Various aspects of creative telescoping: a personal WZ story

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Named after two complex variables

Suppose we have a function $F(n,k)$ of (complex) variables such that both

$$\frac{F(n+1,k)}{F(n,k)} \quad \text{and} \quad \frac{F(n,k+1)}{F(n,k)}$$

are rational functions with complete factorisations into linear factors. Such $F(n,k)$ are known as hypergeometric terms.
Then by the Wilf–Zeilberger (WZ) theory there is a difference operator

$$L = a_r(n)N^r + a_{r-1}(n)N^{r-1} + \cdots + a_1(n)N + a_0(n),$$

where N^i acts as $n \mapsto n+i$ (on functions depending on n) and $a_j(n) \in \mathbb{C}[n]$, and a hypergeometric term $G(n,k)$ (in fact, such that $G(n,k)/F(n,k)$ is rational) for which

$$LF(n,k) = G(n,k+1) - G(n,k).$$

Finally, one can use their fantasy to apply this telescoping (in k) relation to showing — among many other things — that $f(n) = \sum_k F(n,k)$ (whenever it makes sense) satisfies the recursion $Lf(n) = 0$.

My WZ goal

There are further multi-variable, integral, q- and other analogues of the WZ algorithm.
For example, instead of dealing with the sum $f(n) = \sum_k F(n, k)$ one may deal with Mellin–Barnes-type integrals

$$\tilde{f}(n) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} F(n, s) w(s) \, \mathrm{d}s$$

where $w(s)$ is a suitable 1-periodic (weight) function.
WZ is a useful treatment but, at the same time, kind of decease as for those who know it (well), it often happens to be the first thing to try, even in the problems not truly designed for WZ applications.
My goal is to WZ infect much of this audience.
One may observe parallels with the q-decease. The two in fact overlap, in quite non-trivial ways. But that is another (q-WZ) story.

WZ pairs & hypergeometric series

The simplest instance of WZ theory deals with the difference operator $\mathcal{L} = N - 1$, in other words, with the (WZ) relation

$$F(n+1,k) - F(n,k) = G(n+1,k) - G(n,k).$$

It may be hard to believe but that there are many such WZ pairs (F, G).
It was observed by Gessel in the 1990s that *all* classical hypergeometric summations come this way.
At this point it is friendly to define a hypergeometric series:

$$_{m+1}F_m\left(\begin{matrix} a_0,\ a_1,\ \ldots,\ a_m \\ b_1,\ \ldots,\ b_m \end{matrix} \,\middle|\, z\right) = \sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k \cdots (a_m)_k}{k!\,(b_1)_k \cdots (b_m)_k}\, z^k,$$

where $(a)_k = \Gamma(a+k)/\Gamma(a) = a(a+1)\cdots(a+k-1)$ denotes the Pochhammer symbol, so that $(1)_k = k!$
Notice that the kth term of the hypergeometric series is a hypergeometric term in n, k for any choice of $n \in \{a_0, a_1, \ldots, a_m, b_1, \ldots, b_m\}$.

Gauss summation

As a (classical!) example consider the Gauss sum

$$\sum_{k\in\mathbb{Z}} \frac{(a)_k(b)_k}{k!\,(c)_k} = {}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix} \,\middle|\, z\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

(I will ignore technical conditions for convergence of the series).
Divide both sides by the right-hand side and denote the kth term,

$$F(a,k) = \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b)}\,\frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(1+k)},$$

so that we want to prove the sum $f(a) = \sum_k F(a,k)$ is constant 1. We take the WZ mate

$$G(a,k) = \frac{k(c+k-1)}{a(a+1-c)}\,F(a,k),$$

meaning that $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$. Summing over k we deduce $f(n+1) = f(n)$, so that $f(n)$ is constant. Finally (also applying a theorem to validate for all $n \in \mathbb{C}$), the constant is $f(0) = 1$.

WZ seeds

In 2024, Au (motivated by Gessel's considerations in the 1990s) introduced a notion of WZ seed $\Phi(a_1, \dots, a_m, k)$: it is a function such that for any choice of $A_1, \dots, A_m \in \mathbb{Z}$ and $K \in \mathbb{Z}$, the expression

$$F(n, k) = \Phi(a_1 + A_1 n, a_2 + A_2 n, \dots, a_m + A_m n, k + Kn)$$

has a WZ mate $G(n, k)$.

If such WZ seeds exist than access to many additional parameters leads to numerous benefits.

But do they exist?

Potential candidates could originate from summation formulae

$$\sum_k H(a_1, \dots, a_m, k) = h(a_1, \dots, a_m)$$

via taking $\Phi(a_1, \dots, a_m, k) = H(a_1, \dots, a_m, k)/h(a_1, \dots, a_m)$.

Classical WZ seeds

For example,

$$\Phi_{\mathrm{Gauss}}(a,b,c,k) = \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b)} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)\Gamma(1+k)}$$

is a potential WZ seed corresponding to the Gauss summation.
Remarkably, as demonstrated by Au, *all* classical summation formulae correspond to WZ seeds.
Furthermore, Au developed a strategy to specialise WZ seeds in order to target specific hypergeometric evaluations. He used them to prove many longstanding conjectural evaluations including

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^7}{n!^7}(168n^3+76n^2+14n+1)\frac{1}{2^{6n}} = \frac{32}{\pi^3}$$

going back to 2002.

A WZ seed phenomenon

In fact, with each WZ pair $(F(n,k),G(n,k))$ one can associate the WZ seed $F(a,k)$. That is, for any choice of $A,K\in\mathbb{Z}$, the bivariate hypergeometric term $F(a+An,k+Kn)$ has a WZ mate, so it leads to infinity of identities.

(Un)fortunately, there are still many similar-looking evaluations left open; an example is Guillera's conjecture

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac18)_n(\frac38)_n(\frac58)_n(\frac78)_n}{n!^5}(1920n^2+304n+15)\frac{1}{7^{4n}}\overset{?}{=}\frac{56\sqrt{7}}{\pi^2}.$$

The seed technology naturally extends for proving 'divergent' hypergeometric evaluations, when sums are replaced with Mellin–Barnes integrals. Particulars of the theory are developed by Peters (master thesis) based on earlier works of Guillera; one notable example is the evaluation

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{(\frac12)_s^5(\frac15)_s(\frac25)_s(\frac35)_s(\frac45)_s}{(1)_s^9}(5532s^4+5600s^3+2275s^2+425s+30)\left(\frac{3125}{1024}\right)^s\frac{\mathrm{d}s}{\sin\pi s}=\frac{1280}{\pi^4},$$

where $(a)_s=\Gamma(a+s)/\Gamma(a)$ (and $3125=5^5$, $1024=2^{10}$).

Holonomic rank

In fact, a proof of the fact that every (multi-parameter) WZ pair gives rise to a WZ seed is quite elementary. It is related to a notion of *holonomic rank* of the hypergeometric sum, which basically means how many *contiguous* sums

$$\sum_k \Phi(a_1+n_1,\dots,a_m+n_m,k) \quad \text{over all } n_1,\dots,n_m\in\mathbb{Z}$$

are linearly independent (over $\mathbb{C}(a_1,\dots,a_m)$, say). The WZ-pair property basically translates into 'the holonomic rank is 1.'

And what if the rank is greater than 1?
For example, it is equal to 2 in the case of generic ${}_2F_1$ hypergeometric sum, with the term

$$\frac{(a)_k(b)_k}{k!\,(c)_k}\,z^k.$$

How to detect (or even design) the situations when the rank drops to 1 (meaning that we will obtain closed-form evaluations)?

Closed forms from long recursions

A recipe for such rank-dropping was offered by Zeilberger in 2004 (though in a slightly different language); it was rediscovered some 10 years later by Ebisu (also in a different setup).
The methodology shares clear similarities with WZ seeds (introduced even later): for a hypergeometric term $\Phi(a_1,\dots,a_m,k;z)$ (we also care of the parameter z), we choose and fix $A_1,\dots,A_m\in\mathbb{Z}$ and record the telescoper (of the order equal to the generic rank of the sums $\sum_k\Phi(\dots;k)$) for

$$\Phi(a_1+A_1t,\dots,a_m+A_mt,k;z)$$

as a difference operator in $T\colon t\mapsto t+1$, where $a_1,\dots,a_m,z$ are symbolic parameters. Then we look for the choice of these parameters to annihilate the leading coefficient(s) in T.

Examples

For example, applying the strategy to $\dfrac{(a+2t)_k(b+2t)_k}{k!\,(c+t)_k}\,z^k$, we find out a first-order recursion for the choice $z=-\frac{1}{8}$, $a=a$, $b=a+\frac{1}{3}$, $c=\frac{a}{2}+\frac{5}{6}$ which leads to the evaluation

$$ {}_2F_1\left(2t, 2t+\frac{1}{3}; t+\frac{5}{6} \;\middle|\; -\frac{1}{8}\right) = \left(\frac{16}{27}\right)^t \frac{\Gamma(t+\frac{5}{6})\Gamma(\frac{2}{3})}{\Gamma(t+\frac{2}{3})\Gamma(\frac{5}{6})}. $$

The methodology is capable of producing rank reductions for higher rank, with an example

$$ t(4t-1)(4t+1)f(t+1)-(2t+1)(10t^2-10t+3)f(t)+(t-1)(2t-1)(2t+1)f(t-1)=0 $$

for

$$ f(t) = {}_3F_2\left(\begin{matrix}\frac{1}{2},\ t,\ 1-t\\ 1,\ \frac{1}{2}+t\end{matrix} \;\middle|\; \frac{1}{4}\right). $$

q-Analogues(?)

Unlike the WZ-seed story, there is no clear q-extension of rank-dropping methodology.
We only know a few sporadic examples that originate from q-series/combinatorics, like

$$\sum_{k=0}^{\infty} \frac{(q^2/a;q^2)_k (a;q)_k}{(q^3;q^3)_k (a^2q;q^2)_k} (-1)^k q^{\binom{k+1}{2}} a^k = \frac{(aq;q^2)_\infty (a^3q^3;q^6)_\infty}{(a^2q;q^2)_\infty (q^3;q^6)_\infty}$$

(due to Shane) which is a q-analogue of Ebisu's ${}_2F_1$-evaluation

$$\sum_{k=0}^{\infty} \frac{(1-a)_k (2a)_k}{k!\,(2a+\frac{1}{2})_k} \left(-\frac{1}{3}\right)^k = \frac{\Gamma(\frac{1}{2})\Gamma(2a+\frac{1}{2})}{3^a \Gamma(a+\frac{1}{2})^2}.$$

Can this methodology be combined with WZ seeds to access challenging evaluations?

(Matrix-valued) orthogonal polynomials

Assume that $W = W(x) = W(x)^*$ (Hermitian) is a positive definite $(N+1) \times (N+1)$ matrix on an interval $(a, b) \subset \mathbb{R}$. With this *weight function* we associate the matrix inner product

$$\langle P(x), Q(x) \rangle = \langle P(x), Q(x) \rangle_W = \int_a^b P(x) W(x) Q(x)^* \, \mathrm{d}x$$

and a related sequence of (matrix-valued!) orthogonal polynomials $P_n(x)$ of degree n, so that

$$\langle P_n(x), P_m(x) \rangle = \begin{cases} \text{non-degenerate matrix} & \text{if } n = m, \\ \text{zero matrix} & \text{otherwise.} \end{cases}$$

One can show then that the leading (nth) coefficient in the expansion $P_n(x) = \Lambda_n x^n + \cdots + \Lambda_1 x + \Lambda_0$ is a non-degenerate matrix (with coefficients in $\mathbb{R}$).

3-term relations

Normalising the sequence $P_0(x) = \mathrm{Id}$ and taking $P_{-1}(x)$ to be the zero matrix, a theorem of Favard extends to the matrix-valued setting and guarantees the existence of a 3-term recurrence relation

$$xP_n(x) = A_{n+1}P_{n+1}(x) + B_nP_n(x) + A_n^*P_{n-1}(x) \quad \text{for } n = 0, 1, 2, \dots .$$

One can also use the monic renormalisation $\widetilde{P}_0(x) = \Lambda_0^{-1}$, which changes the recursion to

$$x\widetilde{P}_n(x) = \widetilde{P}_{n+1}(x) + B_n\widetilde{P}_n(x) + C_n\widetilde{P}_{n-1}(x) \quad \text{for } n = 0, 1, 2, \dots,$$

and the leading coefficient of each $\widetilde{P}_n(x)$ is the identity matrix.
The case of 1×1 matrices corresponds to classical orthogonal polynomials (like Legendre and Jacobi polynomials).

Gegenbauer (*aka* ultraspherical) polynomials

The scalar Gegenbauer polynomials (which form a subfamily of the Jacobi polynomials) can be put in the hypergeometric form

$$C_n^{(\nu)}(x) = \frac{(2\nu)_n}{n!} \cdot {}_2F_1\left(\begin{matrix} -n,\, 2\nu+n \\ \nu+\frac{1}{2} \end{matrix} \,\middle|\, \frac{1-x}{2}\right);$$

they are orthogonal on $(-1,1)$ with respect to the measure $(1-x^2)^{\nu-\frac{1}{2}}$, satisfy the 3-term recurrence relation

$$2(n+\nu)\,x\,C_n^{(\nu)}(x) = (n+1)\,C_{n+1}^{(\nu)}(x) + (n+2\nu-1)\,C_{n-1}^{(\nu)}(x),$$

and can be generated via the *algebraic* generating function

$$\sum_{n=0}^{\infty} C_n^{(\lambda)}(x)t^n = \frac{1}{(1-2xt+t^2)^{\lambda}}.$$

They also satisfy the connection formula

$$C_m^{(\nu)}(x) = \sum_{s=0}^{\lfloor m/2 \rfloor} \frac{(\lambda+m-2s)\,(\nu)_{m-s}}{(\lambda)_{m-s+1}}\,\frac{(\nu-\lambda)_s}{s!}\,C_{m-2s}^{(\lambda)}(x),$$

which in the particular case $\lambda = \nu + N$ for N non-negative integer, is for $s \leqslant N$.

Matrix weight

For $\ell \in \frac{1}{2}\mathbb{Z}_+$ and $\nu > 0$, the matrix-valued weight inspired by considerations from Representation Theory is the $(2\ell+1)\times(2\ell+1)$ matrix-valued function $W(x) = W^{(\nu)}(x)$ given by

$$\big(W(x)\big)_{i,j} = (1-x^2)^{\nu-\frac{1}{2}} \sum_{k=\max\{0,i+j-2\ell\}}^{\min\{i,j\}} \alpha_k(i,j)\, C^{(\nu)}_{i+j-2k}(x),$$

$$\alpha_k(i,j) = (-1)^k \frac{i!\, j!\, (i+j-2k)!}{k!\,(2\nu)_{i+j-2k}\,(\nu)_{i+j-k}} \frac{(\nu)_{i-k}\,(\nu)_{j-k}}{(i-k)!\,(j-k)!} \frac{i+j-2k+\nu}{i+j-k+\nu}$$
$$\times \frac{(2\ell-i)!\,(2\ell-j)!}{(2\ell+k-i-j)!}\,(-2\ell-\nu)_k\,\frac{2\ell+\nu}{(2\ell)!},$$

where $i, j \in \{0, 1, \dots, 2\ell\}$.

The *monic* matrix-valued Gegenbauer polynomials $P_n^{(\nu)}(x)$, $n = 0, 1, \dots,$ are orthogonal with respect to $W^{(\nu)}(x)$.

3-term for MV Gegenbauers

The polynomials were originally constructed by Koelink, de los Ríos and Román (2017); they gave explicit formulae for them (quite involved triple binomial-sum expressions) and deduced the 3-term relation

$$xP_n^{(\nu)}(x) = P_{n+1}^{(\nu)}(x) + B_n^{(\nu)} P_n^{(\nu)}(x) + C_n^{(\nu)} P_{n-1}^{(\nu)}(x),$$

where the matrices $B_n^{(\nu)}$ and $C_n^{(\nu)}$ are given by

$$\begin{aligned} B_n^{(\nu)} &= \sum_{j=1}^{2\ell} \frac{j(j+\nu-1)}{2(j+n+\nu-1)(j+n+\nu)} E_{j,j-1} \\ &\quad + \sum_{j=0}^{2\ell-1} \frac{(2\ell-j)(2\ell-j+\nu-1)}{2(2\ell-j+n+\nu-1)(2\ell+n-j+\nu)} E_{j,j+1}, \\ C_n^{(\nu)} &= \sum_{j=0}^{2\ell} \frac{n(n+\nu-1)(2\ell+n+\nu)(2\ell+n+2\nu-1)}{4(2\ell+n+\nu-j-1)(2\ell+n+\nu-j)(j+n+\nu-1)(j+n+\nu)} E_{j,j}. \end{aligned}$$

Here $E_{i,j}$ denotes the matrix with 1 on the (i,j)-entry and 0 elsewhere.

Renormalisation

Renormalising the monic polynomials,

$$\hat{P}_n^{(\nu)}(x) = \binom{2\ell}{i} \frac{(\nu+n)_i}{(\nu+n+2\ell-i)_i} \cdot P_n^{(\nu)}(x),$$

we obtain the family satisfying a dimension-free property

$$\frac{\mathrm{d}\hat{P}_n^{(\nu)}}{\mathrm{d}x}(x) = n\hat{P}_{n-1}^{(\nu+1)}(x).$$

In our joint work with Koelink and Román, we give different — and more 'transparent' — formulae for the matrix-valued Gegenbauer polynomials

$$\hat{P}_n^{(\nu)}(x) = \sum_{k=0}^{n} F_{k,n}^{(\nu)}\, C_{n-k}^{(\nu+2\ell)}(x),$$

and we also expressed the classical Gegenbauer polynomials in terms of the matrix-valued ones:

$$C_m^{(\nu)}(x)\,\mathrm{Id} = \sum_{r=0}^{m} G_{r,m}^{(\nu)} \hat{P}_{m-r}^{(\nu)}(x).$$

Matrix-valued hypergeometry

Explicitly, $(F^{(\nu)}_{k,n})_{i,j} = \frac{n!\,\Gamma(\nu+2\ell)}{2^n}\gamma(\nu+n;i,j,k)$ where for $i,j,k \in \{0,1,\dots,2\ell\}$,

$$\begin{aligned}\gamma(\nu;i,j,k) = \gamma_\ell(\nu;i,j,k)\\ &= (-1)^k(\nu+2\ell)(\nu+2\ell-k)\binom{2\ell}{k}\binom{k}{\frac12(k+i-j)}\binom{2\ell-k}{\frac12(i+j-k)}\\ &\times \frac{\Gamma(\nu-\frac12(k-i-j))\,\Gamma(\nu+2\ell-\frac12(k+i+j))}{\Gamma(\nu)\,\Gamma(\nu+2\ell+1-\frac12(k-i+j))\,\Gamma(\nu+2\ell+1-\frac12(k+i-j))}\end{aligned}$$

if $i+j \equiv k \bmod 2$ and $\gamma(\nu;i,j,k)=0$ otherwise.

Similarly, for the dual situation, $(G^{(\nu)}_{r,m})_{i,j} = \dfrac{2^{m-r}}{(m-r)!\,\Gamma(\nu)}\cdot\phi(\nu+m;i,j,r)$ where

$$\begin{aligned}&\phi(\nu;i,j,r) = \phi_\ell(\nu;i,j,r)\\ &= \binom{2\ell}{r}\binom{r}{\frac12(r+i-j)}\binom{2\ell-r}{\frac12(i+j-r)}\binom{2\ell}{i}^{-1}\binom{2\ell}{j}^{-1}(\nu-r+j)(\nu+2\ell-r-j)\\ &\times \frac{\Gamma(\nu+2\ell-r)\,\Gamma(\nu-\frac12(r-i+j))\,\Gamma(\nu-\frac12(r+i-j))}{\Gamma(\nu+1-\frac12(r-i-j))\,\Gamma(\nu+2\ell+1-\frac12(r+i+j))} \quad \text{if } i+j\equiv r \bmod 2.\end{aligned}$$

Telescoping in matrix setting

Proof of the formula for $\hat{P}_n^{(\nu)}(x)$ modulo the known 3-term relation, by comparing the coefficients of same scalar Gegenbauers, reduces to verification of the rational function identity

$$
\begin{aligned}
&\frac{n-k}{\nu+2\ell+n-k-1}\,\gamma(\nu+n;i,j,k+1)+\frac{2(\nu+2\ell)+n-k}{\nu+2\ell+n-k+1}\,\gamma(\nu+n;i,j,k-1)\\
&\quad=\frac{(n+1)(\nu+n)(\nu+2\ell+n)}{(\nu+n+i)(\nu+2\ell+n-i)}\,\gamma(\nu+n+1;i,j,k+1)\\
&\qquad+\frac{(\nu+i-1)(2\ell-i+1)}{(\nu+n+i)(\nu+2\ell+n-i)}\,\gamma(\nu+n;i-1,j,k)\\
&\qquad+\frac{(i+1)(\nu+2\ell-i-1)}{(\nu+n+i)(\nu+2\ell+n-i)}\,\gamma(\nu+n;i+1,j,k)\\
&\qquad+\frac{(\nu+2\ell+n)(2\nu+2\ell+n-1)}{(\nu+n+i)(\nu+2\ell+n-i)(\nu+2\ell+n-1)}\,\gamma(\nu+n-1;i,j,k-1)=0.
\end{aligned}
$$

The converse relation

Surprisingly, to express $C_m^{(\nu)}(x)\,\mathrm{Id}$ in terms of $\hat{P}_n^{(\nu)}(x)$ we could not find a matching telescoping argument, so our proof makes use of the orthogonality relations and known identities for the classical Gegenbauer polynomials.

The connection formulae lead to several remarkable consequences, in particular, they demonstrate the symmetry (along the principal diagonal) of the polynomials $\hat{P}_n^{(\nu)}(x)$. They also allow us to write generating functions of matrix-valued Gegenbauers and write down new differential operators for them.

Is there a creative telescoping algorithm to work efficiently in matrix (non-commutative!) setting? What are matrix-valued hypergeometric identities?

In a slightly different direction, Gegenbauer polynomials are 'diagonal specialisations' of Jacobi polynomials in the scalar case. Insights from Representation Theory are insufficient for constructing matrix-valued Jacobi polynomials (of size > 2). Is there a way to extrapolate the latter?

π

In a joint project with Zeilberger (2020) we took a careful look at existing constructions of rational approximations to $\pi = 3.1415926\ldots$, in order to improve its irrationality measure.

The best known ones at that moment were Salikhov's approximations

$$\begin{aligned} I_{A,B}(n) &= -i\int_{4-2i}^{4+2i} \frac{(x-4+2i)^{2An}(x-4-2i)^{2An}(x-5)^{2An}(x-6+2i)^{2An}(x-6-2i)^{2An}}{x^{2Bn+1}(x-10)^{2Bn+1}}\,\mathrm{d}x \\ &= a_n\pi - b_n \end{aligned}$$

for some rational $a_n = a_n(A,B)$ and $b_n = b_n(A,B)$.
Salikhov used the integrals with $A = 3$, $B = 5$, and his theoretical framework — control of the size of $I(n)$ and of a_n, b_n and their denominators, as $n \to \infty$, indeed performed best possibly for this choice.

Experimenting in Irrational Number Theory

Computing the integrals

$$I_{A,B}(n) = -i \int_{4-2i}^{4+2i} \frac{(x-4+2i)^{2An}(x-4-2i)^{2An}(x-5)^{2An}(x-6+2i)^{2An}(x-6-2i)^{2An}}{x^{2Bn+1}(x-10)^{2Bn+1}} \, \mathrm{d}x$$

directly when $n > 10$ is not affordable.
Remarkably enough, for any fixed A, B we can run a version of the algorithm of creative telescoping to find out a 4-term recurrence relation for $I(n)$, a_n and b_n. Thus, knowing them for $n = 0, 1, 2$ gives simple access to the sequences for *any* n (well, realistically up to $100,000$ say).
For $A = 2$, $B = 3$ we detected experimentally much better arithmetic properties (compared with those of Salikhov's). Proving them required an invention of new arithmetic tools into the existing methodology. But it was worth that! — The bound $\mu(\pi) \leqslant 7.103205334137\dots$ for the irrationality measure of π remains best possible since 2020.
There are many more instances of this experimental strategy.

WZ Thanks!


Thank you
Merci
Thank you
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Thank you
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