Hypergeometric evaluations of L-values of an elliptic curve

Wadim Zudilin

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Ramanujan's closed forms

One of (so many!) Ramanujan's fames is an enormous production of highly nontrivial closed form evaluations of the values of certain "useful" series and functions.

By a *closed form* here we normally mean identifying the quantities in question with certain algebraic numbers or with values of hypergeometric functions

$$ {}_mF_{m-1}\left(\begin{matrix} a_1,\, a_2,\, \dots,\, a_m \\ b_2,\, \dots,\, b_m \end{matrix}\,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!} $$

where

$$ (a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \prod_{j=0}^{n-1} (a+j) $$

denotes the Pochhammer symbol (the shifted factorial).

Efficient formulae

An elegant "side" effect of such evaluations is computationally efficient formulae for mathematical constants, like

$$\frac{1}{\pi} = 32\sqrt{2}\sum_{n=0}^{\infty}\frac{(4n)!}{n!^4}\,(1103+26390n)\,\frac{1}{396^{4n+2}},$$

$$G = L(\chi_{-4},2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} = \pi\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{(1/4)^{2n+1}}{2n+1}.$$

Catalan's constant G is one of the simplest arithmetic quantities whose irrationality is still unproven.

Zeta values

Similar expressions for zeta values, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ where $s = 2, 3, \dots,$ were obtained more recently by others.
R. Apéry (1978) made use of acceleration formulae

$$\zeta(2) = 3 \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}} \quad \text{and} \quad \zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3 \binom{2n}{n}}$$

in his proof of the irrationality of $\zeta(2)$ and $\zeta(3)$.
The computationally efficient acceleration formula

$$\zeta(3) = \frac{1}{2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{5n^2 + 8(5n-2)^2}{n^5 \binom{2n}{n}^5}$$

is due to T. Amdeberhan and D. Zeilberger (1997).

Gamma values

An example of a slightly different type,

$$\frac{\pi}{5^{1/4}\Gamma(\frac{3}{4})^4} = \sum_{n=0}^{\infty} B_n \left(-\frac{1}{20}\right)^n \qquad \text{where} \quad B_n = \sum_{j=0}^{n} \binom{n}{j}^4,$$

is due to J. Guillera and Z. (2012).
Note that it is, roughly speaking, a "half" of Ramanujan-type formula

$$\frac{5}{2\pi} = \sum_{n=0}^{\infty} B_n \,(1+3n)\left(-\frac{1}{20}\right)^n$$

which is established recently by S. Cooper.

Periods

In order to "unify" such representations, M. Kontsevich and D. Zagier (2001) introduced the numerical class of periods.

A *period* is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in $\mathbb{R}^n$ given by polynomial inequalities with rational coefficients.

Without much harm, the three appearances of the adjective "rational" can be replaced by "algebraic".

The set of periods $\mathcal{P}$ is countable and admits a ring structure. It contains a lot of "important" numbers, mathematical constants like π, Catalan's constant and zeta values.

Extended periods

The extended period ring $\widehat{\mathcal{P}} := \mathcal{P}[1/\pi] = \mathcal{P}[(2\pi i)^{-1}]$ (rather than the period ring $\mathcal{P}$ itself) contains even more natural examples, like values of generalised hypergeometric functions ${}_mF_{m-1}$ at algebraic points and special L-values.

For example, a general theorem due to Beilinson and Deninger–Scholl states that the (non-critical) value of the L-series attached to a cusp form $f(\tau)$ of weight k at a positive integer $m \geq k$ belongs to $\widehat{\mathcal{P}}$.

In spite of the effective nature of the proof of the theorem, computing these L-values as periods remains a difficult problem even for particular examples.

Many such computations are motivated by (conjectural) evaluations of the logarithmic Mahler measures of multi-variate polynomials.

Elliptic curves

In the talk we will limit those "special L-values" to the L-values of elliptic curves.

An elliptic curve can be defined in many different ways.

Usually, it is a plane curve defined by $y^2 = x^3 + ax + b$, a Weierstrass equation.

Although a and b can be treated as real or complex numbers, we will assume for all practical purposes that they are in $\mathbb{Z}$.

Example. $y^2 = x^3 - x$ is an elliptic curve (of conductor 32).

The integrality of a and b makes counting possible, not only over $\mathbb{Z}$ but over any finite field $\mathbb{F}_{p^n}$.

The count can be further related to a Dirichlet-type generating function

$$L(E, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

L-series of elliptic curves

The critical line for the function is $\operatorname{Re} s = 1$, and

$$L(E,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

can be analytically continued to $\mathbb{C}$ where it satisfies a functional equation which relates $L(E,s)$ to $L(E,2-s)$.
Computing the coefficients a_n is not a simple task in general... However, thanks to the modularity theorem due A. Wiles, R. Taylor and others, the L-series can be identified with $L(f,s)$ for a cusp form of weight 2 and level N, the conductor of the elliptic curve.

Example. The L-series of $y^2 = x^3 - x$ (and of any elliptic curve of conductor 32) can be generated by

$$\sum_{n=1}^{\infty} a_n q^n = q \prod_{m=1}^{\infty} (1-q^{4m})^2 (1-q^{8m})^2.$$

Computing L-values

Computing $L(E,1)$ is "easy": it is either 0 or the period of elliptic curve E.
Computing $L(E,k)$ for $k \ge 2$ is highly non-trivial. The already mentioned results of Beilinson generalised later by Denninger–Scholl show that any such L-value can be expressed as a period.
Several examples are explicitly given for $k=2$, mainly motivated by showing particular cases of Beilinson's conjectures in K-theory and Boyd's (conjectural) evaluations of Mahler measures.
In spite of the algorithmic nature of Beilinson's method and in view of its complexity, no examples were produced so far for a single $L(E,3)$.
M. Rogers and Z. in 2010–11 created an elementary alternative to Beilinson–Denninger–Scholl to prove some conjectural Mahler evaluations.

Examples from joint work with Rogers

Because the resulting Mahler measures can be expressed entirely via hypergeometric functions, our joint results with Rogers can be stated as follows:

$$\frac{10}{\pi^2}L(E_{20},2)=\frac{5}{4}\log 2-\frac{3}{64}\,{}_4F_3\left(\begin{matrix}\frac{4}{3},\ \frac{5}{3},\ 1,\ 1\\ 2,\ 2,\ 2\end{matrix}\,\middle|\,\frac{27}{32}\right),$$

$$\frac{12}{\pi^2}L(E_{24},2)={}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ \frac{3}{2}\end{matrix}\,\middle|\,\frac{1}{4}\right)=\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{(1/8)^{2n}}{2n+1},$$

$$\frac{15}{\pi^2}L(E_{15},2)={}_3F_2\left(\begin{matrix}\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\\ 1,\ \frac{3}{2}\end{matrix}\,\middle|\,\frac{1}{16}\right)=\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{(1/16)^{2n}}{2n+1}.$$

The last two formulae resemble Ramanujan's evaluation

$$\frac{4}{\pi}\,G=\sum_{n=0}^{\infty}\binom{2n}{n}^2\frac{(1/4)^{2n}}{2n+1}$$

from one of the first slides.

Hypergeometric evaluations of $L(E_{32}, k)$

Our original method with Rogers was used for $L(E,2)$ only, but it is general enough to serve for $L(E,k)$ with $k \geq 3$.

Theorem

For an elliptic curve E of conductor 32*,*

$$L(E,2) = \frac{\pi}{16}\int_0^1 \frac{1+\sqrt{1-x^2}}{(1-x^2)^{1/4}}\,\mathrm{d}x \int_0^1 \frac{\mathrm{d}y}{1-x^2(1-y^2)}$$

$$= \frac{\pi^{1/2}\Gamma(\frac{1}{4})^2}{96\sqrt{2}}\,{}_3F_2\left(\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{7}{4},\ \frac{3}{2}\end{matrix}\,\middle|\,1\right) + \frac{\pi^{1/2}\Gamma(\frac{3}{4})^2}{8\sqrt{2}}\,{}_3F_2\left(\begin{matrix}1,\ 1,\ \frac{1}{2}\\ \frac{5}{4},\ \frac{3}{2}\end{matrix}\,\middle|\,1\right),$$

$$L(E,3) = \frac{\pi^2}{128}\int_0^1 \frac{(1+\sqrt{1-x^2})^2}{(1-x^2)^{3/4}}\,\mathrm{d}x \int_0^1\int_0^1 \frac{\mathrm{d}y\,\mathrm{d}w}{1-x^2(1-y^2)(1-w^2)}$$

$$= \frac{\pi^{3/2}\Gamma(\frac{1}{4})^2}{768\sqrt{2}}\,{}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ \frac{1}{2}\\ \frac{7}{4},\ \frac{3}{2},\ \frac{3}{2}\end{matrix}\,\middle|\,1\right) + \frac{\pi^{3/2}\Gamma(\frac{3}{4})^2}{32\sqrt{2}}\,{}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ \frac{1}{2}\\ \frac{5}{4},\ \frac{3}{2},\ \frac{3}{2}\end{matrix}\,\middle|\,1\right)$$

$$+ \frac{\pi^{3/2}\Gamma(\frac{1}{4})^2}{256\sqrt{2}}\,{}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ \frac{1}{2}\\ \frac{3}{4},\ \frac{3}{2},\ \frac{3}{2}\end{matrix}\,\middle|\,1\right).$$

Dedekind's eta-function

Below we sketch the hardest (and newest) case of $L(E,3)$.

As mentioned earlier, the L-series of an elliptic curve of conductor 32 coincides with the L-series attached to the cusp form

$$f(\tau) = \sum_{n=1}^{\infty} a_n q^n = q \prod_{m=1}^{\infty} (1-q^{4m})^2 (1-q^{8m})^2 = \eta_4^2 \eta_8^2,$$

where $q = e^{2\pi i \tau}$ for τ from the upper half-plane $\operatorname{Im}\tau > 0$,

$$\eta(\tau) := q^{1/24} \prod_{m=1}^{\infty} (1-q^m) = \sum_{n=-\infty}^{\infty} (-1)^n q^{(6n+1)^2/24}$$

is Dedekind's eta-function with its modular involution

$$\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau),$$

and $\eta_k = \eta(k\tau)$ for short.

Integral for $L(E,3)$

Taking the differential operator

$$\delta = \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau} = q\frac{\mathrm{d}}{\mathrm{d}q}$$

and it inverse

$$\delta^{-1}\colon f \mapsto \int_0^q f\,\frac{\mathrm{d}q}{q}$$

(normalised by 0 at $\tau = i\infty$ or $q = 0$), we write

$$\begin{aligned} L(E,3) = L(f,3) = \sum_{n=1}^{\infty}\frac{a_n}{n^3} = (\delta^{-3}f)|_{q=1} = \frac{1}{2}\int_0^1 f\,\log^2 q\,\frac{\mathrm{d}q}{q} \\ = 4\pi^3\int_0^{\infty} f(it)t^2\,\mathrm{d}t. \end{aligned}$$

Eisenstein-series decomposition

Note the (Lambert series) expansion

$$\frac{\eta_8^4}{\eta_4^2} = \sum_{m\ge1}\left(\frac{-4}{m}\right)\frac{q^m}{1-q^{2m}} = \sum_{\substack{m,n\ge1\\ n\text{ odd}}}\left(\frac{-4}{m}\right)q^{mn} = \sum_{m,n\ge1} a(m)b(n)q^{mn},$$

$$\text{where}\quad a(m) := \left(\frac{-4}{m}\right), \quad b(n) := n \bmod 2,$$

and $\left(\frac{-4}{m}\right)$ denotes the quadratic residue character modulo 4.
Then

$$f(it) = \eta_4^2\eta_8^2\big|_{\tau=it} = \frac{\eta_8^4}{\eta_4^2}\,\frac{\eta_4^4}{\eta_8^2}\bigg|_{\tau=it} = \frac{\eta_8^4}{\eta_4^2}\bigg|_{\tau=it}\cdot\frac{1}{2t}\,\frac{\eta_8^4}{\eta_4^2}\bigg|_{\tau=i/(32t)}$$

$$= \frac{1}{2t}\sum_{m_1,n_1\ge1} b(m_1)a(n_1)e^{-2\pi m_1n_1t}\sum_{m_2,n_2\ge1} b(m_2)a(n_2)e^{-2\pi m_2n_2/(32t)}.$$

where $t>0$ and the modular involution of Dedekind's eta-function was used.

Principal trick

Furthermore,

$$\begin{aligned} L(E,3) &= 2\pi^3 \int_0^\infty \sum_{m_1,n_1,m_2,n_2\ge1} b(m_1)a(n_1)b(m_2)a(n_2) \\ &\qquad \times \exp\left(-2\pi\left(m_1n_1t+\frac{m_2n_2}{32t}\right)\right) t\,\mathrm{d}t \\ &= 2\pi^3 \sum_{m_1,n_1,m_2,n_2\ge1} b(m_1)a(n_1)b(m_2)a(n_2) \\ &\qquad \times \int_0^\infty \exp\left(-2\pi\left(m_1n_1t+\frac{m_2n_2}{32t}\right)\right) t\,\mathrm{d}t. \end{aligned}$$

Here comes the crucial transformation of purely analytical origin: we make the change of variable $t = n_2u/n_1$.

This does not change the form of the exponential factor but affects the differential, and we obtain. . .

Principal trick (continued)

...and we obtain

$$L(E,3) = 2\pi^3 \sum_{m_1,n_1,m_2,n_2\ge 1} b(m_1)a(n_1)b(m_2)a(n_2) \times \int_0^\infty \exp\left(-2\pi\left(m_1n_1t + \frac{m_2n_2}{32t}\right)\right) t\,\mathrm{d}t$$

$$= 2\pi^3 \sum_{m_1,n_1,m_2,n_2\ge 1} \frac{b(m_1)a(n_1)b(m_2)a(n_2)n_2^2}{n_1^2} \times \int_0^\infty \exp\left(-2\pi\left(m_1n_2u + \frac{m_2n_1}{32u}\right)\right) u\,\mathrm{d}u$$

$$= 2\pi^3 \int_0^\infty \sum_{m_1,n_2\ge 1} b(m_1)a(n_2)n_2^2 e^{-2\pi m_1n_2u} \times \sum_{m_2,n_1\ge 1} \frac{b(m_2)a(n_1)}{n_1^2} e^{-2\pi m_2n_1/(32u)}\, u\,\mathrm{d}u.$$

More Eisenstein series

Furthermore,

$$\sum_{m,n\ge1} b(m)a(n)n^2q^{mn} = \sum_{\substack{m,n\ge1\\ m \text{ odd}}} \left(\frac{-4}{n}\right) n^2 q^{mn} = \frac{\eta_2^8\eta_8^4}{\eta_4^6},$$

$$\sum_{m,n\ge1} b(m)a(n)m^2q^{mn} = \sum_{\substack{m,n\ge1\\ m \text{ odd}}} \left(\frac{-4}{n}\right) m^2 q^{mn} = \frac{\eta_4^{18}}{\eta_2^8\eta_8^4},$$

so that

$$r(\tau) = \sum_{m,n\ge1} \frac{b(m)a(n)}{n^2}\, q^{mn} = \delta^{-2}\left(\frac{\eta_4^{18}}{\eta_2^8\eta_8^4}\right).$$

Back to $L(E,3)$

Continuing the previous computation,

$$L(E,3) = 2\pi^3 \int_0^\infty \frac{\eta_2^8\eta_8^4}{\eta_4^6}\bigg|_{\tau=iu} \cdot r(i/(32u))\, u\,\mathrm{d}u$$

(we apply the involution to the eta quotient)

$$= \frac{\pi^3}{8} \int_0^\infty \frac{\eta_4^4\eta_{16}^8}{\eta_8^6}\, r(\tau)\bigg|_{\tau=i/(32u)} \frac{\mathrm{d}u}{u^2}$$

(we change the variable $u = 1/(32v)$)

$$= 4\pi^3 \int_0^\infty \frac{\eta_4^4\eta_{16}^8}{\eta_8^6}\, r(\tau)\bigg|_{\tau=iv} \mathrm{d}v.$$

The real challenge of the latter expression is the Eisenstein series $r(\tau)$ of weight -1.

Ramanujan's formula

There is a standard recipe of expressing Eisenstein series of negative weight via solutions of non-homogeneous linear differential equations. It is an efficient way to write $r(\tau)$ as a "period", however a complicated way.
Accidentally, the Eisenstein series $r(\tau)$ of weight -1 possesses a different treatment because of a special formula due to Ramanujan:

$$r(\tau) = \sum_{\substack{m,n\ge1 \\ m \text{ odd}}} \left(\frac{-4}{n}\right) \frac{q^{mn}}{n^2} = \frac{\tilde{x}\, G(-\tilde{x}^2)}{4F(-\tilde{x}^2)},$$

where $\tilde{x}(\tau) = 4\eta_8^4/\eta_2^4$,

$$F(-\tilde{x}^2) = {}_2F_1\left(\begin{matrix} \frac12, \frac12 \\ 1 \end{matrix} \,\middle|\, -\tilde{x}^2\right) = \frac{2}{\pi}\int_0^1 \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1+\tilde{x}^2y^2)}} = \frac{\eta_2^4}{\eta_4^2}$$

and

$$G(z) = {}_3F_2\left(\begin{matrix} 1, 1, 1 \\ \frac32, \frac32 \end{matrix} \,\middle|\, z\right) = \int_0^1\int_0^1 \frac{\mathrm{d}y\,\mathrm{d}w}{1-z(1-y^2)(1-w^2)}.$$

$L(E,3)$ as a period

Choosing the modular function $x(\tau) = 4\eta_2^4\eta_8^8/\eta_4^{12}$ to parameterise everything and noting that $\tilde{x} = x/\sqrt{1-x^2}$ we may now write $L(E,3)$ as

$$L(E,3) = \frac{\pi^3}{64}\int_0^\infty \frac{s(x(\tau))\,x(\tau)}{1-x(\tau)^2}\,G\biggl(-\frac{x(\tau)^2}{1-x(\tau)^2}\biggr)\delta x\bigg|_{\tau=iv}\mathrm{d}v,$$

where

$$s(x) = \frac{16\eta_4^{10}\eta_{16}^8}{\eta_2^8\eta_8^{10}} = \frac{(1-\sqrt{1-x^2})^2}{x(1-x^2)^{3/4}}.$$

After performing the modular substitution $x = x(\tau)$ we finally arrive at

$$L(E,3) = \frac{\pi^2}{128}\int_0^1 \frac{(1-\sqrt{1-x^2})^2}{(1-x^2)^{3/4}}\,\mathrm{d}x\int_0^1\!\!\int_0^1 \frac{\mathrm{d}y\,\mathrm{d}w}{1-x^2(1-(1-y^2)(1-w^2))}.$$

There is still some work to do in order to identify the resulted integral with the linear combination of hypergeometric functions in the theorem. □

Hypergeometric evaluations of $L(E_{32}, k)$

Theorem

For an elliptic curve E of conductor 32,

$$L(E,2) = \frac{\pi}{16}\int_0^1 \frac{1+\sqrt{1-x^2}}{(1-x^2)^{1/4}}\,\mathrm{d}x \int_0^1 \frac{\mathrm{d}y}{1-x^2(1-y^2)}$$
$$= \frac{\pi^{1/2}\Gamma(\frac14)^2}{96\sqrt{2}}\,{}_3F_2\left(\begin{matrix}1,\ 1,\ \frac12\\ \frac74,\ \frac32\end{matrix}\,\middle|\,1\right) + \frac{\pi^{1/2}\Gamma(\frac34)^2}{8\sqrt{2}}\,{}_3F_2\left(\begin{matrix}1,\ 1,\ \frac12\\ \frac54,\ \frac32\end{matrix}\,\middle|\,1\right),$$

$$L(E,3) = \frac{\pi^2}{128}\int_0^1 \frac{(1+\sqrt{1-x^2})^2}{(1-x^2)^{3/4}}\,\mathrm{d}x \int_0^1\int_0^1 \frac{\mathrm{d}y\,\mathrm{d}w}{1-x^2(1-y^2)(1-w^2)}$$
$$= \frac{\pi^{3/2}\Gamma(\frac14)^2}{768\sqrt{2}}\,{}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ \frac12\\ \frac74,\ \frac32,\ \frac32\end{matrix}\,\middle|\,1\right) + \frac{\pi^{3/2}\Gamma(\frac34)^2}{32\sqrt{2}}\,{}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ \frac12\\ \frac54,\ \frac32,\ \frac32\end{matrix}\,\middle|\,1\right)$$
$$+ \frac{\pi^{3/2}\Gamma(\frac14)^2}{256\sqrt{2}}\,{}_4F_3\left(\begin{matrix}1,\ 1,\ 1,\ \frac12\\ \frac34,\ \frac32,\ \frac32\end{matrix}\,\middle|\,1\right).$$

A general formula?

The theorem, in fact, produces amazingly similar hypergeometric forms of $L(E,2)$ and $L(E,3)$. In the notation

$$F_k(a) := \frac{\pi^{k-1/2}\Gamma(a)}{2^{3k-1}\Gamma(a+\frac12)}\;{}_{k+1}F_k\left(\begin{matrix}\overbrace{1,\,\dots,\,1}^{k\text{ times}},\,\frac12\\ a+\frac12,\,\underbrace{\frac32,\,\dots,\,\frac32}_{k-1\text{ times}}\end{matrix}\,\middle|\,1\right),$$

relations for $L(E,2)$ and $L(E,3)$ can be alternatively written as

$$L(E,2)=F_2(\tfrac54)+F_2(\tfrac34)\quad\text{and}\quad L(E,3)=F_3(\tfrac54)+2F_3(\tfrac34)+F_3(\tfrac14).$$

In view of the known formula

$$L(E,1)=\frac{\pi^{-1/2}\Gamma(\frac14)^2}{8\sqrt2}=\frac{\pi^{-1/2}\Gamma(\frac14)^2}{24\sqrt2}\;{}_3F_2\left(\begin{matrix}1,\,\frac12\\ \frac74\end{matrix}\,\middle|\,1\right)=2F_1(\tfrac54),$$

we can conclude that, for $k=1$, 2 or 3, the L-value $L(E,k)$ can be written as a (simple) $\mathbb{Q}$-linear combination of $F_k(\frac74-\frac m2)$ for $m=1,\dots,k$. However this pattern does not seem to work for $k>3$.

Generalisations

The potentials of our method with Rogers are still "in press."
One of the latest news is period evaluations of Ramanujan's zeta function $L(\Delta, s)$ by Rogers, where

$$\Delta(\tau) = \eta(\tau)^{24} = q \prod_{m=1}^{\infty} (1 - q^m)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n,$$

for $s = k \geq 12$.
For example, he shows that

$$L(\Delta, 12) = -\frac{128\pi^{11}}{8241 \cdot 11!} \int_0^1 F(z)^5 F(1-z)^5 \times \frac{2 + 251z + 876z^2 + 251z^3 + 2z^4}{1-z} \log z \, \mathrm{d}z,$$

where as before

$$F(z) = {}_2F_1\left(\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ 1 \end{matrix} \,\middle|\, z\right) = \frac{2}{\pi} \int_0^1 \frac{\mathrm{d}y}{\sqrt{(1-y^2)(1-zy^2)}}.$$

And there are still many more conjectures on Boyd's list...

Merci

Thank you!