta} \rangle| = |K| \cdot |\mathfrak{S}_5| = 1920$, whence

$$\boldsymbol{\Phi} = \langle \boldsymbol{\varphi},\, \boldsymbol{\chi},\, \boldsymbol{\vartheta},\, \boldsymbol{\sigma} \rangle = \langle \boldsymbol{\varphi},\, \boldsymbol{\vartheta} \rangle.$$

Although 1920 divides $8!$, one can prove that no subgroup of the symmetric group $\mathfrak{S}_8$ has order 1920, and we are indebted to R. Dvornicich for pointing this out to us. Thus $\boldsymbol{\Phi}$ cannot be embedded in $\mathfrak{S}_8$, and the above discussion based on the embedding of $\boldsymbol{\Phi}$ in $\mathfrak{A}_{10}$ appears to be the natural one.

We now return to considering the actions of the above permutations on the 16 integers (2.7) and (2.8). As in [4], pp. 38–39, it is clear that if we apply to

$$\frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!} \tag{4.3}$$

any product ϱ of integral transformations φ, χ, ϑ and σ, we get the transformation formula

$$\begin{aligned} &\frac{I(h,j,k,l,m,q,r,s)}{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!} \\ &\qquad = \frac{I(\boldsymbol{\varrho}(h),\boldsymbol{\varrho}(j),\boldsymbol{\varrho}(k),\boldsymbol{\varrho}(l),\boldsymbol{\varrho}(m),\boldsymbol{\varrho}(q),\boldsymbol{\varrho}(r),\boldsymbol{\varrho}(s))}{\boldsymbol{\varrho}(h)!\,\boldsymbol{\varrho}(j)!\,\boldsymbol{\varrho}(k)!\,\boldsymbol{\varrho}(l)!\,\boldsymbol{\varrho}(m)!\,\boldsymbol{\varrho}(q)!\,\boldsymbol{\varrho}(r)!\,\boldsymbol{\varrho}(s)!}, \end{aligned} \tag{4.4}$$

where $\boldsymbol{\varrho}$ is the corresponding product of permutations $\boldsymbol{\varphi}$, $\boldsymbol{\chi}$, $\boldsymbol{\vartheta}$ and $\boldsymbol{\sigma}$ in reverse order. Thus the value of (4.3) is invariant under the action of the permutation group $\boldsymbol{\Phi}$. It is natural to associate with any $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ the quotient

$$\frac{h!\,j!\,k!\,l!\,m!\,q!\,r!\,s!}{\boldsymbol{\varrho}(h)!\,\boldsymbol{\varrho}(j)!\,\boldsymbol{\varrho}(k)!\,\boldsymbol{\varrho}(l)!\,\boldsymbol{\varrho}(m)!\,\boldsymbol{\varrho}(q)!\,\boldsymbol{\varrho}(r)!\,\boldsymbol{\varrho}(s)!} \tag{4.5}$$

resulting from the transformation formula (4.4) for $I(h,j,k,l,m,q,r,s)$. Note that $\boldsymbol{\varrho}(h)+\boldsymbol{\varrho}(m) = \boldsymbol{\varrho}(k)+\boldsymbol{\varrho}(r)$ and $\boldsymbol{\varrho}(j)+\boldsymbol{\varrho}(q) = \boldsymbol{\varrho}(l)+\boldsymbol{\varrho}(s)$, since this property holds for the generators of the group $\boldsymbol{\Phi}$. Therefore, Theorem 2.1 is applicable to the integral

$$I(\boldsymbol{\varrho}(h),\boldsymbol{\varrho}(j),\boldsymbol{\varrho}(k),\boldsymbol{\varrho}(l),\boldsymbol{\varrho}(m),\boldsymbol{\varrho}(q),\boldsymbol{\varrho}(r),\boldsymbol{\varrho}(s)).$$

Moreover

$$\begin{aligned} &h+j+k+l+m+q+r+s \\ &\qquad = \boldsymbol{\varrho}(h)+\boldsymbol{\varrho}(j)+\boldsymbol{\varrho}(k)+\boldsymbol{\varrho}(l)+\boldsymbol{\varrho}(m)+\boldsymbol{\varrho}(q)+\boldsymbol{\varrho}(r)+\boldsymbol{\varrho}(s), \end{aligned} \tag{4.6}$$

again because (4.6) holds for the generators of $\boldsymbol{\Phi}$, and therefore holds for any $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ by linearity.

If $\boldsymbol{\varrho}$ and $\boldsymbol{\varrho}'$ are left-equivalent modulo the subgroup $\boldsymbol{\Theta} = \langle \boldsymbol{\vartheta}, \boldsymbol{\sigma} \rangle$ of $\boldsymbol{\Phi}$, i.e., if $\boldsymbol{\varrho}$ and $\boldsymbol{\varrho}'$ lie in the same left coset of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$, the integers $\boldsymbol{\varrho}'(h)$, $\boldsymbol{\varrho}'(j)$, $\boldsymbol{\varrho}'(k)$, $\boldsymbol{\varrho}'(l)$, $\boldsymbol{\varrho}'(m)$, $\boldsymbol{\varrho}'(q)$, $\boldsymbol{\varrho}'(r)$, $\boldsymbol{\varrho}'(s)$ coincide with $\boldsymbol{\varrho}(h)$, $\boldsymbol{\varrho}(j)$, $\boldsymbol{\varrho}(k)$, $\boldsymbol{\varrho}(l)$, $\boldsymbol{\varrho}(m)$, $\boldsymbol{\varrho}(q)$, $\boldsymbol{\varrho}(r)$, $\boldsymbol{\varrho}(s)$, up to a permutation. Hence the quotient (4.5) for $\boldsymbol{\varrho}$ equals the analogous quotient for $\boldsymbol{\varrho}'$. Thus with each left coset of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$ we associate the quotient (4.5), where $\boldsymbol{\varrho}$ is any representative of the coset considered.

For any $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ we simplify the corresponding quotient (4.5) by removing the factorials appearing both in the numerator and in the denominator. Then, by (4.6), the quotient of factorials thus obtained has the following properties:

(i) The numerator and the denominator are products of the same number of factorials.

(ii) The integers appearing in the numerator are among (2.7), and the integers in the denominator are among (2.8).

(iii) The sum of the integers in the numerator equals the sum of the integers in the denominator.

Since $|\boldsymbol{\Phi}| = 1920$ and $|\boldsymbol{\Theta}| = 16$, there are 120 left cosets of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$. It is not difficult to list explicitly 120 permutations in $\boldsymbol{\Phi}$ yielding distinct quotients of factorials, and hence representatives of all the 120 left cosets of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$. To shorten our notation, we henceforth denote the integers (2.8) by

$$\begin{aligned} h' &= h + l - j = h + q - s, \\ j' &= j + m - k = j + r - h, \\ k' &= k + q - l = k + s - j, \\ l' &= l + r - m = l + h - k, \\ m' &= m + s - q = m + j - l, \\ q' &= q + h - r = q + k - m, \\ r' &= r + j - s = r + l - q, \\ s' &= s + k - h = s + m - r. \end{aligned} \tag{4.7}$$

We say that a left coset of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$ is *of level* v, or that a permutation $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ is of level v, if the corresponding quotient of factorials has v factorials in the numerator and v in the denominator. In other words, $\boldsymbol{\varrho}$ is of level v if the intersection of the sets

$$\{\boldsymbol{\varrho}(h),\ \boldsymbol{\varrho}(j),\ \boldsymbol{\varrho}(k),\ \boldsymbol{\varrho}(l),\ \boldsymbol{\varrho}(m),\ \boldsymbol{\varrho}(q),\ \boldsymbol{\varrho}(r),\ \boldsymbol{\varrho}(s)\}$$

and

$$\{h',\ j',\ k',\ l',\ m',\ q',\ r',\ s'\}$$

contains v elements. Then, as we shall show below, the 120 left cosets of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$ can be classified as follows:

$$\begin{aligned}
1 &\text{ coset } \;\text{ of level } 0,\\
12 &\text{ cosets of level } 2,\\
32 &\text{ cosets of level } 3,\\
30 &\text{ cosets of level } 4,\\
32 &\text{ cosets of level } 5,\\
12 &\text{ cosets of level } 6,\\
1 &\text{ coset } \;\text{ of level } 8.
\end{aligned}$$

The coset of level 0 is the group $\boldsymbol{\Theta}$, and the coset of level 8 is represented, e.g., by the permutation $\boldsymbol{\eta}$ defined by

$$\boldsymbol{\eta} = \boldsymbol{\varphi}\boldsymbol{\vartheta}^2\boldsymbol{\varphi}\boldsymbol{\vartheta}^4\boldsymbol{\varphi}\boldsymbol{\vartheta}^2\boldsymbol{\varphi} = (h\;\;k')(j\;\;s')(k\;\;m')(l\;\;j')(m\;\;r')(q\;\;l')(r\;\;h')(s\;\;q').$$

Moreover, it is plain that for any $\boldsymbol{\varrho} \in \boldsymbol{\Phi}$ the quotients of factorials associated with $\boldsymbol{\varrho}$ and with $\boldsymbol{\varrho\eta}$ are complementary (in the obvious sense). For instance, by (4.1), $h!\,l!\,(q'!\,r'!)^{-1}$ is associated with $\boldsymbol{\varphi}$, and $j!\,k!\,m!\,q!\,r!\,s!$ $\times(h'!\,j'!\,k'!\,l'!\,m'!\,s'!)^{-1}$ is associated with $\boldsymbol{\varphi\eta}$. Therefore, the right multiplication by $\boldsymbol{\eta}$ changes a list of permutations representatives of w cosets of level v into a list of representatives of w cosets of level $8-v$, and this argument shows that the number of cosets of level v equals the number of cosets of level $8-v$.

We give here a complete list of representatives of cosets of levels 2, 3 and 4, each with the corresponding quotient of factorials, the quotients of factorials being all distinct.

Level 2:

$$\begin{array}{lll}
\boldsymbol{\vartheta}^\lambda\boldsymbol{\varphi} & \boldsymbol{\vartheta}^\lambda(h)!\,\boldsymbol{\vartheta}^\lambda(l)!\,(\boldsymbol{\vartheta}^\lambda(q')!\,\boldsymbol{\vartheta}^\lambda(r')!)^{-1} & (\lambda = 0,\dots,7),\\
\boldsymbol{\vartheta}^\mu\boldsymbol{\chi} & \boldsymbol{\vartheta}^\mu(j)!\,\boldsymbol{\vartheta}^\mu(q)!\,(\boldsymbol{\vartheta}^\mu(j')!\,\boldsymbol{\vartheta}^\mu(q')!)^{-1} & (\mu = 0,1,2,3).
\end{array}$$

Level 3:

$$\left.\begin{array}{ll}
\boldsymbol{\vartheta}^\lambda\boldsymbol{\varphi}\boldsymbol{\vartheta}\boldsymbol{\varphi} & \boldsymbol{\vartheta}^\lambda(h)!\,\boldsymbol{\vartheta}^\lambda(j)!\,\boldsymbol{\vartheta}^\lambda(m)!\,(\boldsymbol{\vartheta}^\lambda(m')!\,\boldsymbol{\vartheta}^\lambda(q')!\,\boldsymbol{\vartheta}^\lambda(r')!)^{-1}\\
\boldsymbol{\vartheta}^\lambda\boldsymbol{\varphi}\boldsymbol{\vartheta}^5\boldsymbol{\varphi} & \boldsymbol{\vartheta}^\lambda(h)!\,\boldsymbol{\vartheta}^\lambda(l)!\,\boldsymbol{\vartheta}^\lambda(r)!\,(\boldsymbol{\vartheta}^\lambda(k')!\,\boldsymbol{\vartheta}^\lambda(l')!\,\boldsymbol{\vartheta}^\lambda(r')!)^{-1}\\
\boldsymbol{\vartheta}^\lambda\boldsymbol{\varphi}\boldsymbol{\vartheta}\boldsymbol{\chi} & \boldsymbol{\vartheta}^\lambda(h)!\,\boldsymbol{\vartheta}^\lambda(k)!\,\boldsymbol{\vartheta}^\lambda(q)!\,(\boldsymbol{\vartheta}^\lambda(k')!\,\boldsymbol{\vartheta}^\lambda(q')!\,\boldsymbol{\vartheta}^\lambda(r')!)^{-1}\\
\boldsymbol{\vartheta}^\lambda\boldsymbol{\chi}\boldsymbol{\vartheta}\boldsymbol{\varphi} & \boldsymbol{\vartheta}^\lambda(j)!\,\boldsymbol{\vartheta}^\lambda(m)!\,\boldsymbol{\vartheta}^\lambda(q)!\,(\boldsymbol{\vartheta}^\lambda(h')!\,\boldsymbol{\vartheta}^\lambda(j')!\,\boldsymbol{\vartheta}^\lambda(s')!)^{-1}
\end{array}\right\}$$

$$(\lambda = 0,\dots,7).$$

Level 4:

$$\left.\begin{array}{ll}
\boldsymbol{\vartheta}^\lambda\boldsymbol{\varphi}\boldsymbol{\vartheta}^2\boldsymbol{\varphi} & \boldsymbol{\vartheta}^\lambda(h)!\boldsymbol{\vartheta}^\lambda(k)!\boldsymbol{\vartheta}^\lambda(l)!\boldsymbol{\vartheta}^\lambda(r)!(\boldsymbol{\vartheta}^\lambda(h')!\boldsymbol{\vartheta}^\lambda(m')!\boldsymbol{\vartheta}^\lambda(q')!\boldsymbol{\vartheta}^\lambda(r')!)^{-1}\\
\boldsymbol{\vartheta}^\lambda\boldsymbol{\varphi}\boldsymbol{\vartheta}^6\boldsymbol{\varphi} & \boldsymbol{\vartheta}^\lambda(h)!\boldsymbol{\vartheta}^\lambda(j)!\boldsymbol{\vartheta}^\lambda(l)!\boldsymbol{\vartheta}^\lambda(q)!(\boldsymbol{\vartheta}^\lambda(l')!\boldsymbol{\vartheta}^\lambda(q')!\boldsymbol{\vartheta}^\lambda(r')!\boldsymbol{\vartheta}^\lambda(s')!)^{-1}
\end{array}\right\}$$

$$(\lambda = 0,\dots,7),$$

$$\left.\begin{array}{ll}\boldsymbol{\vartheta}^{\mu}\boldsymbol{\varphi}\boldsymbol{\vartheta}^{4}\boldsymbol{\varphi} & \boldsymbol{\vartheta}^{\mu}(h)!\boldsymbol{\vartheta}^{\mu}(l)!\boldsymbol{\vartheta}^{\mu}(m)!\boldsymbol{\vartheta}^{\mu}(s)!(\boldsymbol{\vartheta}^{\mu}(j')!\boldsymbol{\vartheta}^{\mu}(k')!\boldsymbol{\vartheta}^{\mu}(q')!\boldsymbol{\vartheta}^{\mu}(r')!)^{-1} \\ \boldsymbol{\vartheta}^{\mu}\boldsymbol{\chi}\boldsymbol{\vartheta}\boldsymbol{\chi} & \boldsymbol{\vartheta}^{\mu}(j)!\boldsymbol{\vartheta}^{\mu}(k)!\boldsymbol{\vartheta}^{\mu}(q)!\boldsymbol{\vartheta}^{\mu}(r)!(\boldsymbol{\vartheta}^{\mu}(h')!\boldsymbol{\vartheta}^{\mu}(j')!\boldsymbol{\vartheta}^{\mu}(m')!\boldsymbol{\vartheta}^{\mu}(q')!)^{-1} \\ \boldsymbol{\vartheta}^{\mu}\boldsymbol{\chi}\boldsymbol{\vartheta}^{3}\boldsymbol{\chi} & \boldsymbol{\vartheta}^{\mu}(h)!\boldsymbol{\vartheta}^{\mu}(j)!\boldsymbol{\vartheta}^{\mu}(m)!\boldsymbol{\vartheta}^{\mu}(q)!(\boldsymbol{\vartheta}^{\mu}(j')!\boldsymbol{\vartheta}^{\mu}(k')!\boldsymbol{\vartheta}^{\mu}(q')!\boldsymbol{\vartheta}^{\mu}(r')!)^{-1}\end{array}\right\} \quad (\mu = 0,1,2,3),$$

$$\boldsymbol{\vartheta}^{\nu}\boldsymbol{\chi}\boldsymbol{\vartheta}^{2}\boldsymbol{\chi} \qquad \boldsymbol{\vartheta}^{\nu}(j)!\boldsymbol{\vartheta}^{\nu}(l)!\boldsymbol{\vartheta}^{\nu}(q)!\boldsymbol{\vartheta}^{\nu}(s)!(\boldsymbol{\vartheta}^{\nu}(j')!\boldsymbol{\vartheta}^{\nu}(l')!\boldsymbol{\vartheta}^{\nu}(q')!\boldsymbol{\vartheta}^{\nu}(s')!)^{-1} \quad (\nu = 0,1).$$

5. The irrationality measure of $\zeta(3)$**.** In this section we combine the analytic properties of the integrals $I = a + 2b\zeta(3)$ and $\widetilde{I} = b$ considered in Sections 2 and 3 with the arithmetic information on the denominator of a arising from the group-theoretical arguments given in Section 4. Here the discussion is similar to the one developed in Sections 4 and 5 of [4], and therefore most of the details are omitted.

As in Section 4, let

$$h,\ j,\ k,\ l,\ m,\ q,\ r,\ s \tag{5.1}$$

be non-negative integers satisfying $h + m = k + r$ and $j + q = l + s$, and such that the integers (4.7) are also non-negative. With the notation of Theorem 2.1, we consider the integral

$$I_n = I(hn, jn, kn, ln, mn, qn, rn, sn) = a_n + 2b_n\zeta(3) \qquad (n = 1, 2, \ldots). \tag{5.2}$$

Let $\mathcal{T}$ denote the sequence of the integers (5.1) and (4.7):

$$\mathcal{T} = (h, j, k, l, m, q, r, s, h', j', k', l', m', q', r', s').$$

In this section we define

$$M = \max \mathcal{T}, \qquad N = \max{}' \mathcal{T}, \qquad Q = \max{}'' \mathcal{T}. \tag{5.3}$$

By Theorem 2.1 we have $d_{Mn}d_{Nn}d_{Qn}a_n \in \mathbb{Z}$ and $b_n \in \mathbb{Z}$. The 120 transformation formulae of the type (4.4) for I_n, where $\boldsymbol{\varrho}$ ranges over a full set of representatives of the left cosets of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$, give a wealth of information on the p-adic valuation of the integer $A_n = d_{Mn}d_{Nn}d_{Qn}a_n$, and allow one to eliminate divisors of A_n of the types p, p^2 or p^3 for suitable primes p. However, this arithmetic information turns out to be redundant, in the sense that for any numerical choice of h, j, k, l, m, q, r, s one can find a suitable subset of the set of the 120 transformation formulae which suffices to eliminate the divisors of A_n mentioned above. In other words, for the study of the p-adic valuation of A_n several among the 120 transformation formulae can be disregarded, but the set of these depends on the numerical values for h, j, k, l, m, q, r, s.

The best irrationality measure of $\zeta(3)$ we can prove, i.e. $\mu(\zeta(3)) < 5.513890\ldots$, can be obtained from a suitable numerical choice for (h, j, k, l, m, q, r, s), or from any one of the 1920 choices equivalent to it under the action of the permutation group $\boldsymbol{\Phi}$. However, two numerical choices equivalent under the action of $\boldsymbol{\Phi}$, although giving the same irrationality measure of $\zeta(3)$, in general yield different sets of transformation formulae that can be disregarded. The choice we make at the end of this section, namely $h = 16$, $j = 17$, $k = 19$, $l = 15$, $m = 12$, $q = 11$, $r = 9$, $s = 13$, is such that the corresponding transformation formulae needed for the study of the p-adic valuation of A_n are few (only 15 among 120), and moreover have the advantage of being all associated with left cosets of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$ of the same level, i.e. of level 4, which yield divisors of A_n of the types p and p^2 but not p^3, as we shall show. Thus we may treat only permutations of level 4, and the resulting arithmetic discussion turns out to be quite simple.

Consider, e.g., the transformation formula

$$I(h, j, k, l, m, q, r, s) = \frac{h!\,k!\,l!\,r!}{m'!\,r'!\,h'!\,q'!}\, I(m', r', m, h', q, q', s, j) \tag{5.4}$$

corresponding to the permutation

$$\boldsymbol{\varphi}\boldsymbol{\vartheta}^2\boldsymbol{\varphi} = (h\ \ m'\ \ j'\ \ l')(j\ \ r'\ \ r\ \ s)(k\ \ m\ \ q\ \ q')(l\ \ h'\ \ k'\ \ s').$$

Let

$$I'_n = I(m'n, r'n, mn, h'n, qn, q'n, sn, jn) = a'_n + 2b'_n\zeta(3) \qquad (n = 1, 2, \ldots). \tag{5.5}$$

By (5.2), (5.4) and (5.5) we have, for any $n \geq 1$,

$$(m'n)!\,(r'n)!\,(h'n)!\,(q'n)!\,(a_n + 2b_n\zeta(3)) = (hn)!\,(kn)!\,(ln)!\,(rn)!\,(a'_n + 2b'_n\zeta(3)),$$

whence, by the irrationality of $\zeta(3)$,

$$(m'n)!\,(r'n)!\,(h'n)!\,(q'n)!\,a_n = (hn)!\,(kn)!\,(ln)!\,(rn)!\,a'_n. \tag{5.6}$$

Multiplying (5.6) by $d_{Mn}d_{Nn}d_{Qn}$ we obtain

$$(m'n)!\,(r'n)!\,(h'n)!\,(q'n)!\,A_n = (hn)!\,(kn)!\,(ln)!\,(rn)!\,A'_n, \tag{5.7}$$

where $A_n = d_{Mn}d_{Nn}d_{Qn}a_n$ and $A'_n = d_{Mn}d_{Nn}d_{Qn}a'_n$ are integers by (5.3) and Theorem 2.1. For a prime p, let

$$\alpha_p = v_p((m'n)!\,(r'n)!\,(h'n)!\,(q'n)!),$$
$$\beta_p = v_p((hn)!\,(kn)!\,(ln)!\,(rn)!),$$

where $v_p(L)$ denotes the exponential p-adic valuation of the integer $L > 0$, i.e., the exponent of p in the factorization of L into powers of distinct primes. Using the property (iii) in Section 4 we see, as in [4], pp. 44–45, that for any prime p satisfying

$$p > \sqrt{Mn} \tag{5.8}$$

we have

$$\alpha_p - \beta_p = [m'\omega] + [r'\omega] + [h'\omega] + [q'\omega] - [h\omega] - [k\omega] - [l\omega] - [r\omega],$$

where $\omega = \{n/p\} = n/p - [n/p]$ is the fractional part of n/p. Letting

$$V_1 = [m'\omega] + [h'\omega] - [k\omega] - [r\omega],$$
$$V_2 = [r'\omega] + [q'\omega] - [h\omega] - [l\omega],$$

we get $\alpha_p - \beta_p = V_1 + V_2$. Since

$$m' + h' = k + r \tag{5.9}$$

and

$$r' + q' = h + l, \tag{5.10}$$

we have $-1 \le V_1 \le 1$ and $-1 \le V_2 \le 1$ by Lemma 4.1 of [4]. Therefore $-2 \le \alpha_p - \beta_p \le 2$. Hence, removing from (5.7) the primes $p > \sqrt{Mn}$ dividing the factorials on both sides, we obtain

$$(p_1 \ldots p_\lambda)(q_1 \ldots q_\mu)^2 P A_n = (p'_1 \ldots p'_{\lambda'})(q'_1 \ldots q'_{\mu'})^2 P' A'_n, \tag{5.11}$$

where $p_1, \ldots, p_\lambda$; $q_1, \ldots, q_\mu$; $p'_1, \ldots, p'_{\lambda'}$; $q'_1, \ldots, q'_{\mu'}$ are the distinct primes satisfying (5.8) for which $\alpha_p - \beta_p = 1$; 2; -1; -2 respectively, and P, P' are products of primes each of which does not exceed $\sqrt{Mn}$.

By (5.11), $p'_1, \ldots, p'_{\lambda'}$ and $q'^2_1, \ldots, q'^2_{\mu'}$ divide A_n. Thus any prime $p > \sqrt{Mn}$ for which $\alpha_p - \beta_p < 0$, i.e.

$$[m'\omega] + [r'\omega] + [h'\omega] + [q'\omega] < [h\omega] + [k\omega] + [l\omega] + [r\omega], \tag{5.12}$$

divides A_n, and any prime $p > \sqrt{Mn}$ for which $\alpha_p - \beta_p = -2$, i.e. $V_1 = V_2 = -1$, i.e.

$$\begin{cases} [m'\omega] + [h'\omega] < [k\omega] + [r\omega] \\ [r'\omega] + [q'\omega] < [h\omega] + [l\omega], \end{cases} \tag{5.13}$$

is such that p^2 divides A_n.

The above discussion applies to any transformation formula corresponding to a permutation of level 4. Let F be the set of 30 permutations, representatives of the 30 left cosets of $\boldsymbol{\Theta}$ in $\boldsymbol{\Phi}$ of level 4, listed at the end of Section 4, and let E be a subset of F, to be chosen later. For each permutation in E we consider the inequality analogous to (5.12) arising from the quotient of factorials corresponding to that permutation, and we denote by Ω_E the set of real numbers $\omega \in [0, 1)$ satisfying at least one of such inequalities. Plainly every prime $p > \sqrt{Mn}$, such that $\{n/p\} \in \Omega_E$, divides A_n. Similarly, for each permutation in E we consider the pair of simultaneous inequalities analogous to (5.13) arising from the quotient of factorials corresponding to that permutation, where, in analogy with (5.9) and (5.10), the integers appearing in the quotient of factorials are arranged so that, in each inequality of the pair, the sum of the two integers on the left side equals the sum of the two integers on the right side (in fact, equations analogous to (5.9) and (5.10) hold for any quotient of factorials corresponding to a permutation of level 4, as is easily seen by direct inspection of the quotients of factorials listed at the end of Section 4). We denote by Ω'_E the set of real numbers $\omega \in [0,1)$ satisfying both the inequalities in at least one of the pairs analogous to (5.13) thus obtained. Then every prime $p > \sqrt{Mn}$, for which $\{n/p\} \in \Omega'_E$, is such that p^2 divides A_n. Obviously $\Omega'_E \subset \Omega_E$.

LEMMA 5.1. *If $\omega \in \Omega_E$ then $\omega \geq 1/M$. If $\omega \in \Omega'_E$ then $\omega \geq 1/N$.*

Proof. By (5.3) we have $M = \max \mathcal{T}$, whence

$$\frac{1}{M} \leq \min\left\{\frac{1}{h}, \frac{1}{j}, \frac{1}{k}, \frac{1}{l}, \frac{1}{m}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}\right\}.$$

Thus if $\omega < 1/M$ we get

$$[h\omega] = [j\omega] = [k\omega] = [l\omega] = [m\omega] = [q\omega] = [r\omega] = [s\omega] = 0.$$

Therefore all the inequalities analogous to (5.12) are false, whence $\omega \notin \Omega_E$.

If $\omega < 1/N$, at least seven among $[h\omega]$, $[j\omega]$, $[k\omega]$, $[l\omega]$, $[m\omega]$, $[q\omega]$, $[r\omega]$, $[s\omega]$ vanish, because $N = \max' \mathcal{T}$. Hence at least three among $[h\omega]$, $[k\omega]$, $[l\omega]$, $[r\omega]$ vanish, and therefore at least one of the inequalities in (5.13) is false, whence the condition (5.13) itself is false. Similarly, all the conditions analogous to (5.13) are false, whence $\omega \notin \Omega'_E$. ∎

We now proceed as in [4], pp. 50–51. Let, for $n = 1, 2, \ldots,$

$$\Delta_n = \prod_{\substack{p>\sqrt{Mn} \\ \{n/p\}\in\Omega_E}} p, \qquad \Delta'_n = \prod_{\substack{p>\sqrt{Mn} \\ \{n/p\}\in\Omega'_E}} p, \qquad D_n = \frac{d_{Mn} d_{Nn} d_{Qn}}{\Delta_n \Delta'_n}.$$

Since each prime p dividing Δ_n also divides A_n, and each prime p dividing Δ'_n is such that p^2 divides A_n, we have $\Delta_n \Delta'_n \mid A_n$, whence $D_n a_n = A_n/(\Delta_n \Delta'_n) \in \mathbb{Z}$. From Lemma 5.1 we obtain $\Delta_n \mid d_{Mn}$ and $\Delta'_n \mid d_{Nn}$, whence $D_n \in \mathbb{Z}$. Therefore, by (5.2),

$$D_n I_n = D_n a_n + 2 D_n b_n \zeta(3) \in \mathbb{Z} + 2\mathbb{Z}\zeta(3),$$

and in order to get an irrationality measure of $\zeta(3)$ we can apply Lemma 4.3 of [4] to $D_n I_n$. As in [4], p. 51, we see that

$$\lim_{n\to\infty} \frac{1}{n} \log D_n = M + N + Q - \Big(\int_{\Omega_E} d\psi(x) + \int_{\Omega'_E} d\psi(x)\Big), \tag{5.14}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the Euler gamma-function.

To evaluate $\lim_{n\to\infty}(1/n)\log I_n$ and $\limsup_{n\to\infty}(1/n)\log|b_n|$, we now assume the integers (5.1) and (4.7) to be strictly positive. Let

$$f(x,y,z) = \frac{x^h(1-x)^l y^k(1-y)^s z^j(1-z)^q}{(1-(1-xy)z)^{q+h-r}}. \tag{5.15}$$

Clearly

$$\lim_{n\to\infty}\frac{1}{n}\log I_n = \max_{0<x,y,z<1}\log f(x,y,z). \tag{5.16}$$

From Theorem 3.1 we get, for any $\varrho_1,\varrho_2,\varrho_3>0$ and any $n\ge 1$,

$$\begin{aligned}
&\frac{1}{n}\log|b_n| \\
&\le\log\frac{\varrho_1^h(1+\varrho_1)^l(\varrho_1^{-1}+\varrho_2)^k(1+\varrho_1^{-1}+\varrho_2)^s((\varrho_1\varrho_2)^{-1}+\varrho_3)^j(1+(\varrho_1\varrho_2)^{-1}+\varrho_3)^q}{(\varrho_1\varrho_2\varrho_3)^{q+h-r}} \\
&=\log\frac{(1+u)^l(1+v)^k(1+u+v)^s(1+w)^j(1+v+w)^q}{u^{s+k-h}\,v^{j+q}\,w^{q+h-r}},
\end{aligned}$$

where we have put $\varrho_1=u$, $\varrho_1\varrho_2=v$, $\varrho_1\varrho_2\varrho_3=w$. With the change of variables $u=-x$, $v=xy-1$, $w=(1-xy)z-1$, we get

$$\left|\frac{(1+u)^l(1+v)^k(1+u+v)^s(1+w)^j(1+v+w)^q}{u^{s+k-h}\,v^{j+q}\,w^{q+h-r}}\right| = |f(x,y,z)|.$$

Therefore

$$\limsup_{n\to\infty}\frac{1}{n}\log|b_n| \le \min_{\substack{x,y,z<0\\ xy>1\\ z<(1-xy)^{-1}}}\log|f(x,y,z)|. \tag{5.17}$$

A straightforward computation shows that for $x(1-x)y(1-y)z(1-z) \neq 0$ there are exactly two stationary points (x_0,y_0,z_0) and (x_1,y_1,z_1) of the function (5.15), with $0<x_0,y_0,z_0<1$ and $x_1,y_1,z_1<0$, $x_1y_1>1$, $z_1<(1-x_1y_1)^{-1}$. If we use again the notation (4.7), the points (x_0,y_0,z_0) and (x_1,y_1,z_1) are the solutions of the system

$$\begin{cases}
h'l'r'x^2+((h-j)(h-k)(q-r)+(hl+lr+rh)s'-hlr)x-hrs'=0\\[2mm]
y=\dfrac{l'x+k-h}{(l'-s)x+s'}\\[2mm]
z=\dfrac{h'l'x^2-(hh'+(h-s)l'-k(h+q))x-hs'}{(x-1)((h-s)l'x+hs')}.
\end{cases}$$

Hence, by (5.16) and (5.17),

$$\lim_{n\to\infty}\frac{1}{n}\log I_n = \log f(x_0,y_0,z_0), \tag{5.18}$$

$$\limsup_{n\to\infty} \frac{1}{n} \log |b_n| \leq \log |f(x_1, y_1, z_1)|. \tag{5.19}$$

Define

$$c_0 = -\log f(x_0, y_0, z_0), \qquad c_1 = \log |f(x_1, y_1, z_1)|,$$
$$c_2 = M + N + Q - \Big(\int\limits_{\Omega_E} d\psi(x) + \int\limits_{\Omega'_E} d\psi(x) \Big).$$

From (5.14), (5.18), (5.19) and Lemma 4.3 of [4] we obtain the following theorem, similar to Theorem 5.1 of [4].

THEOREM 5.1. *If* $c_0 > c_2$, *then*

$$\mu(\zeta(3)) \leq \frac{c_0 + c_1}{c_0 - c_2}.$$

We have to choose numerical values for h, j, k, l, m, q, r, s with $h+m = k+r$ and $j+q = l+s$, as well as the set E of permutations of level 4. We take

$$h = 16, \ j = 17, \ k = 19, \ l = 15, \ m = 12, \ q = 11, \ r = 9, \ s = 13,$$

whence

$$h' = 14, \ j' = 10, \ k' = 15, \ l' = 12, \ m' = 14, \ q' = 18, \ r' = 13, \ s' = 16,$$

and, by (5.3),

$$M = 19, \qquad N = 18, \qquad Q = 17.$$

For E we choose the set of the following 15 permutations:

$$\boldsymbol{\varphi\vartheta^2\varphi},\ \boldsymbol{\vartheta\varphi\vartheta^2\varphi},\ \boldsymbol{\vartheta^3\varphi\vartheta^2\varphi},\ \boldsymbol{\vartheta^4\varphi\vartheta^2\varphi},\ \boldsymbol{\vartheta^5\varphi\vartheta^2\varphi},$$
$$\boldsymbol{\vartheta^7\varphi\vartheta^2\varphi},\ \boldsymbol{\varphi\vartheta^6\varphi},\ \boldsymbol{\vartheta\varphi\vartheta^6\varphi},\ \boldsymbol{\vartheta^4\varphi\vartheta^6\varphi},\ \boldsymbol{\vartheta^5\varphi\vartheta^6\varphi},$$
$$\boldsymbol{\chi\vartheta^3\chi},\ \boldsymbol{\vartheta\chi\vartheta^3\chi},\ \boldsymbol{\vartheta^2\chi\vartheta^3\chi},\ \boldsymbol{\vartheta^3\chi\vartheta^3\chi},\ \boldsymbol{\vartheta\chi\vartheta^2\chi}.$$

Then the set Ω_E is the union of the intervals

$$\left[\tfrac{1}{19}, \tfrac{1}{10}\right),\ \left[\tfrac{2}{19}, \tfrac{2}{7}\right),\ \left[\tfrac{5}{17}, \tfrac{1}{2}\right),\ \left[\tfrac{10}{19}, \tfrac{5}{7}\right),\ \left[\tfrac{8}{11}, \tfrac{17}{18}\right),$$

whence

$$\int\limits_{\Omega_E} d\psi(x) = 18.04470204\ldots,$$

and the set Ω'_E is the union of the intervals

$$\left[\tfrac{1}{17}, \tfrac{1}{14}\right),\ \left[\tfrac{2}{17}, \tfrac{1}{7}\right),\ \left[\tfrac{2}{11}, \tfrac{1}{5}\right),\ \left[\tfrac{4}{19}, \tfrac{3}{14}\right),\ \left[\tfrac{5}{19}, \tfrac{5}{18}\right),\ \left[\tfrac{6}{17}, \tfrac{5}{14}\right),$$
$$\left[\tfrac{7}{19}, \tfrac{7}{18}\right),\ \left[\tfrac{8}{19}, \tfrac{3}{7}\right),\ \left[\tfrac{8}{17}, \tfrac{1}{2}\right),\ \left[\tfrac{6}{11}, \tfrac{4}{7}\right),\ \left[\tfrac{10}{17}, \tfrac{3}{5}\right),\ \left[\tfrac{7}{11}, \tfrac{9}{14}\right),$$
$$\left[\tfrac{13}{19}, \tfrac{7}{10}\right),\ \left[\tfrac{13}{17}, \tfrac{11}{14}\right),\ \left[\tfrac{14}{17}, \tfrac{5}{6}\right),\ \left[\tfrac{16}{19}, \tfrac{6}{7}\right),\ \left[\tfrac{10}{11}, \tfrac{13}{14}\right),$$

whence

$$\int\limits_{\Omega'_E} d\psi(x) = 6.14298325\ldots,$$

so that

$$c_2 = 29.81231469\ldots$$

The stationary points of the function (5.15) are

$$x_0 = 0.33482274\ldots, \quad y_0 = 0.44799192\ldots, \quad z_0 = 0.85503803\ldots,$$

and

$$x_1 = -3.15075681\ldots, \quad y_1 = -1.81763478\ldots, \quad z_1 = -1.03862011\ldots,$$

and we get

$$c_0 = 47.15472079\ldots, \quad c_1 = 48.46940964\ldots$$

Thus Theorem 5.1 yields

$$\mu(\zeta(3)) < 5.513891.$$

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