PROBLEMS FOR HGM

Problem 1 (general Euler–Pochhammer integral)**.** (a) The Euler–Pochhammer integral

$$ {}_2F_1(\{\alpha,\beta\},\{\gamma\};\lambda)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 x^{\beta-1}(1-x)^{\gamma-\beta-1}(1-\lambda x)^{-\alpha}\,\mathrm{d}x $$

for $\operatorname{Re}\gamma>\operatorname{Re}\beta>0$ and $|\lambda|<1$, generalises to

$$ \begin{aligned} &{}_mF_{m-1}(\{\alpha_1,\alpha_2,\dots,\alpha_m\},\{\beta_2,\dots,\beta_m\};\lambda) \\ &\quad=\prod_{j=2}^m\frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)\Gamma(\beta_j-\alpha_j)}\int_0^1\frac{\prod_{j=2}^m x_j^{\alpha_j-1}(1-x_j)^{\beta_j-\alpha_j-1}}{(1-\lambda x_2\cdots x_m)^{\alpha_1}}\,\mathrm{d}x_2\cdots\mathrm{d}x_m \end{aligned} $$

provided $\operatorname{Re}\beta_j>\operatorname{Re}\alpha_j>0$ for $j=2,\dots,m$ and $|\lambda|<1$.

(b) Under the conditions $\operatorname{Re}\beta_j>\operatorname{Re}\alpha_j>0$ for $j=2,\dots,m$, the integral converges absolutely for any $\lambda\in\mathbb{C}$ with $\arg(1-\lambda)<\pi$. In other words, the integral provides analytic continuation of the ${}_mF_{m-1}$ in question to $\mathbb{C}\setminus[1,+\infty)$.

Problem 2 (hypergeometric arithmetic data)**.** A way to say that the hypergeometric data $\boldsymbol{\alpha}=\{\alpha_1,\dots,\alpha_m\}\in\mathbb{Q}^m$, $\boldsymbol{\beta}=\{\beta_1,\dots,\beta_m\}\in\mathbb{Q}^m$ is Galois, or defined over $\mathbb{Q}$, is via the requirement that the polynomials $\prod_{j=1}^m(x-e^{2\pi i\alpha_j})$ and $\prod_{j=1}^m(x-e^{2\pi i\beta_j})$ have integral coefficients, so that each happens to be a product of cyclotomic polynomials. In this problem we assume that this condition is met.

(a) Show that

$$ \prod_{j=1}^m\frac{x-e^{2\pi i\alpha_j}}{x-e^{2\pi i\beta_j}}=\frac{\prod_{i=1}^u(x^{a_i}-1)}{\prod_{i=1}^v(x^{b_v}-1)} $$

for some positive integers $a_1,\dots,a_u$ and $b_1,\dots,b_v$ such that $a_1+\dots+a_u=b_1+\dots+b_v$.

(b) Prove that

$$ \frac{(\alpha_1)_n\cdots(\alpha_m)_n}{(\beta_1)_n\cdots(\beta_m)_n}=M^{-n}\cdot\frac{(a_1n)!\cdots(a_un)!}{(b_1n)!\cdots(b_vn)!},\quad\text{where } M=\frac{a_1^{a_1}\cdots a_u^{a_u}}{b_1^{b_1}\cdots b_v^{b_v}}. $$

Problem 3 (algebraic hypergeometric series)**.** Apart from the binomial series

$$ (1-\lambda)^{-\alpha}=\sum_{n=0}^\infty\frac{(\alpha)_n}{n!}\lambda^n={}_1F_0\biggl(\begin{matrix}\alpha\\-\end{matrix}\biggm|\lambda\biggr), $$

there are more examples of hypergeometric series that are algebraic. In fact, the problem of listing all possible configurations of parameters for such ${}_2F_1$ algebraic instances was given by Hermann Schwarz (1873); there are 15 cases. The list of *all* algebraic hypergeometric series (equivalently, of hypergeometric differential equations with finite monodromy) was given by Beukers and Heckman (1990).

(a) Show that

$$ {}_2F_1\biggl(\begin{matrix}\alpha,\,\alpha+\frac12\\2\alpha+1\end{matrix}\biggm|\lambda\biggr)=\biggl(\frac{1+\sqrt{1-\lambda}}{2}\biggr)^{-2\alpha}. $$

Date: 11–13 August 2026.

(b) (Villegas's algebraicity criterion). Show that the hypergeometric series

$$ {}_mF_{m-1}\left(\begin{matrix}\alpha_1,\ \dots,\ \alpha_m\\ \beta_2,\ \dots,\ \beta_m\end{matrix}\ \middle|\ \lambda\right) $$

is algebraic if and only if the hypergeometric data $\boldsymbol{\alpha},\boldsymbol{\beta}$ is defined over $\mathbb{Q}$ and for the corresponding arithmetic data (from Problem 2) we have $v=u+1$ and

$$ \frac{(a_1n)!\cdots(a_un)!}{(b_1n)!\cdots(b_vn)!}\in\mathbb{Z}\quad\text{for all } n\ge 0. $$

The latter condition can be replaced with Landau's

$$ \lfloor a_1x\rfloor+\cdots+\lfloor a_ux\rfloor\ge\lfloor b_1x\rfloor+\cdots+\lfloor b_vx\rfloor\quad\text{for } 0\le x<1. $$

(c) The example in this question originates from Chebyshev's deduction of estimates for the prime counting function. Check that

$$ \frac{(30n)!\,n!}{(15n)!\,(10n)!\,(6n)!}\in\mathbb{Z}\quad\text{for all } n\ge 0 $$

and give a hypergeometric expression for the generating function of this sequence. This is an example of an algebraic hypergeometric series (of degree 483 840).

Problem 4 (algebraic transformations and formulas for $1/\pi$)**.** (a) Prove that

$$ \lim_{\lambda\to1^-}\sqrt{1-\lambda}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\frac12)_n(1-\alpha)_n}{n!^3}\,n\lambda^n=\frac{\sin\pi\alpha}{\pi} $$

and, more generally,

$$ \lim_{\lambda\to1^-}\sqrt{1-\lambda}\sum_{n=0}^{\infty}\frac{(\frac12)_n(\alpha_1)_n(1-\alpha_1)_n\cdots(\alpha_m)_n(1-\alpha_m)_n}{n!^{2m+1}}n^m\lambda^n=\prod_{j=1}^{m}\frac{\sin\pi\alpha_j}{\pi}. $$

(b) Verify the following identity:

$$ {}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix}\ \middle|\ \frac{256x}{9(1+3x)^4}\right)=\frac{1+3x}{1+x/3}\,{}_3F_2\left(\begin{matrix}\frac14,\ \frac12,\ \frac34\\ 1,\ 1\end{matrix}\ \middle|\ \frac{256x^3}{9(3+x)^4}\right). $$

(c) Show that

$$ \sum_{n=0}^{\infty}\frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3}\,(40n+3)\,\frac{1}{7^{4n}}=\frac{49\sqrt3}{9\pi}. $$

This is an example of a Ramanujan-type formula for $1/\pi$.

(d) [Guillera's formula for $1/\pi$; *open*] Show that

$$ \sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac18)_n(\frac38)_n(\frac58)_n(\frac78)_n}{n!^5}\,(1920n^2+304n+15)\,\frac{1}{7^{4n}}\stackrel{?}{=}\frac{56\sqrt7}{\pi^2}. $$

Hints. (a) Use

$$ \lim_{n\to\infty}\frac{(\alpha)_n(1-\alpha)_nn}{n!^2}=\frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}=\frac{\sin\pi a}{\pi} $$

and the Stolz–Cesàro theorem.

(b) One way to proceed is by dividing the identity by $1+3x$ and verifying that both sides satisfy the same (linear) differential equation of order 3.

(c) Apply the differential operator $x\,\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides of the identity in part (b) and pass to the limit as $x\to(1/9)^-$ using part (a). $\square$

Problem 5 (RH-likeness of special hypergeometric series)**.** For real $k > 0$ and complex s, consider

$$W(k,s) = \frac{1}{2\pi i}\int_{|z|=1} |k + (z + 1/z)|^s \,\frac{\mathrm{d}z}{z}.$$

(a) Show that

$$W(k,s) = k^s \cdot {}_2F_1\left(\begin{matrix} -s/2,\ (1-s)/2 \\ 1 \end{matrix} \,\middle|\, \frac{4}{k^2}\right) \qquad \text{if } k > 2,$$

$$W(k,s) = \frac{2^s\Gamma(\frac12 + s)}{\Gamma(\frac12 + \frac{s}{2})\Gamma(1 + \frac{s}{2})} \qquad \text{if } k = 2,$$

$$W(k,s) = \frac{2^{2s}\Gamma(\frac12 + \frac{s}{2})^2}{\pi\Gamma(1+s)} \cdot {}_2F_1\left(\begin{matrix} -s/2,\ -s/2 \\ 1/2 \end{matrix} \,\middle|\, \frac{k^2}{4}\right) \qquad \text{if } k < 2.$$

(b) For $k > 2$ and $s \in \mathbb{C}$, we have

$$W(k, -s-1) = (k^2 - 4)^{-s-1/2} W(k,s).$$

(c) For $0 < k < 2$ and $-1 < \operatorname{Re} s < 0$, we have

$$W(k, -s-1) = \cot\left(-\frac{\pi s}{2}\right)(4 - k^2)^{-s-1/2} W(k,s).$$

(d) For both $k > 2$ and $k < 2$ situations, the line of symmetry of the functional equation is $\operatorname{Re} s = -1/2$. The function $W(k,s)$ possesses some *trivial* zeros $s \in \mathbb{R}$. Show that non-trivial (i.e., non-real) zeros all lie on the line of symmetry.

Problem 6 (p-adic interpolation and p-adic gamma function)**.** Assume that we want to extend a function $f\colon S \to \mathbb{Z}_p$ (sometimes $f\colon S \to \mathbb{Q}_p$) given on a dense subset S in $\mathbb{Z}_p$ to $\mathbb{Z}_p$, that is, to construct a *continuous* function $\tilde f\colon \mathbb{Z}_p \to \Omega$ such that its restriction $\tilde f|_S$ on S coincides with f. Observe that $\mathbb{Z}_p$ is a compact (metric space), so also sequentially compact, hence the resulting function $\tilde f\colon \mathbb{Z}_p \to \Omega$ should be also unformly continuous: given $\varepsilon > 0$, there exists $N > 0$ such that whenever $x, y \in \mathbb{Z}_p$ satisfy $|x - y|_p \le p^{-N}$ then $|\tilde f(x) - \tilde f(y)|_p < \epsilon$. (We deliberately replace $|x - y|_p < \delta$ with $|x - y|_p \le p^{-N}$. The latter can be stated as $x \equiv y \bmod p^N$.)

First notice that the extension $\tilde f\colon \mathbb{Z}_p \to \Omega$ of a mapping $f\colon S \to \mathbb{Z}_p$ exists if and only if f is uniformly continuous on S. Indeed, if $\tilde f$ exists then it is uniformly continuous on $\mathbb{Z}_p$, hence its restriction f to S is. In the other direction, given $a \in \mathbb{Z}_p$, take a sequence $s_1, s_2, \dots \in S$ such that $s_n \to a$ as $n \to \infty$. Then $\lim_{n\to\infty} f(s_n)$ exists and does not depend on the choice of sequence — all as a consequence of the uniform continuity of f. Hence $\tilde f(a) = \lim_{n\to\infty} f(s_n)$ is a well-defined quantity, and $\tilde f$ is continuous on its definition domain.

An illustration of such interpolation is given in the following example. Let $p > 2$, and an integer $m \not\equiv 0 \bmod p$ be fixed. Consider $S = (p-1)\mathbb{Z}$ and the function $f\colon S \to \mathbb{Q}_p$ given by $(p-1)a \mapsto m^{(p-1)a} = (m^{p-1})^a$ for $a \in \mathbb{Z}$; in other wors, $f\colon s \mapsto m^s$. (The image is $\equiv 1 \bmod p$, because $m^{p-1} \equiv 1 \bmod p$.) As S is dense in $\mathbb{Z}_p$, the extension $\tilde f\colon \mathbb{Z}_p \to \mathbb{Q}_p$ of the map is given by the formula

$$\tilde f\colon s \mapsto \exp\left(\frac{s}{p-1} \cdot \log_p(m^{p-1})\right).$$

Note that the argument of the exponential function here is within the convergence domain of the function. Indeed, $m^{p-1} \in 1 + p\mathbb{Z}_p \subset D_1\big((p^{-1/(p-1)})^-\big)$ (the open disk centred at 1 of radius $p^{-1/(p-1)}$) for $p > 2$, so that the image of m^{p-1} under $\log_p$ is inside the disk $D_0\big((p^{-1/(p-1)})^-\big)$.

Alternatively, the interpolation of $s \mapsto m^s$ can be done on the set $S' = 1 + (p-1)\mathbb{Z}$ (or any other shift of $(p-1)\mathbb{Z}$), which is clearly dense in $\mathbb{Z}_p$ for $p > 2$. This time it works as $1 + (p-1)a \mapsto m \cdot (m^{p-1})^a$. Though the extension of this mapping again interpolates $s \mapsto m^s$, this extension is different!

We move on to the p-adic analogue of the gamma-function $\Gamma(s)$, namely, a function $\Gamma_p(s)$ which interpolates the factorial function. In the Archimedean situation (for simplicity we can think of Γ to be a real function defined on $s > 0$), we have $\Gamma(n+1) = n!$ which is implied by $\Gamma(s+1) = s\Gamma(s)$ and $\Gamma(1) = 1$. There are some other nice properties it satisfies, like

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} = \frac{2\pi i \cdot e^{\pi i s}}{e^{2\pi i s} - 1}$$

(Euler's reflection formula) and

$$\prod_{h=0}^{m-1} \Gamma\biggl(\frac{s+h}{m}\biggr) = (2\pi)^{(m-1)/2} m^{1/2-s} \Gamma(s)$$

(Gauss's multiplication formula). Substituting $s = 1$ in the latter and then dividing it by the result leads to

$$\frac{\displaystyle\prod_{h=0}^{m-1} \Gamma\biggl(\frac{s+h}{m}\biggr)}{\displaystyle\Gamma(s) \prod_{h=1}^{m-1} \Gamma\biggl(\frac{h}{m}\biggr)} = m^{1-s}.$$

In our consideration of Γ_p below, we restrict ourselves to $p > 2$ (for $p = 2$ some slight modifications are required). We would like to construct a function $f\colon \mathbb{Z}_p \to \Omega$ which interpolates $\Gamma(n) = \prod_{k<n} k$ on positive integers. However, the mapping $n \to \Gamma(n)$ is not uniformly continuous on positive integers. Indeed, the condition $m \equiv n \bmod p^{-N}$ does not imply that $\Gamma(m)$ and $\Gamma(n)$ are p-adically close: the smallness of $|\Gamma(m) - \Gamma(n)|_p$ (when $m \neq n$) only depends on $\min\{m, n\}$, not on $|m - n|_p$. A standard way to improve the situation is to eliminate all k divisible by p from the product $\prod_{k<n} k$, in other words, interpolating $\Gamma^*(n) = \prod_{k<n,\, p\nmid k} k$ instead. Since $\Gamma^*(n)$ are then p-adic units, the closeness of $\Gamma^*(m)$ and $\Gamma^*(n)$ means that $\Gamma^*(m)/\Gamma^*(n) \equiv 1 \bmod p^M$, and the latter is to be implied by $m \equiv n \bmod p^N$ for some $N = N(M)$. Unfortunately,

$$\frac{\Gamma^*(n+p^N)}{\Gamma^*(n)} = \prod_{\substack{n \le k < n+p^N \\ p \nmid k}} k \equiv -1 \pmod{p^N}, \tag{1}$$

so the sign gets wrong already for the particular instance $m = n + p^N$. (This is Wilson's theorem again, corresponding to the $N = 1$ choice. It can be easily extended from $(\mathbb{Z}/p\mathbb{Z})^\times$ to $(\mathbb{Z}/p^N\mathbb{Z})^\times$ to establish (1) in its generality.) Finally, we moderate the sign and define

$$\Gamma_p(n) = (-1)^n \prod_{\substack{k<n \\ p \nmid k}} k \quad \text{for } n = 1, 2, 3, \dots,$$

and then

$$\Gamma_p(s) = \lim_{n \to s} \Gamma_p(n) \quad \text{for } s \in \mathbb{Z}_p.$$

Since for n positive integers, $\Gamma_p(n) \in \mathbb{Z}$ and even $\Gamma_p(n) \in \mathbb{Z}_p^\times$, where the latter set is closed, we also have $\Gamma_p(s) \in \mathbb{Z}_p^\times$ for all $s \in \mathbb{Z}_p$. With the above extension of Wilson's theorem in mind, we get the following statement: If $s_1 \equiv s_2 \bmod p^N$ then $\Gamma_p(s_1) \equiv \Gamma_p(s_2) \bmod p^N$. In particular, $\Gamma_p\colon \mathbb{Z}_p \to \mathbb{Z}_p^\times$ is a continuous function.

We can now give related properties of the p-adic gamma function which resemble the corresponding properties of Euler's $\Gamma(s)$.

(a) We have $\Gamma_p(0)=1$ and

$$\frac{\Gamma_p(s+1)}{\Gamma_p(s)}=\begin{cases}-s & \text{if } s\in\mathbb{Z}_p^\times,\\ -1 & \text{if } s\in p\mathbb{Z}_p.\end{cases}$$

(b) Write $s\in\mathbb{Z}_p$ as $s=s_0+s_1p$, where $s_0\in\{1,2,\dots,p\}$. Then

$$\Gamma_p(s)\Gamma_p(1-s)=(-1)^{s_0}.$$

(c) In the recording $s=s_0+s_1p$, for a positive integer $m\not\equiv 0 \bmod p$ show that

$$\frac{\prod_{h=0}^{m-1}\Gamma_p\left(\dfrac{s+h}{m}\right)}{\Gamma_p(s)\prod_{h=1}^{m-1}\Gamma_p\left(\dfrac{h}{m}\right)}=m^{1-s_0}(m^{-(p-1)})^{s_1}.$$

As we have already discussed, the latter factor is well defined for $s_1\in\mathbb{Z}_p$.

Problem 7 (Integrality of q-expansions)**.** Many combinatorial generating functions

$$f(x)=\sum_{n=0}^\infty A_nx^n$$

arise in the form from which $f(x)\in\mathbb{Z}[[x]]$ follows easily from the fact that A_n have some counting meaning. But this is not always the case! And proving the integrality may be a difficult task. It happens that sometimes one can (try to) prove $f(x)\in\mathbb{Z}_p[[x]]$ for all primes p instead; this, of course, is equivalent to $f(x)\in\mathbb{Z}[[x]]$. The following criterion of Dwork shows to be particularly useful in such situations.

(a) Assume that $f(x)=1+\sum_{n=1}^\infty A_nx^n\in\mathbb{Q}_p[[x]]$ (so that $A_0=1$). Prove that $f(x)\in\mathbb{Z}_p[[x]]$ if and only if $f(x^p)/f(x)^p\in 1+px\mathbb{Z}_p[[x]]$.

The principle behind Dwork's lemma is a commuting property of $f(x)$ under $x\mapsto x^p$: $f(x^p)$ and $f(x)^p$ agree modulo p in the sense $f(x^p)/f(x)^p=1+p\sum_{k=1}^\infty (p\text{-adic integers})\cdot x^k$.

(b) Let $f(x)\in x\mathbb{Q}_p[[x]]$. Show that $\exp(f(x))\in 1+x\mathbb{Z}_p[[x]]$ if and only if $f(x^p)-pf(x)\in px\mathbb{Z}_p[[x]]$.

One application of the latter criterion is related to the integrality of $q(z)=z\exp(g(z)/f(z))$ where

$$f(z)={}_2F_1\left(\begin{matrix}\frac12,\frac12\\1\end{matrix}\,\middle|\,16z\right)=\sum_{n=0}^\infty\binom{2n}{n}^2z^n\in\mathbb{Z}[[z]]$$

is the (normalised) period of the Legendre family expressed as a hypergeometric function. Another solution of the very same equation, originating from the Frobenius basis at $z=0$, is $\hat f(z)=g(z)+f(z)\log(16z)$, which defines

$$g(z)=4\sum_{n=1}^\infty\binom{2n}{n}^2\left(\sum_{k=1}^{2n}\frac{(-1)^{k-1}}{k}\right)z^n$$

(clearly *not* in $\mathbb{Z}[[z]]$).

(c) Verify that $\hat f(z)=g(z)+f(z)\log(16z)$ satisfies the same (second order) hypergeometric differential equation as $f(z)$.

(d) Show that the power series expansion

$$\exp\left(\frac{g(z)}{f(z)}\right)$$

has integral coefficients. In particular, for the quotient $t(z) = \hat{f}(z)/f(z) = \log(16z) + g(z)/f(z)$ we have $q(z) = \frac{1}{16}\exp(t(z)) = z\exp(g(z)/f(z)) \in z\mathbb{Z}[[z]]$.

If we invert the formal power series

$$q(z) = z + 8z^2 + 84z^3 + 992z^4 + 12514z^5 + 164688z^6 + 2232200z^7 + \cdots,$$

then the resulting (formal) power series

$$z(q) = q - 8q^2 + 44q^3 - 192q^4 + 718q^5 - 2400q^6 + 7352q^7 - 20992q^8 + \cdots$$

clearly has integral coefficients as well. In this particular situation, the inversion can be recognised as modular through very classical formulae due to Jacobi:

$$z(q) = \frac{q\big(\sum_{n=0}^\infty q^{n^2+n}\big)^4}{\big(1+2\sum_{n=1}^\infty q^{n^2}\big)^4} = \frac{1}{16}\bigg(\frac{\sum_{n\in\mathbb{Z}} q^{(n+1/2)^2}}{\sum_{n\in\mathbb{Z}} q^{n^2}}\bigg)^4;$$

and in fact we further have

$$f(z(q)) = \sum_{n=0}^\infty \binom{2n}{n}^2 \bigg(\frac{\sum_{n\in\mathbb{Z}} q^{(n+1/2)^2}}{\sum_{n\in\mathbb{Z}} q^{n^2}}\bigg)^{4n} = \bigg(\sum_{n\in\mathbb{Z}} q^{n^2}\bigg)^2.$$

These equalities can be approached using the corresponding hypergeometric differential equation.

Problem 8. We define the generalised bilateral hypergeometric sum

$${}_m\mathcal{H}_m\left(\begin{matrix}\alpha_1, \dots, \alpha_m\\ \beta_1, \dots, \beta_m\end{matrix}\,\middle|\, \lambda, \varepsilon\right) = \frac{\prod_{j=1}^m \Gamma(\beta_j)}{\prod_{j=1}^m \Gamma(\alpha_j)} \sum_{n=-\infty}^\infty \frac{\prod_{j=1}^m \Gamma(\alpha_j+\varepsilon+n)}{\prod_{j=1}^m \Gamma(\beta_j+\varepsilon+n)}\,\lambda^{n+\varepsilon},$$

and in this problem we focus on the choice $\beta_1 = \cdots = \beta_m = 1$ (corresponding to the maximal unipotent monodromy of the hypergeometric differential equation at $\lambda = 0$).

(a) For $m = 2$ set $\alpha_1 = r$, $\alpha_2 = 1 - r$ with $0 < r < 1$ and $\beta_1 = \beta_2 = 1$:

$$\begin{aligned} F(\lambda;\varepsilon) &= \frac{1}{\Gamma(r)\,\Gamma(1-r)} \sum_{n=-\infty}^\infty \frac{\Gamma(r+\varepsilon+n)\,\Gamma(1-r+\varepsilon+n)}{\Gamma(1+\varepsilon+n)^2}\,\lambda^{n+\varepsilon} \\ &= \frac{1}{\Gamma(r)\,\Gamma(1-r)} \sum_{n=0}^\infty \frac{\Gamma(r+\varepsilon+n)\,\Gamma(1-r+\varepsilon+n)}{\Gamma(1+\varepsilon+n)^2}\,\lambda^{n+\varepsilon} + O(\varepsilon^2) \\ &= F_0(\lambda) + F_1(\lambda)\varepsilon + O(\varepsilon^2) \quad \text{as } \varepsilon \to 0, \end{aligned}$$

where

$$F_0(\lambda) = {}_2F_1\left(\begin{matrix}r,\ 1-r\\ 1\end{matrix}\,\middle|\, \lambda\right).$$

Show that

$$F_1(\lambda) = -\Gamma(r)\Gamma(1-r)\,F_0(1-\lambda) = -\frac{\pi}{\sin\pi r}\,F_0(1-\lambda)$$

for λ from the cut $\mathbb{C}$-plane $\mathbb{C}\setminus(-\infty,0]\cup[1,\infty)$. This is also true along the respective banks of the cut; in particular, for the real parts, the identity

$$\operatorname{Re} F_1(\lambda) = -\frac{\pi}{\sin\pi r}\,\operatorname{Re} F_0(1-\lambda)$$

is true for any complex $\lambda \ne 0, 1$.

(b) Verify that, for $r = 1/2$, we have

$$\left(\frac{-4}{p}\right)\sum_{k=0}^{p-1} \frac{(\frac12)_k^2}{k!^2}\,\lambda^k \equiv \sum_{k=0}^{p-1} \frac{(\frac12)_k^2}{k!^2}\,(1-\lambda)^k \pmod p$$

whenever primes $p>2$ do not divide the denominator of λ. Similar congruences, but for different choices of the quadratic character, hold when $r=\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{6}$.

(c) The corresponding Frobenius traces $a(p)$ are computed via

$$a(p) \equiv \sum_{k=0}^{p-1} \frac{(\frac{1}{2})_k^2}{k!^2}\,\lambda^k \pmod p$$

(for almost all primes) and the estimates $|a(p)| \le 2\sqrt{p}$. If

$$L(\lambda,s)=\prod_p (1-a(p)p^{-s}+\epsilon_p p^{1-2s})^{-1}=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

is the associated L-function, then show that

$$\frac{L(\lambda,1)}{\operatorname{Re} F_1(\lambda)} \in \mathbb{Q},$$

where $F_1(\lambda)$ originates from the ε-expansion above.

Problem 9. A connection with modular forms (Hecke eigen forms) $f(q)=\sum_{n=1}^{\infty} a(n)q^n$ in the cases when corresponding algebraic varieties determined by the hypergeometric data are known to be modular is seen by the hypergeometric series in several ways. Problem 8 discusses such connections for the cases when ynderlying varieties are families of elliptic curves (parametrised by λ). The instances in this problem are rigid and correspond to $\lambda=1$ of the series ${}_m\mathcal{H}_m(\{r,1-r,t,1-t\},\{1,1,1,1\};\lambda)$ where the collection of fourteen $\boldsymbol{\alpha}=\{r,1-r,t,1-t\}$ and the related eigen forms $f(q)$ are reproduced in the following table:

(r,t)	$f_{\boldsymbol{\alpha}}(\tau)$	level	LMFDB tag
$(\frac{1}{2},\frac{1}{2})$	$\eta_2^4\eta_4^4$	$8=2^3$	`8.4.a.a`
$(\frac{1}{2},\frac{1}{3})$	$\eta_6^{14}/(\eta_2^3\eta_{18}^3)-3\eta_2^3\eta_6^2\eta_{18}^3$	$36=2^2\cdot 3^2$	`36.4.a.a`
$(\frac{1}{2},\frac{1}{4})$	$\eta_4^{16}/(\eta_2^4\eta_8^4)$	$16=2^4$	`16.4.a.a`
$(\frac{1}{2},\frac{1}{6})$	$\eta_4^3\eta_6^6\eta_{18}^2/(\eta_{12}^2\eta_{36})-3\eta_2^2\eta_6^6\eta_{36}^3/(\eta_4\eta_{12}^2)$ $+8\eta_2^3\eta_{12}^6\eta_{36}^2/(\eta_6^2\eta_{18})-16\eta_{12}^{12}/\eta_6^4$	$72=2^3\cdot 3^2$	`72.4.a.b`
$(\frac{1}{3},\frac{1}{3})$	$\eta_1^3\eta_3^4\eta_9-27\eta_3\eta_9^4\eta_{27}^3$	$27=3^3$	`27.4.a.a`
$(\frac{1}{3},\frac{1}{4})$	η_3^8	$9=3^2$	`9.4.a.a`
$(\frac{1}{3},\frac{1}{6})$	$\eta_6^{10}/\eta_{18}^2-27\eta_{18}^{10}/\eta_6^2+9\eta_6^7\eta_{54}^3/\eta_{18}^2-9\eta_2^3\eta_{18}^7/\eta_6^2$	$108=2^2\cdot 3^3$	`108.4.a.a`
$(\frac{1}{4},\frac{1}{4})$	$\eta_4^{10}/\eta_8^2-8\eta_8^{10}/\eta_4^2$	$32=2^5$	`32.4.a.a`
$(\frac{1}{4},\frac{1}{6})$	$\eta_{12}^{32}/(\eta_6^{12}\eta_{24}^{12})+16\eta_6^4\eta_{24}^4$	$144=2^4\cdot 3^2$	`144.4.a.f`
$(\frac{1}{6},\frac{1}{6})$		$216=2^3\cdot 3^3$	`216.4.a.c`
$(\frac{1}{5},\frac{2}{5})$	$\eta_5^{10}/(\eta_1\eta_{25})+5\eta_1^2\eta_5^4\eta_{25}^2$	$25=5^2$	`25.4.a.b`
$(\frac{1}{8},\frac{3}{8})$	$\eta_1^2\eta_2\eta_4^3\eta_8^3/\eta_{16}+2\eta_2^2\eta_4^3\eta_8^2\eta_{16}+8\eta_4^2\eta_8^2\eta_{16}^2\eta_{32}^2-24\eta_4^2\eta_{16}^2\eta_{32}^4$ $-16\eta_2^2\eta_4\eta_8^2\eta_{64}^4/\eta_{16}-64\eta_8\eta_{16}^2\eta_{32}^3\eta_{64}^2+32\eta_{16}^2\eta_{32}^5\eta_{64}^2/\eta_8$ $-32\eta_4^4\eta_8\eta_{32}\eta_{64}\eta_{128}^2/\eta_{16}-64\eta_8\eta_{16}^3\eta_{32}\eta_{64}\eta_{128}^2$ $-256\eta_{16}^3\eta_{32}^3\eta_{64}\eta_{128}^2/\eta_8+128\eta_4^2\eta_{32}^2\eta_{64}^3\eta_{128}^2/\eta_{16}$ $-256\eta_4^4\eta_{32}\eta_{128}^4/\eta_8-128\eta_{16}^4\eta_{32}\eta_{128}^4/\eta_8-512\eta_4^2\eta_{64}^2\eta_{128}^4$	$128=2^7$	`128.4.a.b`
$(\frac{1}{10},\frac{3}{10})$		$200=2^3\cdot 5^2$	`200.4.a.f`
$(\frac{1}{12},\frac{5}{12})$		$864=2^5\cdot 3^3$	`864.4.a.a`

in the notation $\eta_j=\eta(j\tau)=q^{j/24}\prod_{n=1}^\infty(1-q^{jn})$ with $q=\exp(2\pi i\tau)$.

Note that $|a(p)|\le 2p^{3/2}$ for primes p, where $a(n)$ are the Fourier coefficients of $f(q)$.

(a) Write

$$\begin{aligned} {}_m\mathcal{H}_m\biggl(\begin{matrix}\alpha_1,\,\dots,\,\alpha_m\\ \beta_1,\,\dots,\,\beta_m\end{matrix}\biggm|\lambda,\varepsilon\biggr) &= \frac{1}{\Gamma(r)\,\Gamma(1-r)\,\Gamma(t)\,\Gamma(1-t)} \\ &\quad\times\sum_{n=-\infty}^{\infty}\frac{\Gamma(r+\varepsilon+n)\,\Gamma(1-r+\varepsilon+n)\,\Gamma(t+\varepsilon+n)\,\Gamma(1-t+\varepsilon+n)}{\Gamma(1+\varepsilon+n)^4}\,\lambda^{n+\varepsilon} \\ &= F_0(\lambda)+F_1(\lambda)\varepsilon+F_2(\lambda)\varepsilon^2+F_3(\lambda)\varepsilon^3+O(\varepsilon^4) \qquad \text{as} \quad \varepsilon\to 0, \end{aligned}$$

so that

$$F_0(\lambda)={}_4F_3\biggl(\begin{matrix}r,\,1-r,\,t,\,1-t\\ 1,\,1,\,1\end{matrix}\biggm|\lambda\biggr).$$

Let $L(s)=L_{r,t}(s)$ denote the L-series corresponding to the modular form $f(q)$ of weight 4. Verify (numerically) that

$$\frac{L(f,1)}{F_1(1)}\in\mathbb{Q}, \quad \frac{L(f,2)}{F_2(1)}\in\mathbb{Q} \quad \text{and} \quad \frac{L(f,3)}{F_3(1)}\in\mathbb{Q}$$

for the critical L-values $L(f,1)$, $L(f,2)$ and $L(f,3)$.

(b) Show that

$$\sum_{k=0}^{p-1}\frac{(r)_k(1-r)_k(t)_k(1-t)_k}{k!^4}\equiv a(p) \pmod{p^3}$$

for primes $p>3$ not dividing the denominators of r,t.

The last part gives an explicit recipe for reconstructing $a(p)$ (with an exception of $a(2)$, $a(3)$ and possibly $a(5)$).

Problem 10 (a hypergeometric deformation of modular forms; *open*)**.** Many hypergeometric evaluations, like

$${}_3F_2\biggl(\begin{matrix}\frac12,\,\frac12,\,\frac12\\ 1,\,1\end{matrix}\biggm|\frac{1}{64}\biggr)=\frac{2}{7\pi}\prod_{j=1}^{7}\Gamma\Bigl(\frac{j}{7}\Bigr)^{\left(\frac{j}{7}\right)},$$

where $\left(\frac{\cdot}{N}\right)$ denotes the Legendre symbol, originate from modular-function parametrisations. For example, the first entry is a CM instance of

$${}_3F_2\biggl(\begin{matrix}\frac12,\,\frac12,\,\frac12\\ 1,\,1\end{matrix}\biggm|z(\tau)\biggr)=f(\tau), \tag{2}$$

where

$$z(\tau)=\frac{64\eta_1^{24}\eta_4^{24}}{\eta_2^{48}} \quad \text{and} \quad f(\tau)=\frac{\eta_2^{20}}{\eta_1^8\eta_4^8}.$$

Some other (simpler!) consequences of identity (2) can be generalised to possess more parameters. For example,

$${}_3F_2\biggl(\begin{matrix}\frac12,\,\frac12,\,\frac12\\ 1,\,1\end{matrix}\biggm|\frac14\biggr)=\frac{2^{1/3}}{3\pi}\biggl(\frac{\Gamma(\frac13)}{\Gamma(\frac23)}\biggr)^3$$

happens to be a special case $t=-\frac14$ of

$${}_3F_2\biggl(\begin{matrix}-2t,\,6t+2,\,2t+1\\ 4t+2,\,2t+\frac32\end{matrix}\biggm|\frac14\biggr)=\frac{2^{8t+4}\Gamma(2t+\frac32)^2\Gamma(\frac12)^2}{3^{6t+3}\Gamma(2t+\frac43)^2\Gamma(\frac23)^2}.$$

One may ask what corresponding deformations of modular forms are, like what

$$ {}_3F_2\left(\begin{matrix} -2t,\, 6t+2,\, 2t+1 \\ 4t+2,\, 2t+\frac{3}{2} \end{matrix} \,\middle|\, \frac{64\eta_1^{24}\eta_4^{24}}{\eta_2^{48}}\right) $$

in our example, for $t \ne -\frac{1}{4}$ (but in a 'neighbourhood' of the point). This can be re-phrased, by taking $t = -\frac{1}{4} + \frac{1}{2}\varepsilon$, to the question about

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}-\varepsilon,\, \frac{1}{2}+3\varepsilon,\, \frac{1}{2}+\varepsilon \\ 1+2\varepsilon,\, 1+\varepsilon \end{matrix} \,\middle|\, \frac{64\eta_1^{24}\eta_4^{24}}{\eta_2^{48}}\right); $$

for example, what kind of modularity properties does it have, or what kind of modularity do the coefficients of its ε-expansion possess?

Already for a simpler-looking deformation

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2},\, \frac{1}{2}+\varepsilon,\, \frac{1}{2}-\varepsilon \\ 1,\, 1 \end{matrix} \,\middle|\, \frac{64\eta_1^{24}\eta_4^{24}}{\eta_2^{48}}\right) = \sum_{k=0}^{\infty} f_k(\tau)\varepsilon^{2k} $$

this is an open question. As indicated before, $f_0(\tau) = f(\tau)$ is a weight 2 modular form. What we found out is that

$$ f_1(\tau) = f(\tau) \cdot \delta^{-2}\left(-\frac{32\eta_1^8\eta_4^8}{\eta_2^8}\right), $$

where

$$ \delta^{-1}\sum_{n=1}^{\infty} a_n q^n = \sum_{n=1}^{\infty} \frac{a_n}{n}\, q^n $$

is the former cuspidal primitive. In other words, the second δ-derivative of $f_1(\tau)/f(\tau)$ is the weight 4 modular form $-32\eta_1^8\eta_4^8/\eta_2^8$. Does this pattern extend to further coefficients $f_2(\tau), f_3(\tau), \dots$? What is a general shape of such deformations?