Ramanujan-type formulae for $1/\pi$: A second wind?*

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Abstract

In 1914 S. Ramanujan recorded a list of 17 series for $1/\pi$. We survey the methods of proofs of Ramanujan's formulae and indicate recently discovered generalisations, some of which are not yet proven.

Let us start with two significant events of the 20th century, in the opposite historical order. At first glance, the stories might be thought of a different nature.

In 1978, R. Apéry showed the irrationality of $\zeta(3)$ (see [4] and [13]). His rational approximations to the number in question (known nowadays as the *Apéry constant*) have the form $v_n/u_n \in \mathbb{Q}$ for $n = 0, 1, 2, \dots$, where the denominators $\{u_n\} = \{u_n\}_{n=0,1,\dots}$ and numerators $\{v_n\} = \{v_n\}_{n=0,1,\dots}$ satisfy the same polynomial recurrence

$$(n+1)^3 u_{n+1} - (2n+1)(17n^2 + 17n + 5)u_n + n^3 u_{n-1} = 0 \tag{1}$$

with the initial data

$$u_0 = 1, \quad u_1 = 5, \qquad v_0 = 0, \quad v_1 = 6. \tag{2}$$

Then

$$\lim_{n\to\infty} \frac{v_n}{u_n} = \zeta(3) \tag{3}$$

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and, surprisingly, the denominators $\{u_n\}$ are integers:

$$u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \in \mathbb{Z}, \qquad n = 0, 1, 2, \dots, \tag{4}$$

while the numerators $\{v_n\}$ are 'close' to being integer.

In 1914, S. Ramanujan [14] recorded a list of 17 series for $1/\pi$, from which we exclude the simplest one

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{n!^3}(4n+1) \cdot (-1)^n = \frac{2}{\pi} \tag{5}$$

and also two quite impressive examples

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3}(21460n+1123) \cdot \frac{(-1)^n}{882^{2n+1}} = \frac{4}{\pi}, \tag{6}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3}(26390n+1103) \cdot \frac{1}{99^{4n+2}} = \frac{1}{2\pi\sqrt{2}} \tag{7}$$

allowing to produce fast approaching (rational) approximations to π. Here

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1) & \text{for } n \geq 1, \\ 1 & \text{for } n = 0, \end{cases}$$

denotes the Pochhammer symbol (the shifted factorial). The Pochhammer products occuring in all formulae of this type may be written by means of the binomial coefficients:

$$\frac{(\frac{1}{2})_n^3}{n!^3} = 2^{-6n}\binom{2n}{n}^3, \qquad \frac{(\frac{1}{3})_n(\frac{1}{2})_n(\frac{2}{3})_n}{n!^3} = 2^{-2n}3^{-3n}\binom{2n}{n}\frac{(3n)!}{n!^3},$$

$$\frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!^3} = 2^{-8n}\frac{(4n)!}{n!^4}, \qquad \frac{(\frac{1}{6})_n(\frac{1}{2})_n(\frac{5}{6})_n}{n!^3} = 12^{-3n}\frac{(6n)!}{n!(2n)!(3n)!}.$$

Ramanujan's original list has been later extended by several other series, which we plan to touch later; for the moment we indicate two more celebrating examples:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{3})_n(\frac{1}{2})_n(\frac{2}{3})_n}{n!^3}(14151n+827) \cdot \frac{(-1)^n}{500^{2n+1}} = \frac{3\sqrt{3}}{\pi}, \tag{8}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{6})_n(\frac{1}{2})_n(\frac{5}{6})_n}{n!^3}(545140134n+13591409) \cdot \frac{(-1)^n}{53360^{3n+2}} = \frac{3}{2\pi\sqrt{10005}}. \tag{9}$$

Formula (8) is proven by H. H. Chan, W.-C. Liaw and V. Tan [7] and (9) is famous Chudnovskys' formula [8], which enabled them to hold the record in the calculation of π in 1989–94. On the left-hand side of each formula (5)–(9) we have linear combinations of a (generalised) hypergeometric series

$$ {}_mF_{m-1}\left(\begin{matrix} a_1, a_2, \dots, a_m \\ b_2, \dots, b_m \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_m)_n}{(b_2)_n \cdots (b_m)_n} \frac{z^n}{n!} $$

and its derivative at a point close to the origin. The fast convergence of the series in (6)–(9) may be used for proving the quantitative irrationality of the numbers $\pi\sqrt{d}$ with $d \in \mathbb{N}$ (see [16] for details).

In both Ramanujan's and Apéry's cases, there were just hints on how the things might be proven. Rigorous proofs appeared somewhen later. We will not discuss proofs of Apéry's theorem and its further generalisations, just concentrating on the things around remarkable Ramanujan-type series. Nevertheless, both Ramanujan's and Apéry's discoveries have several common grounds, which will be underlined during the talk.

Although Ramanujan did not indicate how he arrived at his series, he hinted the belonging of these series to what is now known as 'the theories of elliptic functions to alternative bases'. First rigorous mathematical proofs of Ramanujan's series and their generalisations were given by Borweins [6] and Chudnovskys [8]. Let us sketch, following [8], basic ideas of that very first proofs.

One starts with an elliptic curve $y^2 = 4x^3 - g_2x - g_3$ over $\overline{\mathbb{Q}}$ with fundamental periods ω_1, ω_2 (where $\operatorname{Im}(\omega_2/\omega_1) > 0$) and corresponding quasi-periods η_1, η_2. Besides the Legendre relation

$$ \eta_1\omega_2 - \eta_2\omega_1 = 2\pi i, \tag{10} $$

in the CM (complex multiplication) case, i.e. when $\tau = \omega_2/\omega_1 \in \mathbb{Q}[\sqrt{-d}]$ for some $d \in \mathbb{N}$, the following *linear* relations between $\omega_1, \omega_2, \eta_1, \eta_2$ over $\overline{\mathbb{Q}}$ are available:

$$ \omega_2 - \tau\omega_1 = 0, \qquad A\tau\eta_2 - C\eta_1 + (2A\tau + B)\alpha\omega_1 = 0, \tag{11} $$

where integers A, B and C come from the equation $A\tau^2 + B\tau + C = 0$ defining the quadratic number τ and $\alpha \in \mathbb{Q}(\tau, g_2, g_3) \subset \overline{\mathbb{Q}}$. Equations (11) allow one to express ω_2, η_2 by means of ω_1, η_1 only. Substituting these expressions into (11), and using the hypergeometric formulae for ω_1, η_1 and also for $\omega_1^2, \omega_1\eta_1$ (followed from Clausen's identity) one finally arrives at a formula of the Ramanujan type. An important (and complicated) problem in the proof is computing the algebraic number

$$\alpha = \frac{2\pi^2}{9}\left(E_2(\tau) - \frac{3}{\pi \operatorname{Im}\tau}\right), \qquad \text{where} \quad E_2(\tau) = 1 - 24\sum_{n=1}^{\infty} e^{2\pi i n\tau}\sum_{d|n} d.$$

Note that α viewed as the function of τ is a non-holomorphic modular form of weight 2. Chudnovskys attribute the knowledge of the fact that $\alpha(\tau)$ takes values in the Hilbert class field $\mathbb{Q}(\tau, j(\tau))$ of $\mathbb{Q}(\tau)$ to Kronecker (in Weil's presentation [15]).

Understanding of complication of the above proof came in 2002 with T. Sato's discovery of the remarkable formula

$$\sum_{n=0}^{\infty} u_n \cdot (20n + 10 - 3\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^{12n} = \frac{20\sqrt{3} + 9\sqrt{15}}{6\pi} \tag{12}$$

of Ramanujan type, involving Apéry's numbers (4). The modular argument was essentially simplified by Heng Huat Chan with his collaborators and later by Yi-Fan Yang to produce a lot of new identities like (12) based on a not necessary hypergeometric series $F(z) = \sum_{n=0}^{\infty} u_n z^n$. Examples are

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}\binom{2n-2k}{n-k} \cdot (5n+1)\frac{(-1)^n}{64^n} = \frac{8}{\pi\sqrt{3}}$$

due to H. H. Chan, S. H. Chan and Z.-G. Liu (2003);

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/3]} (-1)^{n-k} 3^{n-3k} \frac{(3k)!}{k!^3}\binom{n}{3k}\binom{n+k}{k} \cdot (4n+1)\frac{1}{81^n} = \frac{3\sqrt{3}}{2\pi}$$

due to H. H. Chan and H. Verrill (2005);

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k}^4 \cdot (4n+1)\frac{1}{36^n} = \frac{18}{\pi\sqrt{15}}$$

due to Y. Yang (2005).

Remark. It should be now mentioned that the Picard–Fuchs differential equations (of order 3) satisfied by the series $F(z)$ always have very nice arithmetic properties. Therefore, it is not surprising that $F(z)$ admits a modular parametrization: $f(\tau) = F(z(\tau))$ is a modular form of weight 2 for a modular (uniformizing) substitution $z = z(\tau)$.

Let us follow Yang's argument to show basic ideas of the new proof on the example of (12). Our choice is

$$z(\tau)=\left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^{12}, \qquad f(\tau)=\frac{\eta(2\tau)^7\eta(3\tau)^7}{\eta(\tau)^5\eta(6\tau)^5},$$

which are modular forms of level 6. The function $g(\tau)=(2\pi i)^{-1}f'(\tau)/f(\tau)$ satisfies the functional equation

$$g(\gamma\tau)=\frac{c(c\tau+d)}{\pi i}+(c\tau+d)^2g(\tau) \qquad \text{for} \quad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(6)+w_6,$$

where w_6 denotes the Atkin–Lehner involution. Taking

$$\gamma=\frac{1}{\sqrt{6}}\begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}, \qquad \tau=\tau_0=\frac{i}{\sqrt{30}}$$

we obtain

$$g(\tau_0)+5g(5\tau_0)=\frac{\sqrt{30}}{\pi}. \tag{13}$$

On the other hand, $h(\tau)=g(\tau)-5g(5\tau)$ is a modular form of weight 2 and level 30 (on $\Gamma_0(30)+\langle w_5,w_6\rangle$). This implies that $h(\tau)/f(\tau)$ is an algebraic function of $z(\tau)$ and after explicit evaluations at $\tau=\tau_0$ we arrive at

$$g(\tau_0)-5g(5\tau_0)=h(\tau_0)=\frac{900\sqrt{2}-402\sqrt{10}}{5}f(\tau_0)=(900\sqrt{2}-402\sqrt{10})f(5\tau_0). \tag{14}$$

Thus combining (13) and (14) we deduce that

$$\frac{\sqrt{30}}{\pi}=(900\sqrt{2}-402\sqrt{10})f(5\tau_0)+10g(5\tau_0),$$

and it remains to use the expansion

$$g(5\tau_0)=z\frac{\mathrm{d}f/\mathrm{d}z}{f}\cdot\frac{1}{2\pi i}\frac{z'(\tau)}{z(\tau)}\bigg|_{\tau=5\tau_0}=(108\sqrt{2}-48\sqrt{10})\sum_{n=0}^{\infty}nu_n\cdot z(5\tau_0)^n$$

and evaluation

$$z(5\tau_0)=z(\tau_0)=161-72\sqrt{5}=\left(\frac{\sqrt{5}-1}{2}\right)^{12}.$$

There is yet another proof but available for a small amount of Ramanujan-type series. It is based on the algorithm of creative telescoping, due to Gosper–Zeilberger, that was a crucial point in the first proof of Apéry's theorem [13]. D. Zeilberger (and his automatic collaborator S. Ekhad) could prove the simplest Ramanujan's identity (5) in the following way. One verifies the (terminating) identity

$$\sum_{n=0}^{\infty} \frac{(1/2)_n^2(-k)_n}{n!^2(3/2+k)_n}(4n+1)(-1)^n = \frac{\Gamma(3/2+k)}{\Gamma(3/2)\Gamma(1+k)} \tag{15}$$

for all *non-negative* integers k. To do this, divide both sides of (15) by the right-hand side and denote the summand on the left by $F(n,k)$; then take

$$G(n,k) = \frac{(2n+1)^2}{(2n+2k+3)(4n+1)} F(n,k)$$

with the motive $F(n,k+1) - F(n,k) = G(n,k) - G(n-1,k)$, hence $\sum_n F(n,k)$ is a constant, which is seen to be 1 by plugging $k=0$. To deduce finally (5) one takes $k=-1/2$, which is legitimate in view of Carlson's theorem.

If one wishes to use the latter proof for another Ramanujan-type formula, then an ingenuity for putting the new parameter k on a right place is required. This was done only recently by J. Guillera [10, 12], who proved by the method some other Ramanujan's identies (in those cases when z has only 2 and 3 in its prime decomposition). If one now doubts on the applicability of the method, then take into account that a pure hypergeometric origin of the method and its independence on the modular stuff allowed Guillera [10, 11, 12] to prove new generalisations of Ramanujan-type series, namely,

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{n!^5}(20n^2+8n+1)\frac{(-1)^n}{2^{2n}} = \frac{8}{\pi^2}, \tag{16}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{n!^5}(820n^2+180n+1)\frac{(-1)^n}{2^{10n}} = \frac{128}{\pi^2}, \tag{17}$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3(\frac{1}{4})_n(\frac{3}{4})_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}} = \frac{32}{\pi^2}, \tag{18}$$

and also to find experimentally [11] four additional formulae

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n(\frac16)_n(\frac56)_n}{n!^5}(1640n^2+278n+15)\frac{(-1)^n}{2^{10n}}=\frac{256\sqrt3}{3\pi^2}, \tag{19}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n(\frac13)_n(\frac23)_n}{n!^5}(252n^2+63n+5)\frac{(-1)^n}{48^n}=\frac{48}{\pi^2}, \tag{20}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac13)_n(\frac23)_n(\frac16)_n(\frac56)_n}{n!^5}(5418n^2+693n+29)\frac{(-1)^n}{80^{3n}}=\frac{128\sqrt5}{\pi^2}, \tag{21}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac18)_n(\frac38)_n(\frac58)_n(\frac78)_n}{n!^5}(1920n^2+304n+15)\frac{1}{7^{4n}}=\frac{56\sqrt7}{\pi^2}. \tag{22}$$

As Guillera notes, the series in (20)–(22) are closely related to the ('modular' proven) series

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(28n+3)\frac{(-1)^n}{48^n}=\frac{16}{\pi\sqrt3},$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac16)_n(\frac56)_n}{n!^3}(5418n+263)\frac{(-1)^n}{80^{3n}}=\frac{640\sqrt{15}}{3\pi},$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(40n+3)\frac{1}{7^{4n}}=\frac{49}{3\pi\sqrt3},$$

respectively. However, there is no obvious way to deduce any of formulae (16)–(22) by modular means; a problem lies upon the fact that the (Zariski closure of the) projective monodromy group for corresponding series $F(z)=\sum_{n=0}^{\infty}u_nz^n$ is always $O_5(\mathbb{R})$ that is essentially richer than $O_3(\mathbb{R})$ for classical Ramanujan's series. There exists also the following higher-dimensional identity

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^7}{n!^7}(168n^3+76n^2+14n+1)\frac{1}{2^{6n}}=\frac{32}{\pi^3},$$

discovered by B. Gourevich in 2002 (by using an integer relations algorithm). Guillera also found experimentally an analogue of Sato's series:

$$\sum_{n=0}^{\infty}v_n\cdot(36n^2+12n+1)\frac{1}{2^{10n}}=\frac{32}{\pi^2},$$
$$\text{where}\quad v_n=\binom{2n}{n}^2\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2n-2k}{n-k}^2. \tag{23}$$

Using the quadratic transformation $z \mapsto -4z/(1-z)^2$ of the hypergeometric series we were able to produce from (16), (17) two more series of the latter type [17]:

$$\sum_{n=0}^{\infty} w_n \frac{(4n)!}{n!^2(2n)!}(18n^2 - 10n - 3)\frac{1}{(2^8 5^2)^n} = \frac{10\sqrt{5}}{\pi^2}, \tag{24}$$

$$\sum_{n=0}^{\infty} w_n \frac{(4n)!}{n!^2(2n)!}(1046529n^2 + 227104n + 16032)\frac{1}{(5^4 41^2)^n} = \frac{5^4 41\sqrt{41}}{\pi^2}, \tag{25}$$

where the sequence of integers

$$w_n = \sum_{k=0}^{n} \binom{2k}{k}^3 \binom{2n-2k}{n-k} 2^{4(n-k)}, \qquad n = 0, 1, 2, \dots,$$

satisfies the recurrence relation

$$(n+1)^3 w_{n+1} - 8(2n+1)(8n^2+8n+5)w_n + 4096n^3 w_{n-1} = 0, \qquad n = 1, 2, \dots .$$

It is worth mentioning that identities like (15) are valid for all non-negative *real* values of k. This fact has several other curious implications; for instance, the series

$$G(k) = \sum_{n=0}^{\infty} \frac{(1/2+k)_n^5}{(1+k)_n^5}\bigl(820(n+k)^2 + 180(n+k) + 13\bigr)\frac{(-1)^n}{2^{10n}}, \tag{26}$$

has closed-form evaluation at $k = 0$ and $k = 1/2$:

$$G(0) = \frac{128}{\pi^2} \qquad \text{and} \qquad G(\tfrac{1}{2}) = 256\zeta(3),$$

where the first formula follows from (17), while the second one was given by T. Amdeberhan and D. Zeilberger [3]. Guillera has (proven and conjectured) evaluations for the series like (26) viewed as a function of the continious (complex or real) parameter k.

It seems to be a challenge to develop a modular-like theory for proving Guillera's identities and finding out a (more or less) general pattern of them. For the moment, we have only speculations in this respect on a relationship to the mirror symmetry, namely, to the linear differential equations for the periods of certain Calabi–Yau threefolds. A standard example here is the hypergeometric series (cf. (16) and (17))

$$F(z) = {}_5F_4\left(\begin{matrix} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 1, 1, 1, 1 \end{matrix} \,\middle|\, z\right) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^5}{n!^5} z^n,$$

which satisfies the 5th-order linear differential equation

$$\bigl(\theta^5 - z(\theta+\tfrac12)^5\bigr)Y = 0, \qquad \text{where} \quad \theta = z\frac{\mathrm d}{\mathrm dz}.$$

If $G(z)$ is another solution of the latter equation, then

$$\widetilde F(z) = (1-z)^{-1/2}\det\begin{pmatrix} F & G \\ \theta F & \theta G \end{pmatrix}^{1/2}$$

(the sharp normalization factor $(1-z)^{-1/2}$ is due to Y. Yang) satisfies the 4th-order equation

$$\Bigl(\theta^4 - \frac{z}{64}(128\theta^4+256\theta^3+304\theta^2+176\theta+39) + z^2(\theta+1)^4\Bigr)Y = 0,$$

which imitates all properties of a differential equation for the periods of a Calabi–Yau threefold (this is entrance #204 in [2], Table A). Are there analogues of Hilbert class fields for this and similar situations?

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